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On new class of analytic functions defined by using differintegral operator

Research Article

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Abstract: Making use a fractional differintegral operator .We introduce a new class of univalent analytic functions in the unit disk U.Among the results investigated for this class of functions include the coefficient inequalities ,starlikenees,convexity,close -to-convexity,extreme points and Integral means inequalities.

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1. Introduction

Let $\Im$ denote the class of analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$,and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (n \in N = \{1,2,3,\dots\}) \tag{1}$$

Let$H$ denote the subclass of the class $\Im$ which is the functions defined by:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0, n \in N) \tag{2}$$

A function $f \in H$ is said to be in the class of starlike functions of order $\beta (0 \leq \beta < 1)$ in U, denoted by $\delta^*(\beta)$,if

$$Re\left\{\frac{z f'(z)}{f(z)}\right\} > \beta \qquad (z \in U) \tag{3}$$

For $\beta = 0$ the class $\delta^*(0) = \delta^*$ is the class of starlike functions in U. (for details, see [1, 2])

If f and g are two analytic in U , then we say that the function $f(z)$ is subordinate to $g(z)$ in U and write $f(z) \prec g(z)$,if there exits a Schwarz function $w(z)$ (analytic in U with $w(0) = 0, and\ |w(z)| < 1$),such that $f(z) = g(w(z)), z \in U$. In particular .if the function $g(z)$ is univalent in U , the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. [3–5]

We recall here the following definition of fractional calculus (that is fractional integral and fractional derivative of an arbitrary order)consider by Owa [6]. See also [7–10]

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Definition 1.1.
The fractional integral of order $\lambda(\lambda > 0)$ is defined for a function $f(z)$ analytic in a simply-connected region of the complex plane containing the origin by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi, \tag{4}$$

where the multiplicity $(z-\xi)^{\lambda-1}$ is removed by requiring $log(z-\xi)$ to be real when $(z-\xi > 0)$.

Definition 1.2.
Under the hypothesis of definition 1.2 the fractional derivative operator of order $\lambda(\lambda \geq 0)$ is defined by :

$$D_z^{\lambda} f(z) = \begin{cases} \dfrac{1}{\Gamma(\lambda)} \dfrac{d}{dz} \displaystyle\int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi & (0 \leqslant \lambda < 1) \\ \dfrac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n-1 \leqslant \lambda < n; n \in N_0 = N \cup \{0\}) \end{cases} \tag{5}$$

where the multiplicity $(z-\xi)^{-\lambda}$ is removed as in definition 1.2

For the purpose of this paper, we define here a fractional differintegral operator

$$\Omega_z^{\lambda} : H \to H \qquad (-\infty < \lambda < 2; n \in N),$$

for a function $f(z)$ of the form (2) by:

$$\Omega_z^{\lambda} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n z^n = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \quad (-\infty < \lambda < 2; z \in U), \tag{6}$$

Definition 1.3.
A function $f(z)$ in H is in the class $H(\lambda, \gamma, \beta, \alpha, \mu)$ if it satisfies the condition

$$Re\left\{(1-\gamma) - \frac{\gamma\left(\beta\Omega_z^{\lambda} f(z) + (1-\beta)z\right) + (1-\gamma)z^2\left(\Omega_z^{\lambda} f(z)\right)''}{z\left(\Omega_z^{\lambda} f(z)\right)'}\right\} > \frac{1}{2}(\alpha+\mu), \tag{7}$$

where $0 \leq \alpha\mu < 1; 0 \leq \beta, \gamma \leq 1; -\infty < \lambda < 2; z \in U$.

Lemma 1.1.
see [11] If ζ is any complex number, then $Re(\zeta) > \tau$ if and only if $|\zeta - (1+\tau)| < |\zeta + (1-\tau)|$ where $\tau \geq 0$.

Some of the following properties have been found on other classes in [12–14].

2. Main results

Theorem 2.1.
Let the functions $f \in H$ be given by (2). Then $f \in H(\lambda, \gamma, \beta, \alpha, \mu)$ if and only if

$$\sum_{n=2}^{\infty} \left[n\left((1-\gamma)n - \frac{1}{2}(\alpha+\mu)\right) + \gamma\beta\right] \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)} a_n \leq 1 - \frac{1}{2}(\alpha+\mu). \tag{8}$$

The result is sharp for the function

$$f(z) = z - \frac{1 - \frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n - \frac{1}{2}(\alpha+\mu)\right) + \gamma\beta\right] \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}} z^n, \quad n \geqslant 2 \tag{9}$$

Proof. Suppose that $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$by Using Lemma 1.1 and letting

$$\zeta=\frac{(1-\gamma)z\left(\Omega_z^\lambda f(z)\right)'-\gamma\left(\beta(\Omega_z^\lambda f(z))+(1-\beta)z\right)+(1-\gamma)z^2\left(\Omega_z^\lambda f(z)\right)''}{z\left(\Omega_z^\lambda f(z)\right)'}=\frac{A(z)}{B(z)}.$$

Then it is sufficient prove that

$$\left|A(z)-\left(1+\frac{1}{2}(\alpha+\mu)\right)B(z)\right|<\left|A(z)+\left(1-\frac{1}{2}(\alpha+\mu)\right)B(z)\right|$$
$$=\left|-\frac{1}{2}(\alpha+\mu)z-\sum_{n=2}^{\infty}a_nz^n\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}\left[n^2(1-\gamma)+\gamma\beta-n\left(1+\frac{1}{2}(\alpha+\mu)\right)\right]\right|$$
$$<\left|\left(2-\frac{1}{2}(\alpha+\mu)\right)z-\sum_{n=2}^{\infty}a_nz^n\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}\left[n^2(1-\gamma)+\gamma\beta+n\left(1-\frac{1}{2}(\alpha+\mu)\right)\right]\right|$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n-\left(1-\frac{1}{2}(\alpha+\mu)\right)\le 0.$$

Conversely, assume that

$$Re\left\{(1-\gamma)-\frac{\gamma\left(\beta\,\Omega_z^\lambda f(z)+(1-\beta)z\right)+(1-\gamma)z^2\left(\Omega_z^\lambda f(z)\right)''}{z\left(\Omega_z^\lambda f(z)\right)'}\right\}$$
$$=Re\left\{\frac{(1-\gamma)z\left(\Omega_z^\lambda f(z)\right)'-\gamma\left(\beta(\Omega_z^\lambda f(z))+(1-\beta)z\right)+(1-\gamma)z^2\left(\Omega_z^\lambda f(z)\right)''}{z\left(\Omega_z^\lambda f(z)\right)'}\right\}\qquad(10)$$
$$=Re\left\{\frac{z-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_nz^n\left[n^2(1-\gamma)+\gamma\beta\right]}{z-\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}na_nz^n}\right\}$$

We can choose the value 0f z on the real axis , so that $z\left(\Omega_z^\lambda f(z)\right)'$is real .Let$z\to 1$. through real value , so we can write (10) as

$$\sum_{n=2}^{\infty}\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_n\le 1-\frac{1}{2}(\alpha+\mu).$$

Finally, sharpness follows if we take

$$f(z)=z-\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^n,n\ge 2.\qquad(11)$$

The proof is complete. □

Corollary 2.1.
Let the function $f\in H$ be given by (2). If $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$, then

$$a_n\le\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^n,n\ge 2.\quad(n\in N)\qquad(12)$$

The result is sharp for the function given by (9).

3. Radius of starlikeness and convexity and close-to-convexity

Theorem 3.1.
Let $f \in H(\lambda,\gamma,\beta,\alpha,\mu)$. Then f is starlike of order $\rho, 0 \le \rho < 1$ in $|z| < r = r_1(\lambda,\gamma,\beta,\alpha,\mu,\rho)$, where

$$r_1(\lambda,\gamma,\beta,\alpha,\mu,\rho) = inf_n \left\{ \frac{(1-\rho)\left[n\left((1-\gamma)n - \frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)(n-\rho)} \right\}^{\frac{1}{n-1}}, n = 2,3,\ldots.$$

The estimate is sharp for the function

$$f(z) = z - \frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}} z^n, n \ge 2 \quad (13)$$

Proof. f is starlike of order $\rho, 0 \le \rho < 1$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (14)$$

that is if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\rho, \quad (15)$$

which simplies to

$$\sum_{n=2}^{\infty} \frac{(n-\rho)a_n|z|^{n-1}}{(1-\rho)} \le 1. \quad (16)$$

By Theorem 2.1, we have

$$a_n \le \frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}, n \ge 2. \quad (17)$$

Using (16) and (17), we get

$$|z|^{n-1} \le \frac{(1-\rho)\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)(n-\rho)} \quad (18)$$

thus

$$|z| < r_1(\lambda,\gamma,\beta,\alpha,\mu,\rho) = inf_n \left\{ \frac{(1-\rho)\left[n\left((1-\gamma)n - \frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)(n-\rho)} \right\}^{\frac{1}{n-1}}, n = 2,3,\ldots.$$

□

Theorem 3.2.
Let $f \in H(\lambda,\gamma,\beta,\alpha,\mu)$. Then f is convex of order $\rho, 0 \le \rho < 1$ in $|z| < r = r_2(\lambda,\gamma,\beta,\alpha,\mu,\rho)$, where

$$r_2(\lambda,\gamma,\beta,\alpha,\mu,\rho) = inf_n \left\{ \frac{(1-\rho)\left[n\left((1-\gamma)n - \frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)n(n-\rho)} \right\}^{\frac{1}{n-1}}, n = 2,3,\ldots.$$

The estimate is sharp for the function

$$f(z) = z - \frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}} z^n, n \ge 2 \quad (19)$$

Proof. $f \in H(\lambda,\gamma,\beta,\alpha,\mu)$ is convex of order $\rho, 0 \le \rho < 1$ if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho \tag{20}$$

that is if

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\rho, \tag{21}$$

which simplies to

$$\sum_{n=2}^{\infty}\frac{n(n-\rho)a_n|z|^{n-1}}{(1-\rho)} \le 1. \tag{22}$$

By Theorem 2.1, we have

$$a_n \le \frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}, n \ge 2. \tag{23}$$

Using (22) and (23), we get

$$|z|^{n-1} \le \frac{(1-\rho)\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)n(n-\rho)} \tag{24}$$

thus

$$|z| < r_2(\lambda,\gamma,\beta,\alpha,\mu,\rho) = inf_n\left\{\frac{(1-\rho)\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)n(n-\rho)}\right\}^{\frac{1}{n-1}}, n = 2,3,\ldots.$$

□

Theorem 3.3.
Let $f \in H(\lambda,\gamma,\beta,\alpha,\mu)$. Then f is close -to-convex of order $\rho, 0 \le \rho < 1$ in $|z| < r = r_3(\lambda,\gamma,\beta,\alpha,\mu,\rho)$, where

$$r_3(\lambda,\gamma,\beta,\alpha,\mu,\rho) = inf_n\left\{\frac{(1-\rho)\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)}\right\}^{\frac{1}{n-1}}, n = 2,3,\ldots. \tag{25}$$

The estimate is sharp for the function

$$f(z) = z - \frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^n, n \ge 2 \tag{26}$$

Proof. Let $f \in H(\lambda,\gamma,\beta,\alpha,\mu)$.Then by Theorem 2.1,

$$\sum_{n=2}^{\infty}\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}a_n \le 1, \tag{27}$$

for $0 \le \rho < 1$, we need to show that $\left|f'(z)-1\right| \le 1-\rho$ for $|z| < r = r_3(\lambda,\gamma,\beta,\alpha,\mu,\rho)$, when is given by (25).
Now

$$\left|f'(z)-1\right| = \left|\sum_{n=2}^{\infty}na_nz^{n-1}\right| \le \sum_{n=2}^{\infty}na_n|z|^{n-1}. \tag{28}$$

Thus $\left|f'(z)-1\right| \le 1-\rho$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\rho}a_n|z|^{n-1} \le 1, \tag{29}$$

but by Theorem 2.1 above inequality holds true if

$$|z|^{n-1} \le \left\{\frac{(1-\rho)\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{\left(1-\frac{1}{2}(\alpha+\mu)\right)}\right\}, n = 2,3,\ldots,$$

and this completes the proof. □

4. Extreme points for the function class $H(\lambda,\gamma,\beta,\alpha,\mu)$

Theorem 4.1.
Let

$$f_1(z)=z \tag{30}$$

and

$$f_n(z)=z-\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^n,(n\in N\backslash\{1\}). \tag{31}$$

Then $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$ if and only if it can be expressed in the following form :

$$f(z)=\sum_{n=2}^{\infty}\chi_n f_n(z)$$

where $\chi_n\geq 0$ and $\sum_{n=2}^{\infty}\chi_n=1$.

Proof. Suppose that

$$f(z)=\sum_{n=2}^{\infty}\chi_n f_n(z)=z-\sum_{n=2}^{\infty}\chi_n\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^n.$$

The from Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}\chi_n$$

$$=\left[1-\frac{1}{2}(\alpha+\mu)\right]\sum_{n=2}^{\infty}\chi_n=\left[1-\frac{1}{2}(\alpha+\mu)\right](1-\chi_n)\leq 1-\frac{1}{2}(\alpha+\mu).$$

Thus ,in view of Theorem 2.1, we find that $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$.

Conversely,let us suppose that $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$.
Then, since

$$a_n\leq\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}\qquad(n\in N\backslash\{1\}),$$

we may set

$$\chi_n=\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}a_n,\qquad(n\in N\backslash\{1\})$$

and $\chi_n=1-\sum_{n=2}^{\infty}\chi_n$. Thus clearly we have

$$f(z)=z-\sum_{n=2}^{\infty}\chi_n f_n(z).$$

This completed the proof of Theorem 4.1 □

Corollary 4.1.
The extreme points of the function class$H(\lambda,\gamma,\beta,\alpha,\mu)$ are given by equations (30) and (31).

5. Integral means inequalities for the function class $H(\lambda,\gamma,\beta,\alpha,\mu)$

In the year 1925, Littlewood [15] prove the following subordination theorem.

Theorem 5.1.
If the functions f and g are analytic in U with $f(z)\prec g(z),(z\in U)$, then ,for $\delta>0$ and $z=re^{i\vartheta},(0<r<1)$,

$$\int_0^{2\Pi}\left|f(z)\right|^{\delta}d\vartheta\le\int_0^{2\Pi}\left|g(z)\right|^{\delta}d\vartheta.$$

We now make use of Theorem 5.1 to prove Theorem 5.2 below.

Theorem 5.2.
Let $f\in H(\lambda,\gamma,\beta,\alpha,\mu)$.Suppose also that f_n is defined be equation (31). If there exists an analytic function $w(z)$ given by

$$[w(z)]^{n-1}=\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}\sum_{n=2}^{\infty}a_nz^{n-1},$$

there for $z=re^{i\vartheta}$ and $(0<r<1)$,

$$\int_0^{2\Pi}\left|f(re^{i\vartheta})\right|^{\delta}d\vartheta\le\int_0^{2\Pi}\left|f_n(re^{i\vartheta})\right|^{\delta}d\vartheta\qquad(\delta>0).$$

Proof. We must show that

$$\int_0^{2\Pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^{\delta}d\vartheta\le\int_0^{2\Pi}\left|1-\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^{n-1}\right|^{\delta}d\vartheta.$$

By applying Littlewoodś subordination theorem (Theorem 5.1 above), it would suffice to show that

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec1-\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}z^{n-1}\qquad(z\in U).$$

By setting

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}=1-\frac{1-\frac{1}{2}(\alpha+\mu)}{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}[w(z)]^{n-1},$$

we find that

$$[w(z)]^{n-1}=\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}\sum_{n=2}^{\infty}a_nz^{n-1},$$

which readily yields $w(0)=0$.
Next by using equation (9), we obtain

$$|w(z)|^{n-1}\le\left|\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}\sum_{n=2}^{\infty}a_nz^{n-1}\right|$$

$$\le\frac{\left[n\left((1-\gamma)n-\frac{1}{2}(\alpha+\mu)\right)+\gamma\beta\right]\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}}{1-\frac{1}{2}(\alpha+\mu)}\sum_{n=2}^{\infty}a_n\left|z^{n-1}\right|\le|z|^{n-1}<1.$$

This completes the proof of Theorem 5.2. □

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