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A new modification of the Adomian decomposition method for nonlinear integral equations

Communication

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Abstract: In this paper, we present a new modification of Adomian decomposition method (ADM) for finding exact solutions of nonlinear integral equations. In [Appl. Math. Comput. 170 (2005) 570-583], the author introduced a modification of ADM, namely two step Adomian decomposition method (TSADM) that facilitates the calculations. However, there is a weakness in TSADM which we invistigate in this paper. We propose a new reliable modification of the ADM and apply this new modified method to solve the Volterra and Fredholm integral equations. Some examples are given to illustrate the ability and reliability of this new modified method. The results reveal that our modified method is very simple and effective.

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1. Introduction

Adomian [1, 2] has presented and developed a so-called decomposition method for solving algebraic, differential, integro-differential, differential-delay, and partial differential equations. The solution is found as an infinite series which converges rapidly to accurate solutions. This method has proven successful in dealing with both linear as well as nonlinear problems, as it yields analytical solutions and offers certain advantages over standard numerical methods. It is free from rounding off errors since it does not involve discretization, and is computationally inexpensive.

The method has many advantages over the classical techniques, mainly, it makes the linearization unnecessary, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously.

In recent decades, there has been a great deal of interest in the Adomian decomposition method. The method was successfully applied to a large amount of equations which have applications in applied sciences. For more details about the method and its application, see [1, 13] and the references cited there. Recently, a modification of the Adomian decomposition method was proposed in [11]. The modified decomposition method needs only a slight variation from the standard Adomian method and has been shown to be computationally efficient in several examples. However, the modified method was established based on the assumption that the function f can be divided into two parts, and its success depends mainly on the proper choice of the parts f_1 and f_2. Moreover, the criterion of dividing the function f into two practical parts, and the case when f consists only of one term remains unsolved

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so far. Furthermore, as will be seen from the examples below, the modified method does not always minimize the size of calculations needed and even requires much more calculations than the standard Adomian method. In [8] a two step Adomian decomposition method (TSADM) was introduced to facilitate the calculations. However, there is a weakness in TSADM which we discuss in this paper. The modifed Adomian decomposition method(MADM) provides the solution by using one iteration only if compared with the standard Adomian method. A comparative study between the MADM and TSADM is conducted to illustrate the efficiency of the TSADM [8].
In this paper, we present an efficient modification of the ADM to solve nonlinear integral equations. It is important to note that the new modification reduces the size of calculations compared to the standard Adomian decomposition method. This paper is organized as follows:
In Section 2, we recall the ADM, MADM and TSADM. The analysis of the proposed new modifed Adomian decomposition method is given in Section 3. Then in Section 4, the proposed method is implemented to some examples. Finally, concluding remarks are presented in Section 5.

2. Description of methods

Consider the differential equation

$$Lu + Ru + Nu = g, \tag{1}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms and g is the source term. From Eq. (1), we have

$$Lu = g - Ru - Nu. \tag{2}$$

Since L is invertible, applying the inverse operator L^{-1} to both sides of Eq. (2), and using the given conditions we obtain

$$u = \Phi + L^{-1}g - L^{-1}(Ru) - L^{-1}(Nu), \tag{3}$$

where the function Φ represents the terms arising from using the given conditions, all of which are assumed to be prescribed. For nonlinear equations, the nonlinear operator $Nu = F(u)$ is usually represented by an infinite series of the so-called Adomian polynomials

$$F(u) = \sum_{k=0}^{\infty} A_k, \tag{4}$$

where A_k are generated for all kinds of nonlinearity. Specific algorithms were set in [4–7, 9] to formulate Adomian polynomials. Employing the Adomian method the series solution u is defined as

$$u = \sum_{n=0}^{\infty} u_n, \tag{5}$$

where the components $u_0, u_1, u_2, \ldots$ are usually determined recursively by:

$$\begin{cases} u_0 = \Phi + L^{-1}g, \\ u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), \quad k \geqslant 0. \end{cases} \tag{6}$$

2.1. The modified Adomian decomposition method [11, 14, 15]

It is important to note that the modified decomposition method was established based on the assumption that the function $f = \Phi + L^{-1}g$ can be divided into two parts, namely f_1 and f_2. Under this assumption we set

$$f = f_1 + f_2. \tag{7}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the zeroth component u_0, whereas the remaining part f_2 be combined with the other terms given in (6) to define u_1. Consequently, the modified recursive relation:

$$\begin{cases} u_0 = f_1, \\ u_1 = f_2 - L^{-1}(Ru_0) - L^{-1}(A_0), \\ u_{k+2} = -L^{-1}(Ru_{k+1}) - L^{-1}(A_{k+1}), \quad k \geqslant 0 \end{cases} \tag{8}$$

was developed.

Remark 2.1.
We point out that the necessary condition for the modified method given in [11] is that the exact solution u must appear as a part of f among other terms and we consider it as f_1.

2.2. Two step Adomian decomposition method [8]

Another modification of ADM, namely two-step Adomian decomposition method was introduced in [8]. In this method, Eq. (3)

$$\varphi = \Phi + L^{-1}g,$$

and it is assumed that the exact solution u appears in φ. Then φ is written as

$$\varphi = \varphi_0 + \varphi_1 + ... + \varphi_m, \tag{9}$$

where, $\varphi_0, \varphi_1, ..., \varphi_m$ are the terms arising from integrating the source term g and from using the given conditions. Based on this, u_0 is defined as

$$u_0 = \varphi_k + \varphi_{k+1} + ... + \varphi_{k+s}, \tag{10}$$

where $k = 0, 1, ..., m$, $s = 0, 1, ..., m - k$. If u_0 satisfied the original equation and the given conditions by substitution, then u_0 is the exact solution. Otherwise, the SADM shoud be used.

Remark 2.2.
We point out that the advantage of TSADM compared to MADM [11] is that this method finds suitable f_1 in first instance. But like MADM, in this method also the exact solution u must appear as parts of f among other terms.

We will address this problem in this paper. In this paper we solve this problem. We assigned $f = f_1 + f_2$, where $u_0 = f_1 = \sum_{m=0}^{N} \alpha_m v_m(x)$, that this concept is improved (TSADM).

3. A new modified Adomian decomposition method

The modified Adomian decomposition method provides the exact solution by using one iteration only. In this section we present a new method which we call new modified Adomian decomposition method (NMADM). We discuss only u_0 and u_1 which after one iteration gives exact solution(s). The solution is usually a unique solution, but in this work we will present an example that give two solutions. In this method the rate of convergence is accelerated. The ADM usually gives one solution among other solutions. However, as we will see in an example below, our method can give more than one solution. To achieve our goal, we first rewrite Eq. (3) as:

$$u = \sum_{m=0}^{N} \alpha_m v_m(x) - \sum_{m=0}^{N} \alpha_m v_m(x) + \Phi + L^{-1}g - L^{-1}(Ru) - L^{-1}(Nu), \tag{11}$$

where $\alpha_m, m = 0, 1, 2, ..., N$ are called the accelerating components of the parameter, and $v_m(x), m = 0, 1, 2, ..., N$ are selective functions. Further more, the number of the terms in u_0, namely N, is small in many practical problems.

Remark 3.1.
We get $\alpha_m, m = 0, 1, 2, ..., N$, and $v_m(x), m = 0, 1, 2, ..., N$ where $v_m(x)$ is form of function $g(x)$ accordingly we will obtain the exact solution, if we could not find $\alpha_m, m = 0, 1, 2, ..., N$ with $v_m(x), m = 0, 1, 2, ..., N$ so this method is not effective and this is a weakness in [10]. But if we increased N, $N \longrightarrow \infty$, we will obtain exact solution by Taylor series method.

We discussed an example (3) about this case.
We recall the modified decomposition method was established based on the assumption that the function $f = \sum_{m=0}^{N} \alpha_m v_m(x) - \sum_{m=0}^{N} \alpha_m v_m(x) + \Phi + L^{-1}g$ can be divided into two parts, namely f_1 and f_2. Under this assumption we set

$$f = f_1 + f_2, \tag{12}$$

where $f_1 = \sum_{m=0}^{N} \alpha_m v_m(x)$ and $f_2 = -\sum_{m=0}^{N} \alpha_m v_m(x) + \Phi + L^{-1}g$. Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the zeroth component u_0, whereas the remaining part f_2 be combined with the other terms given in (6) to define u_1. Consequently, the modified recursive relation:

$$\begin{cases} u_0 = \sum_{m=0}^{N} \alpha_m v_m(x), \\ u_1 = -\sum_{m=0}^{N} \alpha_m v_m(x) + \Phi + L^{-1}g - L^{-1}(Ru_0) - L^{-1}(A_0), \\ u_{k+2} = -L^{-1}(Ru_{k+1}) - L^{-1}(A_{k+1}), \quad k \geqslant 0, \end{cases} \tag{13}$$

was developed.

4. Numerical examples

This section contains some examples of nonlinear Volterra and Fredholm integral equations.

Example 4.1.
[14] Consider the following nonlinear Fredholm integral equation

$$y(x)=\frac{7}{8}x+\frac{1}{2}\int_0^1 xty^2(t)\,dt \tag{14}$$

with the exact solutions $y(x)=x,\ 7x$.

In this example TSADM is not effective because we can not define Eq. (9), which is a weakness in this method. We apply our new modified adomian decomposition method and we see that this weakness is eliminated. We get $N=1$ then

$$y(x)=\sum_{m=0}^{1}\alpha_m x^m-\sum_{m=0}^{1}\alpha_m x^m+\frac{7}{8}x+\frac{1}{2}\int_0^1 xty^2(t)\,dt. \tag{15}$$

In view of Eq. (13) we have

$$\begin{cases} y_0(x)=\sum_{m=0}^{1}\alpha_m x^m,\\ y_1(x)=-\sum_{m=0}^{1}\alpha_m x^m+\frac{7}{8}x+\frac{1}{2}\int_0^1 xty_0^2\,dt=0. \end{cases} \tag{16}$$

Now we find $\alpha_m,\ m=0,1$ in such a way that $y_1=0$. If $y_1=0$ then $y_2=y_3=...=0$, and the exact solution will be obtained as $y(x)=y_0(x)$. Hence for all values of x we have

$$\begin{cases} -\alpha_0=0,\\ \frac{1}{24}(21-24\alpha_1+6\alpha_0^2+8\alpha_0\alpha_1+3\alpha_1^2)=0. \end{cases}$$

Solving the above algebric equations we have $\alpha_0=0$ and $\alpha_1=1,7$. So, the solutions will be as follows

$$y(x)=x,\quad y(x)=7x,$$

which are the same as the exact solutions.

Example 4.2.
[15] Consider the following system of nonlinear Volterra integral equations

$$u(x)=2e^x-\frac{1}{2}e^{-2x}-\frac{1}{2}-\int_0^x\left\{u(t)+v^2(t)\right\}dt$$

$$v(x)=\frac{1}{2}e^{2x}+\frac{1}{2}-\int_0^x\left\{u^2(t)+v(t)\right\}dt$$

with the exact solutions $(u(x),v(x))=(e^x,e^{-x})$.

In this example TSADM is not effective because we can not define Eq. (10). This is a weakness in this method. But when we apply the NMADM, the weakness is eliminated. We get $N=2$ then

$$u(x)=\alpha_0+\alpha_1e^x+\alpha_2e^{-x}-\left(\alpha_0+\alpha_1e^x+\alpha_2e^{-x}\right)+2e^x-\frac{1}{2}e^{-2x}-\frac{1}{2}-\int_0^x\left\{u(t)+v^2(t)\right\}dt$$

$$v(x)=\beta_0+\beta_1e^x+\beta_2e^{-x}-\left(\beta_0+\beta_1e^x+\beta_2e^{-x}\right)+\frac{1}{2}e^{2x}+\frac{1}{2}-\int_0^x\left\{u^2(t)+v(t)\right\}dt$$

In view of Eq. (13) we have

$$u_0(x)=\alpha_0+\alpha_1e^x+\alpha_2e^{-x},$$

$$u_1(x)=-\left(\alpha_0+\alpha_1e^x+\alpha_2e^{-x}\right)+2e^x-\frac{1}{2}e^{-2x}-\frac{1}{2}-\int_0^x\left\{u_0(t)+v_0^2(t)\right\}dt$$

$$v_0(x)=\beta_0+\beta_1e^x+\beta_2e^{-x},$$

$$v_1(x)=-\left(\beta_0+\beta_1e^x+\beta_2e^{-x}\right)+\frac{1}{2}e^{2x}+\frac{1}{2}-\int_0^x\left\{u_0^2(t)+v_0(t)\right\}dt$$

Now we find α_m and β_m, $m=0,1,2$ in such a way that $u_1$and $v_1=0$. If $u_1=v_1=0$ then all components $u_i=0$ and $v_i=0$, $i=1,2,\cdots$, and the exact solution will be obtained as $u(x)=u_0(x)$ and $v(x)=v_0(x)$, hence for all values of x we should have

$$\begin{cases} -\frac{1}{2}\alpha_2^2=0, \quad \frac{1}{2}\beta_1^2=0, \quad \frac{1}{2}-\frac{1}{2}\alpha_1^2=0, \quad -\frac{1}{2}+\frac{1}{2}\beta_2^2=0, \\ \alpha_0+\beta_0^2+2\beta_1\beta_2=0, \quad -\alpha_1+2-(\alpha_1+2\beta_0\beta_1)=0, \\ -\alpha_0-\frac{1}{2}-(\alpha_2+2\beta_0\beta_2)+(\alpha_1+2\beta_0\beta_1)-\frac{1}{2}\beta_2^2+\frac{1}{2}\beta 1^2=0 \\ -\alpha_2-(\alpha_2+2\beta_0\beta_2)=0, \quad \beta_0+\alpha_0^2+2\alpha_1\alpha_2=0, \\ -\beta_0+\frac{1}{2}-((\beta_2+2\alpha_0\alpha_2)-(\beta_1+2\alpha_0\alpha_1)+\frac{1}{2}\alpha_2^2-\frac{1}{2}\alpha_1^2)=0 \\ \beta_1-(\beta_1+2\alpha_0\alpha_1)=0, \quad \beta_2-(\beta_2+2\alpha_0\alpha_2)=0 \end{cases}$$

Solving the above algebraic equations we have $\alpha_0=\alpha_2=\beta_0=\beta_1=0$, and $\alpha_1=\beta_2=1$. So, the solution is as follows

$$(u(x),v(x))=(e^x,e^{-x}),$$

which is the same as the exact solution.

Example 4.3.

[14] Consider the following nonlinear Volterra integral equation

$$y(x)=1-x+x^2-\frac{2}{3}x^3-\frac{1}{5}x^5+\int_0^x y^2(t)\,dt, \tag{17}$$

with the exact solution $y(x)=1+x^2$.

We apply this new modified adomian decomposition method. We get $N=2$ then

$$y(x)=\sum_{m=0}^{2}\alpha_m x^m-\sum_{m=0}^{2}\alpha_m x^m+1-x+x^2-\frac{2}{3}x^3-\frac{1}{5}x^5+\int_0^x y^2(t)\,dt, \tag{18}$$

substituting Eq. (5) into Eq. (18) we have

$$\begin{cases} y_0(x)=\sum_{m=0}^{2}\alpha_m x^m, \\ y_1(x)=-\sum_{m=0}^{2}\alpha_m x^m+1-x+x^2-\frac{2}{3}x^3-\frac{1}{5}x^5+\int_0^x y_0^2(t)\,dt=0. \end{cases} \tag{19}$$

Now we find α_m, $m=0,1,2$ in such a way that $y_1=0$. If $y_1=0$ then $y_2=y_3=...=0$, and the exact solution will be obtained as $y(x)=y_0(x)$, hence for all values of x we have

$$\begin{cases} 1-\alpha_0=0, \\ -1+\alpha_0^2-\alpha_1=0, \\ 1+\alpha_0\alpha_1-\alpha_2=0, \\ \frac{1}{3}(-2+\alpha_1^2+2\alpha_0\alpha_2)=0, \\ \frac{1}{2}\alpha_1\alpha_2=0, \\ \frac{1}{5}(-1+\alpha_2^2)=0. \end{cases}$$

Solving the above algebraic equations, we have $\alpha_0=1,\alpha_1=0$ and $\alpha_2=1$. So, the solution will be

$$y(x)=y_0(x)=1+x^2,$$

which is the same as the exact solution.

Example 4.4.

[14] Consider the following nonlinear Volterra integral equation

$$y(x)=x+\int_0^x y^2(t)\,dt, \tag{20}$$

with the exact solution $y(x)=tan(x)$.

We apply this new modified adomian decomposition method and get $N=1$. Then

$$y(x)=\sum_{m=0}^{1}\alpha_m x^m-\sum_{m=0}^{1}\alpha_m x^m+x+\int_0^x y^2(t)\,dt, \tag{21}$$

In view of Eq. (13) we have

$$\begin{cases} y_0(x)=\alpha_0+\alpha_1 x, \\ y_1(x)=-\alpha_0-\alpha_1 x+x+\displaystyle\int_0^x (\alpha_0+\alpha_1 t)^2 dt=0. \end{cases} \tag{22}$$

Now we find α_m, $m=0,1$ in such a way that $y_1=0$. If $y_1=0$ then $y_2=y_3=...=0$, and the exact solution will be obtained as $y(x)=y_0(x)$. hence for all values of x we have

$$\begin{cases} -\alpha_0=0, \\ -\alpha_1+1+\alpha_0^2=0, \\ \alpha_0\alpha_1=0, \\ \dfrac{\alpha_1^2}{3}=0. \end{cases}$$

From these algebraic equations we cannot get the value of α_1 because of made a counteraction. So, we increase N and let

$$\begin{cases} y_0(x)=\displaystyle\sum_{m=0}^{\infty}\alpha_m x^m, \\ y_1(x)=-\displaystyle\sum_{m=0}^{\infty}\alpha_m x^m+x+\int_0^x\left(\sum_{m=0}^{\infty}\alpha_m t^m\right)^2 dt. \end{cases} \tag{23}$$

Consequently, we have series solution of the form

$$y(x)=\sum_{m=0}^{\infty}\alpha_m x^m=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots=\tan(x),$$

which is the same as the exact solution.

Remark. We note that in this NMADM, when algebraic equations cannot be solved and N is taken to be infinity, we obtain the series solution method for solving integral equations.

Example 4.5.

[14] Consider the following nonlinear Fredholm integral equation

$$y(x)=\cos x+\sin x-\frac{\pi+2}{8}+\frac{1}{4}\int_0^{\frac{\pi}{2}} y^2(t)\,dt, \tag{24}$$

with the exact solutions $y(x)=\cos x+\sin x$.

Now we apply this new modified adomian decomposition method as follows:

$$y(x)=\alpha\cos x+\beta\sin x+c-\alpha\cos x-\beta\sin x-c+\cos x+\sin x-\frac{\pi+2}{8}+\frac{1}{4}\int_0^{\pi/2} y^2(t)\,dt, \tag{25}$$

and according to this method we choose

$$\begin{cases} y_0(x)=\alpha\cos x+\beta\sin x+c, \\ y_1(x)=-\alpha\cos x-\beta\sin x-c+\cos x+\sin x-\dfrac{\pi+2}{8}+\dfrac{1}{4}\displaystyle\int_0^{\pi/2} y^2(t)\,dt=0. \end{cases} \tag{26}$$

Now we find α and β in such a way that $y_1=0$. If $y_1=0$ then $y_2=y_3=...=0$, and the exact solution will be obtained as $y(x)=y_0(x)$, hence for all values of x we have

$$\begin{cases} 1-\alpha=0, \\ 1-\beta=0, \\ -c+\dfrac{1}{8}(-2-\pi)+\dfrac{1}{16}(2c^2\pi+4\alpha\beta+8c(\alpha+\beta)+\pi(\alpha^2+\beta^2))=0, \end{cases}$$

and by solving the above algebraic equations we have $\alpha=1,\beta=1$ and $c=0$. So, the solution is

$$y(x)=y_0(x)=\cos x+\sin(x),$$

which is the same as the exact solution.

5. Concluding remarks

In this paper, we proposed a new modification of the Adomian decomposition method for finding exact solutions of nonlinear integral equations. the method was applied to solve several Volterra and Fredholm integral equations. These example showed that this new modified method is reliable and effective. The weaknesses of the two pervious modifications of the Adomian decomposition method were also discussed in this paper.

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