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Numerical solution of Bagley-Torvik equation using Chebyshev wavelet operational matrix of fractional derivative

Research Article

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Abstract: In this paper Chebyshev wavelet and their properties are employed for deriving Chebyshev wavelet operational matrix of fractional derivatives and a general procedure for forming this matrix is introduced. Then Chebyshev wavelet expansion along with this operational matrix are used for numerical solution of Bagley-Torvik boundary value problems. The error analysis and convergence properties of the Chebyshev wavelet method are investigated.

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Keywords: Chebyshev wavelet • Fractional derivatives • Operational matrix • Bagley-Torvik equation • Tau method

1. Introduction

The idea of derivatives of noninteger order was initially appeared in a letter from Leibniz to L'Hospital in 1695. For three centuries, studies of the theory of fractional order were mainly constraint to the field of pure theoretical mathematics, which were only useful for mathematicians. In the last several decades, many researchers found that derivatives of non-integer order are very suitable for the description of various physical phenomena such as damping laws, diffusion process, etc. These findings invoked the growing interest of studies of the fractional calculus in various fields such as physics, chemistry and engineering. For these reasons we need reliable and efficient techniques for the solution of fractional differential equations [1–5].
The Bagley-Torvik equation is a kind of fractional differential equation that apeare in the studies on behavior of real material by use of fractional calculus [6, 7]. It has many applications in engineering and applied sciences fields, for mor details see[3]. So, numerical solution of Bagley-Torvik fractional differential equation has attracted many attention and has been studied by many authors. Several methods such as Adomian decomposition method [8, 9], HeâĂŹs variational iteration method [10], Taylor collocation method [11] have been used to solve this fractional differential equation. Diethelm [12] transformed this equation into first-order coupled fractional differential equation and solved the problem with Adams predictor and corrector approach. Podlubny used successive approximation method to solve the equation and recently applied the matrix approach to discretization of fractional derivatives for the same problem [3, 13].
Wavelets theory is a relatively new and an emerging area in mathematical research. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit the accurate representation of a variety of functions and operators [14–16]. In this paper The Chebyshev wavelets are first introduced, then by using shifted Chebyshev polynomial and their properties the operational matrix of derivative and fractional derivative are derived. Then, applications of these operational matrices for solving fractional order Bagley-Torvik boundary value problems are described. Illustrative examples are given to demonstrate the efficiency

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and capability of the proposed method. The error analysis and convergence properties of the Chebyshev wavelet method are investigated.
The article is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory and Chebyshev polynomials. In Section 3 the Chebyshev wavelet and its operational matrix of derivatives and fractional derivative are derived and general procedure for forming these matrices are introduced. Fractional order boundary value problems are introduced in Section 4 and then a method based on Chebyshev wavelet and its operational matrices is establish for solving this fractional boundary value problems. Numerical examples are included in section 5. Finally, a conclusion is given in Section 6.

2. Basic definitions

There are various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition and Riemann-Liouville's definition. The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity [3, 4].

Definition 2.1.

A real function $f(t), t>0$, is said to be in the space C_{μ}, $\mu\in\mathbb{R}$ if there exists a real number $p(>\mu)$ and a function $f_1(t)C[0,\infty)$ such that $f(t)=t^p f_1(t)$, and it is said to be in the space $C_{\mu}^{n}, n\in N$ if $f^{(n)}C$.

Definition 2.2.

The Riemann-Liouville fractional integration operator of order $\alpha\geq 0$ of a function $f\in C_{\mu}, \geq -1$, is defined as:

$$D^{\alpha}f(t)=\begin{cases}\frac{d^n f(t)}{dt^n}, & \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, & 0\leq n-1<\alpha<n.\end{cases} \tag{1}$$

Definition 2.3.

The fractional derivative of order $\alpha>0$ in the Caputo sense is defined as

$$D_*^{\alpha}f(t)=\begin{cases}\frac{d^n f(t)}{dt^n}, & \alpha=n\in\mathbb{N},\\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\ t>0, & 0\leq n-1<\alpha<n.\end{cases} \tag{2}$$

The useful relation between the Riemann-Liouvill operator and Caputo operator is given by the following expression

$$I^{\alpha}D_*^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!},\ t>0,\quad (n-1<\alpha\leq n). \tag{3}$$

Also for the Caputo's derivative we have $D_*^{\alpha}c=0$, in which c is a constant and

$$D_*^{\alpha}t^n=\begin{cases}0, & n<\lceil\alpha\rceil,\ n\in\mathbb{Z}^+\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}, & n\geq\lceil\alpha\rceil,\ n\in\mathbb{Z}^+\end{cases}, \tag{4}$$

where we use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. For more details about fractional calculus and its properties see [3].

2.1. Shifted Chebyshev polynomials

The sequence of Chebyshev polynomials $\{T_n(t)\}_{n=0}^{\infty}$ which are orthogonal with respect to the weight function $w(t)=\frac{1}{\sqrt{1-t^2}}$ defined on the interval $[-1,1]$ and they obey the following recurrence relation [20]

$$T_0(t)=1, T_1(t)=t,$$

$$T_n(t)=2T_{n-1}(t)-T_{n-2}(t),\quad n\geq 2.$$

for useing Chebyshev polynomials on the interval $[0,1]$ we define the so-called shifted Chebyshev polynomials by introducing the change of variable $t=2x-1$. So, the shifted Chebyshev polynomials $T_n^*(x)$ on [0,1] can be obtained as

$$T_n^*(x)=T_n(2x-1).$$

The orthogonality condition for shifted polynomials is

$$\int_0^1 \frac{T_m^*(x)T_n^*(x)}{\sqrt{1-(2x-1)^2}}dx = \begin{cases} \frac{\pi\gamma_m}{4}, & m=n, \\ 0, & m\neq n, \end{cases} \tag{5}$$

where

$$\gamma_m = \begin{cases} 2, & m=0, \\ 1, & m\geq 1. \end{cases}$$

Lemma 2.1.
The analytic form of the shifted Chebyshev polynomial $T_m^*(x)$ of degree m is given by

$$T_m^*(x) = \sum_{i=0}^{m} a_{mi}x^{m-i}, \tag{6}$$

where

$$a_{mi} = (-1)^i 2^{2m-2i}\frac{m(2m-i-1)!}{(i)!(2m-2i)!}. \tag{7}$$

Proof. See [17]. □

3. Chebyshev wavelets

In this section, we introduce Chebyshev wavelet and its operational matrix. Chebyshev wavelets $\psi_{nm}(x) = \psi(k,n,m,x)$ are defined on the interval $[0,1)$ by

$$\psi_{nm}(x) = \begin{cases} 2^{\frac{k+1}{2}}\tilde{T}_m(2^k x-(2n+1)), & \frac{n}{2^k}\leq x\leq\frac{n+1}{2^k} \\ 0, & otherwise \end{cases}, \tag{8}$$

where

$$\tilde{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m=0 \\ \sqrt{\frac{2}{\pi}}T_m(x), & m>0 \end{cases},$$

and $n = 0,1,\ldots,2^k-1$, $k\in\mathbb{N}$, $m\in Z_M = \{0,1,2,...,M-1\}$ for a fixed positive integer M. The Chebyshev wavelets $\{\psi_{nm}(x)|n = 0,1,\ldots,2^k-1, m\in Z_M\}$ forms an orthonormal basis for $L^2_{w_n}[0,1]$ with respect to the weight function $w_n(x) = w(2^{k+1}x-(2n+1))$, in which $w(x) = \frac{1}{\sqrt{1-x^2}}$.
Any square integralable function $f(x)$ defined over $[0,1]$ may be expanded by Chebyshev wavelets as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{9}$$

If the infinite series in (9) is truncated, then it can be written as

$$f(x) \simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{10}$$

where C and $\Psi(x)$ are $2^k M$ column vectors as

$$\begin{aligned} C &= \left[c_{00},\ldots,c_{0(M-1)}|c_{10},\ldots,c_{1(M-1)}|,\ldots,|c_{2^k0},\ldots,c_{2^k(M-1)}\right]^T, \\ \Psi(x) &= \left[\psi_{00},\ldots,\psi_{0(M-1)}|\psi_{10},\ldots,\psi_{1(M-1)}|,\ldots,|\psi_{2^k0},\ldots,\psi_{2^k(M-1)}\right]^T. \end{aligned} \tag{11}$$

Lemma 3.1.
By using the shifted Chebyshev polynomials, any component $\Psi_r(x)$ of (11) can be written as:

$$\Psi_r(x) = \psi_{nm}(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\tilde{T}_m(2^k x-n)\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]},$$

where $r = nM+m+1$, $m = 0,\ldots,M-1$, $n = 0,\ldots,2^k-1$ and $\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$ is the characteristic function defined as:

$$\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}(x) = \begin{cases} 1, & x\in[\frac{n}{2^k},\frac{n+1}{2^k}], \\ 0, & therwise. \end{cases}$$

Now we present an useful theorem about operational matrix of derivative for Chebyshev wavelets.

Theorem 3.1.
Let $\Psi(x)$ be the Chebyshev wavelet vector defined in (11), and $\alpha > 0$ $(N-1 < \alpha \le N)$ then we have

$$D_*^{\alpha}\Psi(x) = D^{\alpha}\Psi(x), \tag{12}$$

where D^{α} is the $(2^k M)\times(2^k M)$ operational matrix of fractional derivative of order α, in the Caputo sense and its (p,q)-th component is

$$[D^{\alpha}]_{pq} = \begin{cases} 0, & nM+1 \le p \le nM+\lceil\alpha\rceil, \\ 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\,w_{mq}^{\alpha}, & nM+\lceil\alpha\rceil+1 \le p \le (n+1)M. \end{cases} \tag{13}$$

in which

$$w_{mq}^{\alpha} = \sum_{i=0}^{m}\sum_{j=0}^{m-i} b_{jq}a_{mi}\binom{m-i}{j}(-1)^{m-i-j}2^{kj}n^{m-i-j} \tag{14}$$

$$b_{jq} = \left(D_*^{\alpha}(x^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}), \Psi_q(x)\right)_{w_n} \tag{15}$$

and a_{mj} is defined in (7).

Proof. For $nM+1 \le p \le nM+\lceil\alpha\rceil$, we have $0 \le m \le \lceil\alpha\rceil-1$, and consequently

$$D_*^{\alpha}\Psi_p(x) = \Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}D_*^{\alpha}T_m^*(2^k x-n) = 0, \quad n = 0,\ldots,2^k-1, \tag{16}$$

for $nM+\lceil\alpha\rceil+1 \le p \le (n+1)M$ we have $\lceil\alpha\rceil \le m \le M-1$, so $D_*^{\alpha}\Psi_p(x) \ne 0$. By using Lemma 3.1 we obtain

$$\Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\sum_{i=0}^{m}a_{mi}(2^k x-n)^{m-i}\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}. \tag{17}$$

this function is zero outside the interval $\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$. So by substitute Newton expansion of $(2^k x-n)^{m-i}$ into (17) we obtain

$$\Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\sum_{i=0}^{m}\sum_{j=0}^{m-i}a_{mi}\binom{m-i}{j}(-1)^{m-i-j}2^{kj}n^{m-i-j}x^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}, \tag{18}$$

Now by apply D_*^{α} on both sides of (18) we have:

$$D_*^{\alpha}\Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\sum_{i=0}^{m}\sum_{j=0}^{m-i}a_{mi}\binom{m-i}{j}(-1)^{m-i-j}2^{kj}n^{m-i-j}D_*^{\alpha}\left(x^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}\right). \tag{19}$$

Eq. (19) can be rewritten as

$$D_*^{\alpha}\Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\sum_{i=0}^{m}\sum_{j=0}^{m-i}a_{mi}\binom{m-i}{j}(-1)^{m-i-j}2^{kj}n^{m-i-j}f_j(x), \tag{20}$$

where

$$f_j(x) = D_*^{\alpha}\left(x^j\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}\right) = \frac{1}{\Gamma(n-\alpha)}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\frac{\frac{d^n}{dt^n}t^j}{(x-t)^{\alpha-n+1}}dt\,\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$$

$$+\int_{\frac{n+1}{2^k}}^{x}\frac{\frac{d^n}{dt^n}t^j}{(x-t)^{\alpha-n+1}}dt\,\chi_{[\frac{n+1}{2^k},1]}, \quad j = 0,\ldots,m-i. \tag{21}$$

Here, by knowing that $f_j(x)$ is zero outside the interval $[\frac{n}{2^k},\frac{n+1}{2^k}]$, it is consequence that Chebyshev wavelets expansion of this function have only components of basis Chebyshev wavelets $\Psi(x)$ that are non-zero in this interval that yield:

$$f_j(x) = \sum_{q=nM}^{(n+1)M} b_{jq}\Psi_q(x), \qquad j = 0,\ldots,m-i, \tag{22}$$

where $b_{jq} = (f_j(x),\Psi_q(x))$. Now by substitute (22) into (20), we obtain:

$$D_*^{\alpha}\Psi_p(x) = 2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\sum_{q=nM}^{(n+1)M}\left(\sum_{i=0}^{m}\sum_{j=0}^{m-i}b_{jq}a_{mi}\binom{m-i}{j}(-1)^{m-i-j}2^{kj}n^{m-i-j}\right)\Psi_q(x), \tag{23}$$

and this leads to desired results. □

Lemma 3.2.
For the integer value $\alpha = 1$ Chebyshev wavelet operational matrix of derivative can be expressed by

$$\frac{d\Psi(x)}{dt} = D\Psi(x) \tag{24}$$

where D is the $2^k M$ operational matrix of derivative defined as follow

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & F \end{pmatrix} \tag{25}$$

In which F is $M \times M$ matrix and its (r,s)-th element is defined as follow

$$F_{r,s} = \begin{cases} 2^{k+2} m \sqrt{\frac{\gamma_{r-1}}{\gamma_{s-1}}} & r = 2,...,M,\ s = 1,...,r-1 \ and\ (r+s)\ odd \\ 0 & o.w \end{cases} \tag{26}$$

Proof. It is an immediate consequence of Theorem 3.1. □

Corollary 3.1.
By using Eq. (24) the operational matrix for n-th derivative can be derived as

$$\frac{d^n\Psi(x)}{dx^n} = D^n\Psi(x), \tag{27}$$

where D^n is the n-th power of matrix D.

3.1. Error analysis

In this section we give the convergence properties and error bound for the Chebyshev wavelets expansion.

Theorem 3.2.
Any square integrable function $f(x)$ in $[0,1)$ with bounded second derivative $|f''(x)| \le B$ can be expressed in terms of Chebyshev wavelets, and the series converges uniformly to $f(x)$, that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{28}$$

where $c_{nm} = (f(x), \psi_{nm}(x))$ and $(.,.)$ denotes the inner product on $L^2_{w_n}[0,1]$.

Proof. Please see [18]. □

Theorem 3.3.
Suppose $f(x)$ be a continuous function defined on $[0,1)$, with bounded second derivative $|f''(x)| \le B$, and $\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)$ be its approximated value using Chebyshev wavelets, then we have the following accuracy estimation

$$\sigma_{M,k} = \frac{B\sqrt{\pi}}{8}\left(\sum_{n=2^k}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(m-1)^4}\right)^{\frac{1}{2}} \tag{29}$$

where

$$\sigma_{k,M} = \left(\int_0^1 \left[f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right]^2 w_n(x)dx\right)^{\frac{1}{2}}$$

Proof. Please see [18]. □

4. Application and results

In this section, in order to show the high importance of operational matrix of derivative, we apply it to solve fractional order Bagley-Torvikboundary value problems. These problems are considered because closed form solutions are available for them, or they have also been solved using other numerical schemes. This allows one to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes. The general Bagley-Torvik boundary value problems of order 2 have the form [19]

$$\left(A_0D^2+A_1D_*^{\frac{3}{2}}+A_2\right)y(t)=f(t), \quad t\in[0,\, T], \tag{30}$$

subject to initial-boundary conditions

$$\begin{cases} \gamma_0 y(0)+\gamma_1 y^{'}(0)=\alpha_0, \\ \gamma_3 y(T)+\gamma_4 y^{'}(T)=\alpha_1, \end{cases} \tag{31}$$

where A_0, A_1, A_2, α_0,γ_0, γ_1, γ_3,γ_4, α_0 and α_1 are constants with $A_0\neq 0$, and $y\in L_1[0,\, T]$. The existence and uniqueness of the exact solution of the solution for these problems are discussed in [3]. Here we introduce a new method based on Chebyshev wavelets expansion and their operational matrices of derivatives. To solving the Bagley-Torvik boundary value problems of the form (30) subject to the conditions (31) we approximate the $y(t)$ and $f(t)$ by the Chebyshev wavelets as

$$y(t)\simeq C^T\Psi(t), \quad f(t)\simeq F^T\Psi(t) \tag{32}$$

where C and F are coefficient vector of Chebyshev wavelets expansion for functions $y(t)$ and $f(t)$. By using theorems 3.1 we have

$$\begin{cases} D^2y(x)\simeq C^TD^2\Psi(x), \\ D^{\frac{3}{2}}y(x)\simeq C^TD^{\frac{3}{2}}\Psi(x), \end{cases} \tag{33}$$

Substituting Eqs. (37)in (30) the residual $R(x)$ can be derived as

$$R(x)\simeq\left(A_0C^TD^2+A_1C^TD^{\frac{3}{2}}+A_2C^T\right)\Psi(x). \tag{34}$$

By using typical tau method [20] we generate 2^kM-2 linear equations by applying

$$\left\langle R_{2^kM}(x),\Psi_j(x)\right\rangle=0, \quad j=0,1,...,2^kM-2 \tag{35}$$

Also, by substituting boundary conditions we get

$$\begin{cases} \gamma_0C^T\Psi(0)+\gamma_1C^TD\Psi(0)=\alpha_0, \\ \gamma_3C^T\Psi(T)+\gamma_4C^TD\Psi(T)=\alpha_1, \end{cases} \tag{36}$$

Eqs. (35) and (36) together generate 2^kM set of linear equations. These linear equations can be solved for unknown coefficients of the vector C. By substituting the derived vector C in Eq. (37) approximation solution $y(x)$ can be obtained.

5. Numerical experiments

In this section we will consider the three fractional order Bagley-Torvik boundary value problems. We used the method described in the Section 4 for solving these problems. The algorithms are performed by Maple 12 with 30 digits precision.

Example 5.1.

Consider the following boundary value problem in the case of the inhomogeneous Bagley-Torvik equation [19]

$$\begin{cases} D^2y(x)+D^{\frac{3}{2}}y(x)+y(x)=t^2+4\sqrt{\frac{t}{\pi}}+2 \\ y(0)=0, \quad y(5)=25. \end{cases} \tag{37}$$

Where the exact solution of this problem is $y(x)=x^2$. We solve this fractional boundary value problem by applying the method described in Section 4 using Chebyshev wavelets expansion and its operational matrices of derivatives with M=4, k=1. Using (35) we obtain four linear equations. And by applying boundary condition we have two linear equations. By solving this linear system we get the unknown vector C. By substituting this vectorin Eq. (37) we have the exact solution.

Table 1. The absolute errors for different values of M and k

x	$(M,k)=(6,1)$	$(M,k)=(6,2)$	$(M,k)=(12,1)$	$(M,k)=(12,2)$
4	1.1×10^{-3}	2.4×10^{-4}	3.9×10^{-6}	6.1×10^{-7}
8	4.4×10^{-4}	3.5×10^{-4}	7.6×10^{-7}	8.8×10^{-9}
12	2.6×10^{-4}	5.4×10^{-5}	8.7×10^{-8}	5.4×10^{-9}
16	2.0×10^{-4}	6.5×10^{-6}	19.5×10^{-8}	7.4×10^{-10}

Example 5.2.

Consider the boundary value problem

$$\begin{cases} D^{\frac{3}{2}}y(x)+y(x)=x^5-x^4+\frac{128}{7\sqrt{\pi}}x^{3.5}-\frac{64}{5\sqrt{\pi}}x^{2.5} \\ y(0)=0\,,\quad y(1)=0. \end{cases} \tag{38}$$

where the exact solution of this problem is $y(x)=x^4(1-x)$. We solve this fractional boundary value problem by applying the method described in Section 4 by using Chebyshev wavelets expansion $M=6, k=2$. Similar to Example 1 by solving linear system derived for this problem we get the exact solution.

Example 5.3.

Consider the boundary value problem

$$\begin{cases} D^2y(x)+\frac{1}{2}D^{\frac{3}{2}}y(x)+\frac{1}{2}y(x)=8\nu(x)-8\nu(x-1) \\ y(0)=0\,,\ y(20)=-1.48433, \end{cases} \tag{39}$$

In which $\nu(x)$ is the Heaviside function. This example was solved theoretically by Podlubny [3] and the compact form of the solution is given by

$$y(x)=\int_0^x 8G(x-t)(\nu(t)-\nu(t-1))dt \tag{40}$$

where G is the fractional Green function defined as

$$G(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\sum_{j=0}^{\infty}\frac{(j+k)!\left(-t^{\frac{1}{2}}\right)^j}{j!2^j\Gamma\left(\frac{1}{2}j+\frac{1}{2}k+\frac{3}{2}k+2\right)} \tag{41}$$

In Eq. (41) is the Gamma function. Here we solve this problem using Chebyshev wavelets method described in Section 4 with $(M,k)=(6,1),(M,k)=(6,2),(M,k)=(12,1)$ and $(M,k)=(12,2)$. Fig. 1 shows the approximate solution and theoretical solution derived by Podlubny[3] for $(M,k)=(12,2)$ in the interval $[0,20]$. The absolute errors for different values in the interval are shown in the Table 1. From Table 1, we see that we can achieve a good approximation with the exact solution by usinga few terms of Chebyshev wavelets.

6. Conclusion

In this article a general formulation for deriving the Chebyshev wavelet operational matrix of derivatives has been derived. Then a numerical method based on Chebyshev wavelets expansion, its operational matrix of fractional order and tau method is introduced for approximate the solution of Bagley-Torvik fractional boundary value problems. Moreover, the convergence and error analysis for the proposed method is considered.

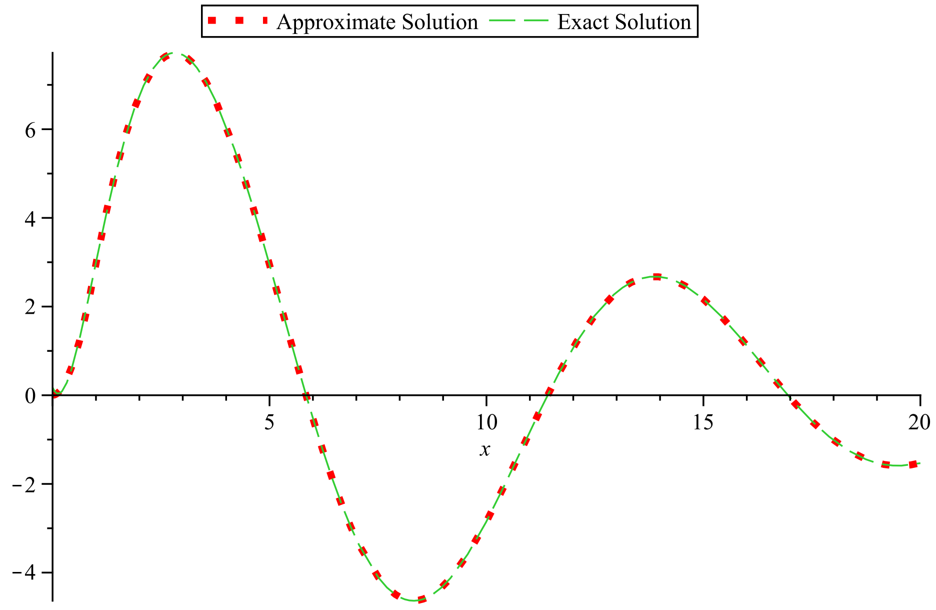

Fig. 1. Approximate solution and exact solution for Example 3 in the interval [0,20]

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