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Controllability result of impulsive stochastic fractional functional differential equation with infinite delay

Research Article

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Abstract: This paper is concerned with the controllability result of mild solution for an impulsive neutral fractional order stochastic integro-differential equation with infinite delay subject to nonlocal conditions. The existence result is obtained by using the fixed point technique on a Hilbert space. At last, we present an example to verify the result.

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1. Introduction

The differential equations with non-integer order have gained considerable importance due to their numerous applications in various fields such as physics, fluid mechanics, viscoelasticity, chemistry, control and porous media etc. The theory of differential equations with fractional order has been extensively studied by many authors [1–9].
The deterministic systems generated by men or nature often fluctuate due to environmental noise which is random, these systems are modeled as stochastic differential equations. Stochastic differential equations with infinite delay play an important role in recent years as a mathematical models of various phenomena in both physical and social sciences. Other than the environmental noise, sometimes we have to consider the impulsive effects which exists in many evolution processes because the impulsive effects may bring an abrupt change at certain moments of time. For the literatures on controllability of stochastic system with impulsive effect, one can see the papers [10–13] and references therein.
One of the basic qualitative behaviors of a dynamical system is controllability, it means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. As a result of its great application, the controllability of such systems all have received more and more attention, we refer the work for more details [14–16].
Some recent survey on the existence of mild solutions is presented here. In [6] Dabas and Chauhan considered the following problem

$$\begin{cases} {}^cD_t^\alpha[x(t)+g(t,x_t)] = A[x(t)+g(t,x_t)] + J_t^{1-\alpha}f(t,x_t,Bx(t)), t\in[0,T], t\neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), k=1,2,\ldots,m, \\ x_0=\phi\in\mathfrak{B}_h, \end{cases}$$

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and established the existence, uniqueness and continuous dependence of mild solution by using the Banach contraction fixed point theorem.
Sakthivel et al. [17] studied the existence of mild solutions by using Banach contraction, Krasnoselskii's, and Schaefer's fixed point theorems of the following problem

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x_t, B_1 x(t)) + \sigma(t, x_t, B_2 x(t))\frac{dw(t)}{dt}, \ t \in [0,T], t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), k = 1,2,\ldots,m, \\ x(t) = \phi(t), \phi(t) \in \mathfrak{B}_h. \end{cases}$$

Recently, Sun et al. [18] studied the following mathematical problem

$$\begin{cases} \frac{{}^cD^\alpha}{dt^\alpha}[x(t) + g(t, x_t)] = -Ax(t) + Bu(t) + \int_0^t \sigma(t,s,x_s)dW(s), t \in [0,b], b > 0, \\ x(t) = \phi(t), t \in (-\infty, 0], \end{cases}$$

and discussed the controllability results of mild solution by fixed point theorems.
Hernández et al. [19] comments on some recent results on exact controllability of abstract differential control systems with a linear part dominated by a sectorial operator. Actually, author's have shown that the abstract control problems considered in cited papers [20–24] not exactly controllable because A is considered as unbounded operator, which implies that the generated $\alpha-$resolvent family is unbounded due to this fact, the results are absurd.
Motivated by the above mention works [6, 17, 18], we consider the existence and uniqueness of mild solution for impulsive fractional stochastic functional integro-differential equation with infinite delay of the form:

$${}^cD_t^\alpha Q(x_t) = AQ(x_t) + \mathbb{J}_t^{1-\alpha}[Bu(t) + f(t, x_t, Kx(t))\frac{dw(t)}{dt}], t \in J, t \neq t_k, \tag{1}$$

$$\Delta x(t_k) = I_k(x(t_k^-)), k = 1,2,...,m, \tag{2}$$

$$x(t) + h(x) = \phi(t) \in \mathfrak{B}_h, t \in (-\infty, 0], \tag{3}$$

where $J = [0,T]$ and ${}^cD_t^\alpha$ stand for the Caputo's fractional derivative of order $\alpha \in (0,1)$. The operator $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is a closed linear operator of sectorial type defined on a Hilbert space $(\mathbb{H}, \|\cdot\|)$. The functions $f : J \times \mathfrak{B}_h \times \mathbb{H} \to \mathbb{H}, Q(x_t) = x(t) + g(t, x_t)$ are given and satisfy some assumptions, where $\mathfrak{B}_h$ is a phase space. The bounded linear operator B defined from U to $\mathbb{H}$, and the control function $u(\cdot)$ takes values in $L^2(J;U)$ of admissible control functions with U as a Hilbert space. The process $\{\mathbb{W}(t) : t \geq 0\}$ is a given U-valued Wiener process with a finite trace nuclear covariance operator $\mathbb{Q} \geq 0$ defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_{t\geq 0}\}, \mathbb{P})$. We assume that $x_t : (-\infty, 0] \to \mathbb{H}$, $x_t(s) = x(t+s), s \leq 0$, belong to an abstract phase space $\mathfrak{B}_h$. The term $Kx(t)$ is given by $Kx(t) = \int_0^t S(t,s)x(s)ds$, where $S \in C(D, R^+)$, the set of all positive functions which are continuous on $D = \{(t,s) \in R^2 : 0 \leq s \leq t < T\}$. Here $0 \leq t_0 < t_1 < \cdots < t_m < t_{m+1} \leq T, I_k \in C(\mathbb{H}, \mathbb{H}), (k = 1,2,\ldots,m)$, are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0} x(t_k - h)$ represent the right and left-hand limits of x(t) at $t = t_k$, respectively, also we take $x(t_i^-) = x(t_i)$. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathscr{F}_0$-measurable, $\mathfrak{B}_h$-valued random variable independent of $w(t)$ with finite second moments.
To the best of our knowledge, the controllability of the system (1.1)-(1.3) with solution operators is an untreated topic yet in the literature and this fact is the motivation of the present work.
This paper is divided into four sections, Second section provides the basic definitions and preliminaries results which are used in proving our main result. In the third section, we obtain the controllability result and the fourth section is concerned with an example.

2. Preliminaries

Let $\mathbb{H}, \mathbb{K}$ be two separable Hilbert spaces and $\mathscr{L}(\mathbb{K}, \mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}, \mathbb{K}$ and $\mathscr{L}(\mathbb{K}, \mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$. $\mathbb{W} = (\mathbb{W}_t)_{t\geq 0}$ be a $\mathbb{Q}$-Wiener process defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator $\mathbb{Q}$ such that $Tr\mathbb{Q} < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $\mathbb{Q}e_k = \lambda_k e_k, \ k = 1,2,\ldots,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t), e \in \mathbb{K}, t \geq 0.$$

Let $\mathscr{L}_2^0 = \mathscr{L}_2(\mathbb{Q}^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $<\varphi, \psi>_{\mathscr{L}_2^0} = Tr[\varphi\mathbb{Q}\psi*]$.

Now, we introduce abstract phase space $\mathfrak{B}_h$. Assume that $h:(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^{0}h(t)dt<\infty$, a continuous function. Abstract space $\mathfrak{B}_h$ defined by
$\mathfrak{B}_h=\{\phi:(-\infty,0]\to\mathbb{H}$, for any $a>0$, $(E|\phi(\theta)|^2)^{1/2}$ is bounded and measurable function on $[-a,0]$ with $\phi(0)=0$ and $\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}(E|\phi(\theta)|^2)^{1/2}ds<\infty\}$.
If $\mathfrak{B}_h$ is endowed with the following norm

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}(E|\phi(\theta)|^2)^{1/2}ds,\ \phi\in\mathfrak{B}_h,$$

then $(\mathfrak{B}_h,\|\cdot\|_{\mathfrak{B}_h})$ is a Banach space [25, 26].
Now we consider the space

$$\mathfrak{B}_{h'}=\{x:(-\infty,T]\to\mathbb{H}\text{ such that }x|_{J_k}\in C(J_k,\mathbb{H})\text{ and there exist }x(t_k^+)\text{ and }x(t_k^-)\text{ with }x(t_k)=x(t_k^-),$$
$$x_0=\phi\in\mathfrak{B}_h,k=1,2,\dots,m\},$$

where $x|_{J_k}$ is the restriction of x to $J_k=(t_k,t_{k+1}]$, $k=0,1,2,\dots,m$. The function $\|\cdot\|_{\mathfrak{B}'_h}$ to be a semi-norm in $\mathfrak{B}'_h$, it is defined by

$$\|x\|_{\mathfrak{B}'_h}=\|\phi\|_{\mathfrak{B}_h}+\sup_{s\in[0,T]}(E\|x(s)\|^2)^{1/2},x\in\mathfrak{B}'_h.$$

Lemma 2.1 ([17]).
Assume that $x\in\mathfrak{B}'_h$, then for $t\in J, x_t\in\mathfrak{B}_h$. Moreover, $l(E\|x(t)\|^2)^{1/2}\le l\sup_{s\in[0,t]}(E\|x(s)\|^2)^{1/2}+\|x_0\|_{\mathfrak{B}_h}$, where $l=\int_{-\infty}^{0}h(s)ds<\infty$.

Definition 2.1.
The Riemann-Liouville fractional integral operator for order $\alpha>0$, of a function $f:\mathbb{R}^+\to\mathbb{R}$ and $f\in L^1(\mathbb{R}^+,X)$ is defined by

$$\mathbb{J}_t^0f(t)=f(t),\ \mathbb{J}_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad t>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2.
Caputo's derivative of order $\alpha>0$ for a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds=\mathbb{J}^{n-\alpha}f^{(n)}(t),$$

for $n-1<\alpha<n,\ n\in\mathbb{N}$. If $0<\alpha<1$, then

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}f^{(1)}(s)ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3.
A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi\iota}\int_c\frac{\mu^{\alpha-\beta}e^\mu}{\mu^\alpha-z}d\mu,\alpha,\beta>0,z\in\mathbb{C},$$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\le|z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag-Lefller functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt=\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\omega},Re\lambda>\omega^{\frac{1}{\alpha}},\omega>0.$$

Definition 2.4.
A measurable $\mathscr{F}_t-$ adapted stochastic process $x:(-\infty,T]\to\mathbb{H}$ is called a mild solution of the system (1)-(3) if $x(0)=\phi(0)-h(x)$ on $(-\infty,0]$, $\Delta x|_{t=t_k}=I_k(x(t_k^-))$, $k=1,2,\ldots,m$ the restriction of $x(\cdot)$ to the interval $[0,T)\backslash t_1,\ldots,t_m$, is continuous and $x(t)$ satisfies the following fractional integral equation

$$x(t)=\begin{cases} S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]-g(t,x_t)+\int_0^t S_\alpha(t-s)Bu(s)ds+ \\ \int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s), & t\in(0,t_1] \\ S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]+S_\alpha(t-t_1)I_1(x(t_1^-))-g(t,x_t) \\ +S_\alpha(t-t_1)[g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})]+ \\ \int_0^t S_\alpha(t-s)Bu(s)ds+\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s), & t\in(t_1,t_2] \\ \cdots \\ S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]+\sum_{i=1}^m S_\alpha(t-t_i)I_i(x(t_i^-)) \\ +\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,x_{t_i}+I_i(x_{t_i^-}))-g(t_i,x_{t_i})]-g(t,x_t) \\ +\int_0^t S_\alpha(t-s)Bu(s)ds+\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s), & t\in(t_m,T] \end{cases} \tag{4}$$

where

$$S_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}d\lambda,$$

is called analytic solutions operator and Γ is a suitable path lying on $\sum_{\theta,\omega}$ for more details one can see [4].

Definition 2.5.
The nonlinear neutral stochastic differential equation (1)-(3) is said to be controllable on the interval J, if for every continuous initial stochastic process $\phi\in\mathfrak{B}_h$ defined on $(-\infty,0]$ there exists a stochastic control $u\in L^2(J,U)$ which is adapted to the filtration $\mathscr{F}_t\geq 0$ such that the solution $x(E)$ of (1)-(3) satisfies $x(T)=m(t_1)$, where $m(t_1)$ and T are preassigned terminal state and time, respectively.

3. Existence and uniqueness result

In order to establish the result, we consider that if $\alpha\in(0,1)$ and $A\in\mathscr{A}^\alpha(\theta_0,\omega_0)$, then for any $x\in\mathbb{H}$ and $t>0$, we have $\|S_\alpha(t)\|\leq Me^{\omega t}, \omega>\omega_0$. Thus, we have $\|S_\alpha(t)\|\leq\widetilde{M}_S$, where $\widetilde{M}_S=\sup_{0\leq t\leq T}\|S_\alpha(t)\|$ (for more details, see[27]). Further we impose the following conditions.

(H_1) There are positive constants $L_{f_1}, L_{f_2}, L_h, L_g$, and L_I, such that

$$\begin{aligned} &\left\|f(t,\gamma,x)-f(t,\psi,y)\right\|_{\mathbb{H}}^2\leqslant L_{f_1}\left\|\gamma-\psi\right\|_{\mathfrak{B}_h}^2+L_{f_2}E\left\|x-y\right\|_{\mathbb{H}}^2, \\ &E\left\|h(x)-h(y)\right\|_{\mathbb{H}}^2\leqslant L_hE\left\|x-y\right\|_{\mathbb{H}}^2, \\ &E\left\|I_k(x)-I_k(y)\right\|_{\mathbb{H}}^2\leqslant L_IE\left\|x-y\right\|_{\mathbb{H}}^2, \\ &E\left\|g(s,z)-g(s,z^*)\right\|_{\mathbb{H}}^2\leqslant L_gE\|z-z^*\|_{\mathbb{H}}^2, \end{aligned}$$

for all $x,y,z,z^*\in\mathbb{H};\gamma,\psi\in\mathfrak{B}_h,\ t\in[0,T]$ and each $k=1,2,\cdots,m$.

(H_2) For $k=1,2,3,....,m+1$, the linear operators $W_k:L^2([t_{k-1},t_k];U)\to\mathbb{H}$ defined by $W_ku=\int_{t_{k-1}}^{t_k}S_\alpha(t-s)Bu(s)ds$; has an invertible operator W_k^{-1} taking values in $L^2([t_{k-1},t_k];U)\setminus ker(W_k)$ and there exists a positive constant M_k such that $\|BW_k^{-1}\|\leq M_k$, and $M=\max\{M_k\}$, $\forall\ k$.

Theorem 3.1.
Let the assumptions (H_1) and (H_2) hold, and

$$\Delta=\begin{cases}(7\widetilde{M}_S^2L_h+7L_gl+7\widetilde{M}_S^2L_I+7\widetilde{M}_S^2L_g[L_Il+l]+7\widetilde{M}_S^2L_gl \\ +7\widetilde{M}_S^2T^2[6M(\widetilde{M}_S^2L_h+m\widetilde{M}_S^2L_I+m\widetilde{M}_S^2L_g(l+L_Il)+m\widetilde{M}_S^2L_gl+L_gl, & <1 \\ +\widetilde{M}_S^2T^2(L_{f_1}l+L_{f_2}K^*))]+7\widetilde{M}_S^2T[L_{f_1}l+L_{f_2}K^*])\end{cases}$$

where $K^*=\sup_{t\in[0,T]}\int_0^t\|S(t,s)\|ds$. Then the system (1)-(3) has a unique mild solutions on J.

Proof. Let $v \in PC(J;\mathbb{H})$ be any arbitrary function and transfer the system (1)-(3) from initial state to $v(T)$, and consider the control

$$u(t)=\begin{cases} W_1^{-1}[v(t_1)-S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]+g(t,x_t)-\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s)](t), & t\in(0,t_1]\\ W_2^{-1}[v(t_2)-S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]-S_\alpha(t-t_1)I_1(x(t_1^-))+g(t,x_t) & \\ -S_\alpha(t-t_1)[g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})]-\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s)](t), & t\in(t_1,t_2]\\ \cdots & \\ W_{m+1}^{-1}[v(T)-S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]-\sum_{i=1}^m S_\alpha(t-t_i)I_i(x(t_i^-)) & \\ -\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,x_{t_i}+I_i(x_{t_i^-}))-g(t_i,x_{t_i})]+g(t,x_t) & \\ -\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s)](t), & t\in(t_m,T]. \end{cases} \tag{5}$$

Consider the operator $P:\mathfrak{B}'_h\to\mathfrak{B}'_h$ defined by

$$P(x)(t)=\begin{cases} S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]-g(t,x_t)+\int_0^t S_\alpha(t-s)Bu(s)ds+ & \\ \int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s), & t\in(0,t_1]\\ S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]+S_\alpha(t-t_1)I_1(x(t_1^-))-g(t,x_t) & \\ +S_\alpha(t-t_1)[g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})]+ & \\ \int_0^t S_\alpha(t-s)Bu(s)ds+\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s), & t\in(t_1,t_2]\\ \cdots & \\ S_\alpha(t)[\phi(0)-h(x)+g(0,\phi)]+\sum_{i=1}^m S_\alpha(t-t_i)I_i(x(t_i^-)) & \\ +\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,x_{t_i}+I_i(x_{t_i^-}))-g(t_i,x_{t_i})]-g(t,x_t) & \\ +\int_0^t S_\alpha(t-s)Bu(s)ds+\int_0^t S_\alpha(t-s)f(s,x_s,Kx(s))dw(s) & t\in(t_m,T]. \end{cases} \tag{6}$$

Let $y(.):(-\infty,T]\to\mathbb{H}$ be the function defined by

$$y(t)=\begin{cases}\phi(t)-h(x),\ t\in(-\infty,0]\\ 0,\ t\in J,\end{cases}\quad \text{then } y(0)=\phi(0)-h(x). \tag{7}$$

For each $z:J\to\mathbb{H}$ with $z|_{t_k}\in C(I_k,\mathbb{H}), k=1,...,m$ and $z(0)=0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}=\begin{cases}0,\ t\in(-\infty,0]\\ z(t),\ t\in J.\end{cases} \tag{8}$$

If $x(\cdot)$ satisfies the system (4), then we can decompose $x(\cdot)$ as $x(t)=y(t)+\overline{z}(t)$, which implies $x_t=y_t+\overline{z}_t$ for $t\in J$ and the function $z(\cdot)$ satisfies

$$z(t)=\begin{cases} S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-g(t,y_t+\overline{z}_t)+\int_0^t S_\alpha(t-s)B\overline{u}(s)ds & \\ +\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s))dw(s), & t\in(0,t_1]\\ S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-))-g(t,y_t+\overline{z}_t) & \\ +S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))-g(t_1,y_{t_1}+\overline{z}_{t_1})]+\int_0^t S_\alpha(t-s)B\overline{u}(s)ds & \\ +\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_1,t_2]\\ \cdots & \\ S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) & \\ +\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,y_{t_i}+\overline{z}_{t_i}+I_i(y_{t_i^-}+\overline{z}_{t_i^-}))-g(t_i,y_{t_i}+\overline{z}_{t_i})]-g(t,y_t+\overline{z}_t) & \\ +\int_0^t S_\alpha(t-s)B\overline{u}(s)ds+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_m,T]. \end{cases} \tag{9}$$

Set $\mathfrak{B}''_h$, such that $z_0=0$ and for any $z\in\mathfrak{B}''_h$, we have

$$\|z\|_{\mathfrak{B}''_h}=\|z_0\|_{\mathfrak{B}_h}+\sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}}=\sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}}.$$

Thus $(\mathfrak{B}''_h,\|\cdot\|_{\mathfrak{B}''_h})$ is a Banach space. Define an operator $N:\mathfrak{B}''_h\to\mathfrak{B}''_h$ by

$$(Nz)(t)=\begin{cases} S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-g(t,y_t+\overline{z}_t)+\int_0^t S_\alpha(t-s)B\overline{u}(s)ds & \\ +\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s))dw(s), & t\in(0,t_1]\\ S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-))-g(t,y_t+\overline{z}_t) & \\ +S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))-g(t_1,y_{t_1}+\overline{z}_{t_1})]+\int_0^t S_\alpha(t-s)B\overline{u}(s)ds & \\ +\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_1,t_2]\\ \cdots & \\ S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) & \\ +\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,y_{t_i}+\overline{z}_{t_i}+I_i(y_{t_i^-}+\overline{z}_{t_i^-}))-g(t_i,y_{t_i}+\overline{z}_{t_i})]-g(t,y_t+\overline{z}_t) & \\ +\int_0^t S_\alpha(t-s)B\overline{u}(s)ds+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_m,T], \end{cases} \tag{10}$$

where

$$
\overline{u}(t)=\begin{cases}
W_1^{-1}[v(t_1)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+g(t,y_t+\overline{z}_t) & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(0,t_1] \\
W_2^{-1}[v(t_2)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-))+g(t,y_t+\overline{z}_t) & \\
-S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))-g(t_1,y_{t_1}+\overline{z}_{t_1})] & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(t_1,t_2] \\
\dots & \\
W_{m+1}^{-1}[v(T)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) & \\
-\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,y_{t_i}+\overline{z}_{t_i}+I_i(y_{t_i^-}+\overline{z}_{t_i^-}))-g(t_i,y_{t_i}+\overline{z}_{t_i})]+g(t,y_t+\overline{z}_t) & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(t_m,T].
\end{cases}
$$

We have

$$
(Nz)(t)=\begin{cases}
S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-g(t,y_t+\overline{z}_t) & \\
+\int_0^t S_\alpha(t-s)D(s,y+\overline{z})ds+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in[0,t_1] \\
S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-))-g(t,y_t+\overline{z}_t)+S_\alpha(t-t_1) & \\
\times[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))-g(t_1,y_{t_1}+\overline{z}_{t_1})]+\int_0^t S_\alpha(t-s)D(s,y+\overline{z})ds & \\
+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_1,t_2] \\
\dots & \\
S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) & \\
+\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,y_{t_i}+\overline{z}_{t_i}+I_i(y_{t_i^-}+\overline{z}_{t_i^-}))-g(t_i,y_{t_i}+\overline{z}_{t_i})]-g(t,y_t+\overline{z}_t) & \\
+\int_0^t S_\alpha(t-s)D(s,y+\overline{z})ds+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s), & t\in(t_m,T],
\end{cases}
\tag{11}
$$

where

$$
D(s,z)=\begin{cases}
BW_1^{-1}[v(t_1)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]+g(t,y_t+\overline{z}_t) & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(0,t_1] \\
BW_2^{-1}[v(t_2)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-)) & \\
+g(t,y_t+\overline{z}_t)-S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))-g(t_1,y_{t_1}+\overline{z}_{t_1})] & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(t_1,t_2] \\
\dots & \\
BW_{m+1}^{-1}[v(T)-S_\alpha(t)[\phi(0)-h(y+\overline{z})+g(0,\phi)]-\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) & \\
+g(t,y_t+\overline{z}_t)-\sum_{i=1}^m S_\alpha(t-t_i)[g(t_i,y_{t_i}+\overline{z}_{t_i}+I_i(y_{t_i^-}+\overline{z}_{t_i^-}))-g(t_i,y_{t_i}+\overline{z}_{t_i})] & \\
-\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))dw(s)](t), & t\in(t_m,T].
\end{cases}
\tag{12}
$$

For convenience, let us take for interval $(0,t_1]$

$$
\begin{aligned}
E\|D(s,z)-D(s,z^*)\|_{\mathbb{H}}^2 \;\le\; & 3E\|BW_1^{-1}S_\alpha(t)[h(y+\overline{z})-h(y+\overline{z}^*)]\|_{\mathbb{H}}^2 \\
& +3E\|BW_1^{-1}[g(t,y_t+\overline{z}_t)-g(t,y_t+\overline{z}_t^*)]\|_{\mathbb{H}}^2+3E\|BW_1^{-1}\int_0^t S_\alpha(t-s) \\
& \times[f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s)))-f(s,y_s+\overline{z}_s^*,K(y(s)+\overline{z}^*(s)))]dw(s)]\|_{\mathbb{H}}^2,
\end{aligned}
$$

from our assumptions, we have

$$
E\|D(s,z)-D(s,z^*)\|_{\mathbb{H}}^2\le 3M_1(\widetilde{M}_S^2L_h+L_gl+\widetilde{M}_S^2T(L_{f_1}l+L_{f_2}K^*))\|z-z^*\|_{\mathfrak{B}_h''}^2.
$$

Similarly for $(t_k, t_{k+1}]$, we get

$$\begin{aligned}
E\|D(s,z)-D(s,z^*)\|^2_{\mathbb{H}} \leqslant\; & 6E\left\|BW^{-1}_{k+1}S_\alpha(t)[h(y+\bar{z})-h(y+\bar{z}^*)]\right\|^2_{\mathbb{H}}\\
&+6E\left\|BW^{-1}_{k+1}\sum_{i=1}^{k}S_\alpha(t-t_i)[I_i(y(t_i^-)+\bar{z}(t_i^-))-I_i(y(t_i^-)+\bar{z}^*(t_i^-))]\right\|^2_{\mathbb{H}}\\
&+6E\left\|\begin{array}{l}BW^{-1}_{k+1}\sum_{i=1}^{k}S_\alpha(t-t_i)[g(t_i,y_{t_i}+\bar{z}_{t_i}+I_i(y_{t_i^-}+\bar{z}_{t_i^-}))\\ -g(t_i,y_{t_i}+\bar{z}^*_{t_i}+I_i(y_{t_i^-}+\bar{z}^*_{t_i^-}))]\end{array}\right\|^2_{\mathbb{H}}\\
&+6E\left\|BW^{-1}_{k+1}\sum_{i=1}^{k}S_\alpha(t-t_i)[g(t_i,y_{t_i}+\bar{z}_{t_i})-g(t_i,y_{t_i}+\bar{z}^*_{t_i})]\right\|^2_{\mathbb{H}}\\
&+6E\left\|BW^{-1}_{k+1}[g(t,y_t+\bar{z}_t)-g(t,y_t+\bar{z}^*_t)]\right\|^2_{\mathbb{H}}\\
&+6E\left\|BW^{-1}_{k+1}\int_0^t S_\alpha(t-s)\times[f(s,y_s+\bar{z}_s,K(y(s)+\bar{z}(s)))-f(s,y_s+\bar{z}^*_s,K(y(s)+\bar{z}^*(s)))]dw(s)]\right\|^2_{\mathbb{H}},
\end{aligned}$$

from our assumptions, we obtain

$$\begin{aligned}
E\|D(s,y+\overline{z})-D(s,y+\overline{z}^*)\|^2_{\mathbb{H}} \leq\; & 6M_{k+1}(\widetilde{M}_S^2L_h+m\widetilde{M}_S^2L_I+m\widetilde{M}_S^2L_g(l+L_Il)+m\widetilde{M}_S^2L_gl+L_gl\\
&+\widetilde{M}_S^2T(L_{f_1}l+L_{f_2}K^*))\|z-z^*\|^2_{\mathfrak{B}''_h}.
\end{aligned}$$

To show the existence results, it is enough to show that N has a unique fixed point. Let $z,z^*\in\mathfrak{B}''_h$ then for $t\in(0,t_1]$, we have

$$\begin{aligned}
E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq\; & 4E\|S_\alpha(t)[h(y+\overline{z})-h(y+\overline{z}^*)\|^2_{\mathbb{H}}+4E\|g(t,y_t+\overline{z}_t)-g(t,y_t+\overline{z}^*_t)\|^2_{\mathbb{H}}\\
&+4E\|\int_0^t S_\alpha(t-s)[f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s))\\
&-f(s,y_s+\overline{z}^*_s,K(y(s)+\overline{z}^*(s))]dw(s)\|^2_{\mathbb{H}}\\
&+4E\|\int_0^t S_\alpha(t-s)[D(s,y+\overline{z})-D(s,y+\overline{z}^*)]ds\|^2_{\mathbb{H}},\\
\leq\; & (4L_h\widetilde{M}_S^2+4L_gl+4\widetilde{M}_S^2T[L_{f_1}l+L_{f_2}K^*]\\
&+12\widetilde{M}_S^2T^2M_1(\widetilde{M}_S^2L_h+L_gl+\widetilde{M}_S^2T(L_{f_1}l+L_{f_2}K^*)))\|z-z^*\|^2_{\mathfrak{B}''_h}.
\end{aligned}$$

For $t\in(t_1,t_2]$, we have

$$\begin{aligned}
E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq\; & 7E\|S_\alpha(t)[h(y+\overline{z})-h(y+\overline{z}^*)\|^2_{\mathbb{H}}+7E\|g(t,y_t+\overline{z}_t)-g(t,y_t+\overline{z}^*_t)\|^2_{\mathbb{H}}\\
&+7E\|S_\alpha(t-t_1)[I_1(y(t_1^-)+\overline{z}(t_1^-))-I_1(y(t_1^-)+\overline{z}^*(t_1^-))\|^2_{\mathbb{H}}\\
&+7E\|S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1}+I_1(y_{t_1^-}+\overline{z}_{t_1^-}))\\
&-g(t_1,y_{t_1}+\overline{z}^*_{t_1}+I_1(y_{t_1^-}+\overline{z}^*_{t_1^-}))]\|^2_{\mathbb{H}}\\
&+7E\|S_\alpha(t-t_1)[g(t_1,y_{t_1}+\overline{z}_{t_1})-g(t_1,y_{t_1}+\overline{z}^*_{t_1})]\|^2_{\mathbb{H}}\\
&7E\|\int_0^t S_\alpha(t-s)[D(s,y+\overline{z})-D(s,y+\overline{z}^*)]ds\|^2_{\mathbb{H}}\\
&+7E\|\int_0^t S_\alpha(t-s)[f(s,y_s+\overline{z}_s,K(y(s)+\overline{z}(s))\\
&-f(s,y_s+\overline{z}^*_s,K(y(s)+\overline{z}^*(s)))]dw(s)\|^2_{\mathbb{H}},\\
\leq\; & (7\widetilde{M}_S^2L_h+7L_gl+7\widetilde{M}_S^2I_1+7\widetilde{M}_S^2L_g[L_Il+l]+7\widetilde{M}_S^2L_gl\\
&+7\widetilde{M}_S^2T^2[6M_2(\widetilde{M}_S^2L_h+m\widetilde{M}_S^2L_I+m\widetilde{M}_S^2L_g(l+L_Il)+m\widetilde{M}_S^2L_gl+L_gl\\
&+\widetilde{M}_S^2T^2(L_{f_1}l+L_{f_2}K^*))]+7\widetilde{M}_S^2T[L_{f_1}l+L_{f_2}K^*])\|z-z^*\|^2_{\mathfrak{B}''_h}.
\end{aligned}$$

Similarly, when $t\in(t_i,t_{i+1}]$, $i=2,\dots,m$, we obtain

$$\begin{aligned}
E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq\; & (7\widetilde{M}_S^2L_h+7L_gl+7\widetilde{M}_S^2I_1+7\widetilde{M}_S^2L_g[L_Il+l]+7\widetilde{M}_S^2L_gl\\
&+7\widetilde{M}_S^2T^2[6M_{i+1}(\widetilde{M}_S^2L_h+m\widetilde{M}_S^2L_I+m\widetilde{M}_S^2L_g(l+L_Il)+m\widetilde{M}_S^2L_gl+L_gl\\
&+\widetilde{M}_S^2T^2(L_{f_1}l+L_{f_2}K^*))]+7\widetilde{M}_S^2T[L_{f_1}l+L_{f_2}K^*])\|z-z^*\|^2_{\mathfrak{B}''_h}.
\end{aligned}$$

Thus for all $t \in [0,T]$, we have

$$
\begin{aligned}
E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \le & (7\widetilde{M}_S^2 L_h + 7L_g l + 7\widetilde{M}_S^2 I_1 + 7\widetilde{M}_S^2 L_g [L_I l + l] + 7\widetilde{M}_S^2 L_g l \\
& + 7\widetilde{M}_S^2 T^2 [6M(\widetilde{M}_S^2 L_h + m\widetilde{M}_S^2 L_I + m\widetilde{M}_S^2 L_g (l + L_I l) + m\widetilde{M}_S^2 L_g l + L_g l \\
& + \widetilde{M}_S^2 T^2 (L_{f_1} l + L_{f_2} K^*))] + 7\widetilde{M}_S^2 T [L_{f_1} l + L_{f_2} K^*])\|z - z^*\|^2_{\mathfrak{B}''_h} \\
& \le \Delta \|z - z^*\|^2_{\mathfrak{B}''_h}.
\end{aligned}
$$

Since $\Delta < 1$, hence, N is a contraction map and therefore it has a unique fixed point $z \in \mathfrak{B}''_h$ which is a mild solution of (1)-(3) on $(-\infty, T]$. This completes the proof of the theorem. □

4. Application

To illustrate the application of the theory we consider the following partial integro-differential equation with fractional derivative of the form

$$
\begin{aligned}
\frac{\partial^q Q(u_t)(x)}{\partial t} = & \frac{\partial^2}{\partial x^2} Q(u_t)(x) + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \\
& \times [Bu(s) + \int_{-\infty}^s H(s,x,\xi - s) Q_2(u(\xi,x)) d\xi \\
& + \int_0^s k(\xi,s) e^{-u(\xi,x)} d\xi] ds \frac{d\beta(t)}{dt}, x \in [0,\pi], t \in [0,b], t \neq t_k
\end{aligned} \tag{13}
$$

$$
u(t,0) = 0 = u(t,\pi), t \ge 0 \tag{14}
$$

$$
\begin{aligned}
u(t,x) + & \int_0^t c(x,\gamma) \cos(1 + \|u(t,\gamma)\|) d\gamma = \phi(t,x), \\
& t \in (-\infty, 0], x \in [0,\pi].
\end{aligned} \tag{15}
$$

$$
\Delta u(t_i)(x) = \int_{-\infty}^{t_i} q_i(t_i - s) u(s,x) ds, x \in [0,\pi], \tag{16}
$$

where $\frac{\partial^q}{\partial t}$ is Caputo's fractional derivative of order $q \in (0,1), 0 < t_1 < t_2 < \cdots < t_n < T$ are prefixed numbers, $\phi \in \mathfrak{B}_h$ and

$$
Q(u_t)(x) = u(t,x) + \int_{-\infty}^t a(t,x,s-t) Q_1(u(s,x)) ds.
$$

Let $\mathbb{H} = L^2[0,\pi]$ and define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $A\omega = \omega''$ with the domain $D(A) := \{\omega \in X : \omega, \omega'$ are absolutely continuous, $\omega'' \in \mathbb{H}, \omega(0) = 0 = \omega(\pi)\}$. Then
$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n) \omega_n, \omega \in D(A)$, where $\omega_n(x) = \sqrt{\frac{2}{\pi}} sin(nx), n \in N$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \ge 0}$ in $\mathbb{H}$ and is given by

$$
T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n) \omega_n, \text{ for all } \omega \in \mathbb{H}, \text{ and every } t > 0.
$$

The subordination principle of solution operator (Theorem 3.1 in Ref. [28]) implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t \ge 0}$. Since $S_\alpha(t)$ is strongly continuous on $[0,\infty)$, by uniformly bounded theorem, there exists a constant $M > 0$ such that $\|S_\alpha(t)\|_{L(\mathbb{H})} \le M$ for $t \in [0,T]$. Let $h(s) = e^{2s}, s < 0$ then $l = \int_{-\infty}^0 h(s) ds = \frac{1}{2}$ and define

$$
\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s) \sup_{\theta \in [s,0]} \|\phi(\theta)\|_{L^2} ds.
$$

Hence for $(t,\phi) \in [0,T] \times \mathfrak{B}_h$, where $\phi(\theta)(x) = \phi(\theta,x), (\theta,x) \in (-\infty,0] \times [0,\pi]$. Set $u(t)(x) = u(t,x)$,

$$
g(t,\phi)(x) = \int_{-\infty}^0 a(t,x,\theta) Q_1(\phi(\theta)(x)) d\theta,
$$

$$
f(t,\phi,Ku(t))(x) = \int_{-\infty}^0 H(t,x,\theta) Q_2(\phi(\theta)(x)) d\theta + Ku(t)(x),
$$

$$
I_i(\phi)(x) = \int_{-\infty}^0 q_i(-\theta) \phi(\theta)(x) d\theta
$$

$$
h(x) = \int_0^t c(x,\gamma) \cos(1 + \|u(t,\gamma)\|) d\gamma,
$$

where $Ku(t)(x) = \int_0^t k(s,t)e^{-u(s,x)}ds$. Then with these settings the equations (13)-(16) can be written in the abstract form of equations (1)-(3). Suppose further that:
(i) The functions $Q_i, i = 1,2$ are continuous and $u(\theta,x)$, $v(\theta,x)$ are continuous in $(-\infty,0]\times[0,\pi]$,

$$0 \leq Q_i(u(\theta)(x)) - Q_i(v(\theta)(x)) \leq \int_{-\infty}^{0} e^{2s}\|u(s,.)-v(s,.)\|_{L^2}ds.$$

(ii) The function $H(t,x,\theta)$, continuous in $[0,T]\times[0,\pi]\times(-\infty,0)$ and satisfying

$$\int_{-\infty}^{0} H^2(t,x,\theta)d\theta < C, C>0.$$

(iii) The functions $q_i \in C(\mathbb{R},\mathbb{R})$ and $d_i = (\int_{-\infty}^{0}\frac{q_i^2(-\theta)}{h(\theta)}d\theta) < \infty$, for $i = 1,...,m$.
Now we can see that: For $(t,\phi,Bu(t)),(t,\psi,Bv(t)) \in [0,T]\times\mathfrak{B}_h,\times X$, we have

$$\begin{aligned}
&\|f(t,\phi,Bu(t)) - f(t,\psi,Bv(t))\|_{L^2} = \\
&\left[\int_0^\pi\left\{\int_{-\infty}^0 H(t,x,\theta)(Q_2(\phi(\theta)(x)) - Q_2(\psi(\theta)(x)))d\theta + Bu(t)(x) - Bv(t)(x)\right\}^2dx\right]^{1/2}\\
&\leqslant \left[\int_0^\pi\left\{\int_{-\infty}^0 H(t,x,\theta)(Q_2(\phi(\theta)(x)) - Q_2(\psi(\theta)(x)))d\theta\right\}^2dx\right]^{1/2} + \left[\int_0^\pi\{Bu(t)(x)-Bv(t)(x)\}^2dx\right]^{1/2}\\
&\leqslant \left[\int_0^\pi\left\{\int_{-\infty}^0 H(t,x,\theta)(\int_{-\infty}^0 e^{2s}\|\phi(s)(.)-\psi(s)(.)\|_{L^2}ds)d\theta\right\}^2dx\right]^{1/2} + \|Bu(t)-Bv(t)\|_{L^2}\\
&\leqslant \left[\int_0^\pi\left\{\int_{-\infty}^0 H(t,x,\theta)(\int_{-\infty}^0 e^{2s}\sup_{s\in[\theta,0]}\|\phi(s)-\psi(s)\|_{L^2}ds)d\theta\right\}^2dx\right]^{1/2} + \|Bu(t)-Bv(t)\|_{L^2}\\
&\leqslant \left[\int_0^\pi\left\{\int_{-\infty}^0 H(t,x,\theta)d\theta\right\}^2dx\right]^{1/2}\|\phi-\psi\|_{\mathfrak{B}_h} + \|Bu(t)-Bv(t)\|_{L^2}\\
&\leqslant C\sqrt{\pi}\|\phi-\psi\|_{\mathfrak{B}_h} + \|Bu(t)-Bv(t)\|_{L^2}.
\end{aligned}$$

Hence function f satisfies (H_1) and in a similar way we can show that g, h and I_i may satisfy (H_1), respectively. All the conditions of Theorem 3.1 have been fulfilled so we deduced that the system (13)-(16) has a mild solution on J.

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