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Stochastic inventory system with two types of services

Research Article

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Abstract: We consider a continuous review $(s;S)$ inventory system at a service facility with two types of service and finite waiting hall. The maximum inventory level is S units. The waiting hall of size is N. We assume that the demands arrive according to a Poisson process with rate $\lambda(>0)$ and demands only single unit at a time. The demanded item is delivered to the customer after a random time of service. We consider that the server provides two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service time following exponential distribution with rate $p_i\mu(>0)$, $i=1,2$. Each arriving customer may choose either type of service. A completion of type 1 service causes one customer to leave the system and type 2 service causes one customer to leave the system and reduces the inventory by one item. The reorder level is fixed as s and an order is placed when the inventory level reaches the reorder level s. The ordering quantity for the commodity is $Q(=S-s>s+1)$ items. The lead time is assumed to be distributed as negative exponential with parameter $\beta(>0)$. The demands that occurred during stock-out periods are lost. We have derived the joint probability distribution of both the inventory level and the number of customers in the system in the steady state case. We also have derived various stationary measures of system performances and computed total expected cost rate. We have provided numerical examples to illustrate the convexity of the total expected cost rate.

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Keywords: Inventory with service time • Continuous review • Markov process

1. Introduction

Research on queueing systems with inventory control has captured much attention of researchers over the past decades. In this system, customers arrive at the service facility one by one and require service. In order to complete the customer service an item from the inventory is needed. A served customer departs immediately from the system and the on-hand inventory decreases by one at the moment of service completion. This system is called queueing -inventory system [1]. For the references on various types of queueing inventory systems with service facility, see Berman et al., [2], Berman and Kim [3, 4], Berman and Sapna [5, 6], Elango [7] and Paul Manuel et. al. [8] and the references therein.Choi and Park [9] investigated an $M/G/1$ retrial queue with two types of customers in which the service distributions for both types of customers are the same.

In this paper, we consider an inventory system with service facility. We assume that the demand realizing according to a Poisson process that are issued to the customer after a random time of service performed on it. The system has two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service time following exponential distribution. An (s,S) ordering policy with positive random lead time is adopted. This type of situation occurs frequently in real time,for example, a product is purchased if and only if the customer is convinced of the service, otherwise he leaves the system without making any purchase.

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Notations

Z	Zero matrix
$\mathbf{e}^T$	$(1,1,\dots,1)$.
I_N	Identity matrix of order N
δ_{ij}	$= \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{otherwise.} \end{cases}$
$\overline{\delta_{ij}}$	$=1\text{-}\delta_{ij}$
$\sum_{j=0}^{i} a^j$	$= \begin{cases} a^0+a^1+\cdots+a^i, & \text{if } i \text{ is non negative integer} \\ 0, & \text{otherwise} \end{cases}$
$[A]_{ij}$	(i,j)–th element of the matrix A
$\underset{i=j}{\overset{k}{\Omega}} c_i$	$= \begin{cases} c_j c_{j-1}\cdots c_k & \text{if } j \ge k \\ 1 & \text{if } j<k \end{cases}$

The remainder of this paper is organized as follows. In Section 2, we present the mathematical model and the notations. Analysis of the model and the steady state solution are given in Section 3. In Section 4, we derive various measures of system performance in steady state. The total expected cost rate is calculated in Section 5. Our numerical study is presented in Section 6. Section 7 has concluding remarks.

2. Formulation of the model

Consider a continuous review inventory system with service facility with maximum inventory of S units. The waiting hall of size is N. We assumed that the demands arrive according to a Poisson process with rate $\lambda(>0)$ and demands only single unit at a time.The demanded item is delivered to the customer after a random time of service. The server provides two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service time following exponential distribution with rate $p_i\mu(>0), i=1,2$. Each arriving customer may choose either type of service. A completion of type 1 service causes one customer to leave the system and type 2 service causes one customer to leave the system and reduces the inventory by one item. The reorder level is fixed as s and an order is placed when the inventory level reaches the reorder level s. The ordering quantity for the commodity is $Q(=S-s>s+1)$ items. The requirement $S-s>s+1$ ensures that after a replenishment the inventory level will be always above the reorder level s. Otherwise it may not be possible to place reorder which leads to perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter $\beta(>0)$. The demands that occurred during stock-out periods are lost.

3. Analysis

Let $L(t)$, $X(t)$, denote, respectively, the inventory level and number of customers (waiting and being served) in the system at time t. From the assumptions made on the input and output processes, it may be shown that

$$(L,X)=\{(L(t),X(t),);t\ge 0\}$$

on the state space $E=\{(i,j)/i=0,1,2,...5, j=0,1,2,...N\}$, is a Markov process. The infinitesimal generator of this process,

$$\tilde{A}=((\ a((i,k);(j,l)),\tag{1}$$

may be obtained by using the following arguments:

- An arrival of a customer causes a transition from (i,k) to $(i,k+1)$, $i=0,1,\dots,S$; $k=0,1,\dots,N$; .The rate for this transition is λ.
- A completion of type-1 service causes one customer to leave the system . Thus, a transition takes place from (i,k) to $(i,k-1)$, $i=1,2,\dots,S$, $k=1,2,\dots,N$, . The rate for this transition is $p_1\mu$.
- A completion of type 2 service causes one customer to leave the system and reduces the inventory by one item. Thus, a transition takes place from (i,k) to $(i-1,k-1)$, $i=1,2,\dots,S$, $k=1,2,\dots,N$,. The rate for this transition is $p_2\mu$.
- A transition from (i,k) to $(i+Q,k)$ for $i=0,1,\dots,S$; $k=0,1,\dots,N$; with intensity β when a replenishment occurs.

- For other transition from (i,k) to (j,l) except $(i,k)\neq(j,l)$, the rate is zero.
- Finally, note that

$$a((i,k);(i,k)) = -\sum_{j}\sum_{\substack{l \\ (j,l)\neq(i,k)}} a((i,k);(j,l)). \tag{2}$$

Hence, $a((i,k);(j,l))$ can be writtten as

$$\begin{cases} \lambda, & j=i, & l=j+1, \\ & i=0,1,2,\cdots,S, & k=0,1,\cdots,N-1 \\ p_1\mu, & j=i, & l=j-1, \\ & i=0,1,\cdots,S, & k=1,2,\cdots,N \\ p_2\mu, & j=i-1, & l=j-1, \\ & i=0,1,\cdots,S, & k=1,2,\cdots,N \\ \beta, & j=i+Q, & l=j, \\ & i=s,s-1,\cdots,0, & k=0,1,\cdots,N \\ -[\,\overline{\delta_{Nj}}\,\lambda+H(s-i)\beta+ & & \\ \overline{\delta_{0j}}\,(p_1\,\mu+\overline{\delta_{0i}}\,p_2\,\mu)\,], & j=i, & l=j, \\ & i=0,1,\cdots,S & k=0,1,\cdots,N \\ 0, & \text{otherwise.} & \end{cases}$$

By ordering the set of states of E as lexicographically, the infinitesimal generator

$$\tilde{A}=(a((i,k),(j,l))),\quad (i,k),(j,l)\in E,$$

can be conveniently expressed in a block partitioned matrix with entries

$$\tilde{A}=\begin{cases} A_0, & j=i, & i=s+1,s+2,\ldots,S \\ A_1, & j=i, & i=s,s-1,\ldots,2,1 \\ A_2, & j=i, & i=0 \\ B, & j=i-1, & i=1,2,\ldots,S \\ C, & j=i+Q, & i=0,1,\ldots,s \\ \mathbf{0}, & \text{otherwise.} & \end{cases}$$

More explicitly,

$$\tilde{A}=\begin{array}{c} S \\ S-1 \\ \vdots \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array}\begin{pmatrix} A_0 & B & & & & & & \\ & A_0 & B & & & & & \\ & & \ddots & \ddots & & & & \\ C & & & A_1 & B & & & \\ & C & & & A_1 & B & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & C & & & A_1 & B \\ & & & & C & & & A_2 \end{pmatrix}$$

where

$$[A_k]_{ij}=\begin{cases} \lambda & \text{if} \quad j=i+1, \quad i=0,1,\cdots,N-1 \\ p_1\mu & \text{if} \quad j=i-1, \quad i=1,2,\cdots,N \\ -[\,\overline{\delta_{Ni}}\,\lambda+\overline{\delta_{0k}}\beta+ & \\ \overline{\delta_{0i}}\,(p_1\,\mu+\overline{\delta_{2k}}\,p_2\,\mu)\,] & \text{if} \quad j=i, \qquad i=0,1,\cdots,N \\ 0 & \text{otherwise.} \end{cases}$$

with k = 0,1,2.

$$[B]_{ij} = \begin{cases} p_2\mu & \text{if} \quad j = i-1, \quad i = 1,2,\cdots,N \\ 0 & \text{otherwise.} \end{cases}$$

$$[C]_{ij} = \begin{cases} \beta & \text{if} \quad j = i, \quad i = 0,1,\cdots,N \\ 0 & \text{otherwise.} \end{cases}$$

3.1. Steady state results

It can be seen from the structure of the rate matrix $\tilde{A}$ that the homogeneous Markov process $\{(L(t), X(t)), t \geq 0\}$ on the finite state space E is irreducible. Hence, the limiting distribution of the Markov process exists. Let $\mathbf{\Phi}$, partitioned as $\mathbf{\Phi} = (\mathbf{\Phi}^{(S)}, \mathbf{\Phi}^{(S-1)}, \ldots, \mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(0)})$, denote the steady state probability vector of $\tilde{A}$. That is, $\mathbf{\Phi}$ satisfies

$$\Phi\tilde{A} = 0 \text{ and } \sum_{(i,j)\in E}\sum \phi^{(i,j)} = 1 \tag{3}$$

The first equation of the above yields the following set of equations:

$$\begin{aligned} \mathbf{\Phi}^{(i)}A_0 + \mathbf{\Phi}^{(i-Q)}C &= \mathbf{0}, \quad i = S \\ \mathbf{\Phi}^{(i+1)}B + \mathbf{\Phi}^{(i)}A_0 + \mathbf{\Phi}^{(i-Q)}C &= \mathbf{0}, \quad i = S-1, S-2, \ldots, Q \\ \mathbf{\Phi}^{(i+1)}B + \mathbf{\Phi}^{(i)}A_0 &= \mathbf{0}, \quad i = Q-1, Q-2, \ldots, s+1 \\ \mathbf{\Phi}^{(i+1)}B + \mathbf{\Phi}^{(i)}A_1 &= \mathbf{0}, \quad i = s, s-1, \ldots, 1 \\ \mathbf{\Phi}^{(i+1)}B + \mathbf{\Phi}^{(i)}A_2 &= \mathbf{0}, \quad i = 0 \end{aligned}$$

After long simplifications, the above equations, except the first one, yield

$$\begin{aligned} \boldsymbol{\phi}^{\boldsymbol{i}} &= \boldsymbol{\phi}^{(\mathbf{0})}(-1/B)^i A_2 A_1^{i-1}, & i = 1,2,\cdots,s+1 \\ &= \boldsymbol{\phi}^{(\mathbf{0})}\left((-1/B)^i A_2 A_1^s A_0^{i-s-1}\right), & i = s+2, s+3, \cdots, Q \\ &= \boldsymbol{\phi}^{(\mathbf{0})}[(-1/B)^i A_2 A_1^s A_0^{i-s-1} + \\ &(-1/B)^{i-Q} C A_0^{i-Q-1} + \\ &(-1/B)^{i-Q} C A_2 (\sum_{j=1}^{i-Q-1} A_1^{j-1} A_0^{i-Q-j-1})] & i = Q+1, Q+2, \cdots, S \end{aligned}$$

where $\boldsymbol{\phi}^{(\mathbf{0})}$ can be obtained by solving,

$$\mathbf{\Phi}^{(S)}A_0 + \mathbf{\Phi}^{(s)}C = 0$$

and

$$\sum_{i=0}^{S} \boldsymbol{\phi}^{(\boldsymbol{i})}\mathbf{e} = 1, \tag{4}$$

that is

$$\phi^{(0)}[(-1/B)^S A_2 A_1^s A_0^Q + (-1/B)^s C A_0^s + (-1/B)^s C A_2 (\sum_{j=1}^{s-1} A_1^{j-1} A_0^{s-j}) + (-1/B)^s A_2 A_1^{s-1} C] = 0,$$

and

$$\boldsymbol{\phi}^{\mathbf{0}}[I+\sum_{i=1}^{s+1}((-1/B)^{i}A_2A_1^{i-1})+\sum_{i=s+2}^{Q}((-1/B)^{i}A_2A_1^{s}A_0^{i-s-1})+\sum_{i=Q+1}^{S}[(-1/B)^{i}A_2A_1^{s}A_0^{i-s-1}+(-1/B)^{i-Q}CA_0^{i-Q-1}+$$
$$(-1/B)^{i-Q}CA_2(\sum_{j=1}^{i-Q-1}A_1^{j-1}A_0^{i-Q-j-1})]]\mathbf{e}=1.$$

4. System performance measures

In this section we derive some stationary performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

4.1. Mean inventory level

Let ζ_1 denote the average inventory level in the steady state. Then we have,

$$\zeta_1=\sum_{i=1}^{S}i\left(\sum_{k=0}^{N}\phi^{(i,k)}\right)$$

4.2. Mean reorder rate

Let ζ_2 denote the mean reorder rate, then we have,

$$\zeta_2=\sum_{k=0}^{N}p_2\mu\,\phi^{(s+1,k)}$$

4.3. Mean balking rate

Let ζ_3 denote the mean balking rate. Then we have,

$$\zeta_3=\lambda\sum_{i=0}^{S}\phi^{(i,N)}$$

4.4. Mean waiting time

Let $\overline{W}$ denote the mean waiting time of the customers. Then, by Little's formula

$$\overline{W}=\frac{\Gamma}{\lambda_a}where,\Gamma=\sum_{k=1}^{N}k\left(\sum_{i=0}^{S}\phi^{(i,k)}\right).$$

The expected arrival rate is given by

$$\lambda_a=\lambda\sum_{i=0}^{S}\sum_{k=0}^{N-1}\phi^{(i,k)}.$$

5. Cost analysis

In order to compute the total expected cost per unit time, we introduce the following notations:

- c_h : The inventory carrying cost per unit per unit time.
- c_s : The setup cost per order.
- c_w : Waiting time cost of a customer per unit time.
- c_B : balking cost per customer per unit time.

Then the long-run expected cost rate is given by

$$TC(S,s,N) = c_h\zeta_1 + c_s\zeta_2 + c_w\overline{W} + c_B\zeta_3.$$

Substituting $\zeta's$ and $\overline{W}$ into the above equation, we obtain

$$TC(S,s,N) = c_h\left(\sum_{i=1}^{S} i\left(\sum_{k=0}^{N}\phi^{(i,k)}\right)\right) + c_s\left(\sum_{k=0}^{N} p_2\mu\phi^{(s+1,k)}\right) + c_w\left(\frac{\Gamma}{\lambda_a}\right) + c_B\left(\lambda\sum_{i=0}^{S}\phi^{(i,N)}\right)$$

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Hence, a detailed computational study of the cost function is carried out in the next section.

6. Numerical analysis

Since we have not shown analytically the convexity of the function $TC(S,s,N)$ we have explored the behaviour of this function by considering it as functions of any two variable by fixing the other one at a constant value.
The Table 1 gives the total expected cost rate for various combinations of S and s when fixed values for other parameters and costs are assumed. They are $N = 14, \lambda = 19, \beta = 2.5, \mu = 17, p_1 = 0.6, p_2 = 0.4,, c_h = 0.5, c_s = 5, c_w = 0.05, c_b = 2.4$.

Table 1. Total expected cost rate for various combination of S and s

S	s			
	1	2	3	4
14	13.63756	**13.62568**	13.76442	14.04218
15	13.62893	**13.60036**	13.71230	13.94981
16	13.65123	**13.61111**	13.70356	13.91142
17	13.69922	**13.65126**	13.72948	13.91518

The Table 2 gives the total expected cost rate for various combinations of S and N. We have assumed constant values for other parameters and costs. Namely, $s = 4, \lambda = 9, \mu = 7, \beta = 2.5, p_1 = 0.6, p_2 = 0.4,, c_h = 0.5, c_s = 5, c_w = 0.05, c_b = 2.4$.

Table 2. Total expected cost rate for various combination of S and N

S	N				
	9	10	11	12	13
10	11.22003	11.17442	**11.15567**	11.15673	11.17262
11	11.13815	11.09196	**11.07268**	11.07326	11.08872
12	11.13834	11.09171	**11.07203**	11.07226	11.08739
13	11.19352	11.14655	11.12655	**11.12650**	11.14138
14	11.28731	11.24008	11.21984	**11.21955**	11.23423

Since, in Table 1 and Table 2, the value which is both underlined and in bold is smaller than the row minima and column minima, we have obtained a (local) optima for the associated cost function of the tables. The numerical values in each table also exhibit the convexity of the cost function in the domain of its arguments.

In Table 3, the impact of arrival rate λ and reorder rate β on the optimal values (S,N) and the corresponding total expected cost rate are studied by fixing the parameters and costs $s = 4, \mu = 7, p_1 = 0.6, p_2 = 0.4, c_h = 0.5, c_s = 5, c_w = 0.05, c_b = 2.4$.We observed that the total expected cost rate increase when λ and β increases.

In Table 4, the impact of service rate μ and reorder rate β on the optimal values (S,N) and the corresponding total expected cost rate are studied by fixing the parameters and costs $s = 4, \lambda = 9, p_1 = 0.6, p_2 = 0.4, c_h = 0.5, c_s = 5, c_w = 0.05, c_b = 2.4$.We observed that the total expected cost rate decrease when μ increases and increase when β increases.

In Table 5, the impact of holding cost c_h and setup cost c_s on the optimal values (S,N) and the corresponding total expected cost rate are studied by fixing the parameters and costs $s = 5, \lambda = 9, \mu = 7, \beta = 2.5, p_1 = 0.6, p_2 = 0.4,, c_w = 0.05, c_b = 2.4$.We observed that the total expected cost rate increase when c_h and c_s increases.

In Table 6, the impact of waiting time cost c_w and setup cost c_s on the optimal values (S,N) and the corresponding total expected cost rate are studied by fixing the parameters and costs $s = 5, \lambda = 9, \mu = 7, \beta = 2.5, p_1 = 0.6, p_2 = 0.4,, c_h = 0.5, c_b = 2.4$. We observed that the total expected cost rate increase when c_w and c_s increases.

Table 3. Effect of demand rate λ and reorder rate β on optimal values

λ	β				
	2.5	2.6	2.7	2.8	2.9
9	12 11 11.07203	12 11 11.087853	12 11 11.10271	12 11 11.34671	12 11 11.11677
9.1	12 11 11.13011	12 11 11.31763	12 11 11.33256	12 11 11.34671	12 11 11.36014
9.2	12 11 11.53350	12 11 11.54919	12 11 11.56405	12 11 11.57816	12 11 11.59154
9.3	12 11 11.76674	12 11 11.78236	12 11 11.79717	12 11 11.81122	12 11 11.82457
9.4	12 11 12.00144	12 11 12.01700	12 11 12.03176	12 11 12.04577	12 11 12.05907

Table 4. Effect of service rate μ and reorder rate β on optimal values

m	β				
	2.5	2.6	2.7	2.8	2.9
6.8	11 11 11.47750	11 11 11.49296	11 11 11.50762	11 11 11.52151	11 11 11.53469
6.9	11 11 11.27340	11 11 11.28898	11 11 11.30375	11 11 11.31776	11 11 11.33105
7.0	12 11 11.07203	12 11 11.08785	12 12 11.10271	12 12 11.11677	12 12 11.13011
7.1	12 12 10.86496	12 12 10.88070	12 12 10.89563	12 12 10.90980	12 12 10.92326
7.2	12 12 10.66132	12 12 10.67718	12 12 10.69223	12 12 10.70652	12 12 10.72009

Table 5. Effect of holding cost c_h and setup cost c_s on optimal values

c_h	c_s				
	6.0	7.0	8.0	9.0	10.0
0.50	14 12 11.91044	14 12 12.22004	14 12 12.52963	15 12 12.80996	16 12 13.08479
0.51	13 12 11.97760	14 12 12.30897	14 12 12.61856	15 12 12.90391	16 12 13.18376
0.52	13 12 12.06150	14 12 12.39790	14 12 12.70749	15 12 12.99786	15 12 13.27664
0.53	13 12 12.14540	14 12 12.48683	14 12 12.79642	15 12 13.09181	15 12 13.370
0.54	13 12 12.22930	14 12 12.57576	14 12 12.88535	15 12 13.18576	15 12 13.46454
0.55	13 12 12.31321	13 12 12.66128	14 12 12.97428	15 12 13.27972	15 12 13.55849

Table 6. Effect of waiting time cost c_w and setup cost c_s on optimal values

c_w	c_s				
	6.0	7.0	8.0	9.0	10.0
0.06	13 11 12.00112	14 11 12.32759	14 11 12.63720	15 11 12.91763	16 11 13.19254
0.07	13 10 12.10641	14 10 12.43305	14 10 12.74269	15 10 13.02392	16 10 13.29827
0.08	13 10 12.20170	14 10 12.52829	14 10 12.83793	15 10 13.11845	16 10 13.39345
0.09	13 10 12.29698	14 10 12.62353	14 10 12.93317	15 10 13.21366	16 10 13.48863

7. Conclusion

In this paper, we discussed a continuous review stochastic (s, S) inventory system at a service facility with two types of service and finite waiting hall. Demands occurring during stock-out periods are lost. We have derived the joint probability distribution of the inventory level and the number of customers in the steady state cases. We also have derived various stationary measures of system performances and computed total expected cost rate. We have provided numerical examples to illustrate the convexity of the total expected cost rate in steady state.

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