Int. J. Adv. Appl. Math. and Mech. 2(1) (2014) 26 - 33
ISSN: 2347-2529
Available online at www.ijaamm.com

International Journal of Advances in **Applied Mathematics and Mechanics**

A priori inequality for majorant Cauchy problem with nonlinear boundary value problems

Research Article

S. Balamuralitharan *

Faculty of Engineering and Technology, Department of Mathematics, SRM University, Kattankulathur, Chennai-603 203, India

Received 14 July 2014; accepted (in revised version) 25 August 2014

Abstract: The aim of this paper is to conclude some of the basic lemmas and theorems, required results in the mathematical literature, related to a priori inequalities. The Cauchy problem equations such as $\dot{x} = Fx$, $x(a) = \alpha$, are derived. Thomas-Fermi problem, Volterra operator in various functions are established. Further results on boundary value problems, Leray Schauder theorem, or Nemytskii operator behavior are considered. Relations with Cauchy problem of a priori inequality are discussed. Some new Cauchy problem main results are given.

MSC: 34A40 • 26D10 • 26D07

Keywords: Cauchy problem • Thomas-Fermi problem • Volterra operator • Leray Schauder • Nemytskii operator

1. Introduction

In the recent papers [1]-[8] the nonlinear boundary value problems of a priori inequality have been considered. It is well known that it is complicate to solve a priori inequality even in case of nonlinear equations [1]. This paper used was that of establishing approximate solutions in Cauchy problem, and then proving a priori inequality of the set of these solutions. The integral Volterra operators obtain the most usual property of the concept of abstract Volterra operator, and their theory is the most developed. Then other types of Volterra operators, not necessarily developing integral inequalities, provide better examples of abstract Volterra operators [5]. It is our hope that this theory of a priori inequality with Volterra operators will establish in the coming years, to the extent of being able to offer the necessary in the several applications of those results in science and technology.

The several applications of a priori inequality operators are obtained to the dynamics of nuclear reactors, to control theory, and to continuous linear programming. The same author has also considered the extension of the results in linear boundary conditions which contain the priori estimate [3]. Schauder observed the possibility of extending the inequality in Leray Schauder theory [3] to more linear and even nonlinear spaces.

The continuity and boundedness properties of nonlinear Nemytskii operators acting in a priori inequality of integrable functions are considered in [6]. It is known that the Cauchy problem as well as the inequality of solutions of operators depends mainly on the unknown function. Our paper is that we obtain a nonlinear boundary value problem of a priori inequalities involving functions and derivatives higher than the first, which would not be the condition if we reduced to differential operators [7]. Thomas-Fermi [2] related theories of atoms and molecules which result may be of some interest in the theory of non-linear boundary value problems.

* Corresponding author.
E-mail address: balamurali.maths@gmail.com

Let $\dot{x} = Fx$ be the reducible to the form

$$\dot{x}(t) = \varphi(t, x(a)). \tag{1}$$

Since the Cauchy problem $\dot{x} = Fx, x(a) = \alpha$ is derived. We consider the boundary value problem

$$\dot{x} = Fx, lx = \beta \tag{2}$$

where

$$\beta \; := \; \Psi x(a) + \int_b^a \Phi(s)\dot{x}(s)ds, det\Psi \neq 0. \tag{3}$$

The general solution of the equation $\dot{x} = Fx$ has the form

$$x(t) = \alpha + \int_a^t \varphi(s,\alpha)ds. \tag{4}$$

Applying the functional ℓ to both sides, we get

$$\gamma \; := \; \Psi^{-1}\beta = \alpha + \Psi^{-1}\int_b^a \Phi(s)\varphi(s,\alpha)ds \; := \; \alpha + \theta\alpha. \tag{5}$$

From equation (2), we have

$$\alpha + \theta\alpha = \gamma \tag{6}$$

with respect to α. Let $|.|$ be the norm function in $\mathbb{R}^n$. We set

$$|Q\alpha| \leq \left\|\Psi^{-1}\right\| \int_a^b \left\|\phi(s)\right\|.\left|\varphi(s,\alpha)\right| ds. \tag{7}$$

Let the function $m(s,\alpha)$ be bounded or

$$\lim_{\alpha\to\infty} \frac{m(s,\alpha)}{\alpha} = 0$$

holds. Then equation (2) is derived.
Therefore, we can derive solvability of the form

$$\left|\varphi(s,\alpha)\right| \leq m(s,\alpha). \tag{8}$$

We consider the boundary value problem

$$\dot{x}(t) = (Fx)(t) \; := \; f(t,(S_h x)(t),(S_g \dot{x})(t)), \qquad lx = \beta. \tag{9}$$

where $t - h(t) > 0$, $t - g(t) > 0$. Then the solution of the Cauchy problem can be developed by the step-by-step method. We get

$$lx = \Psi x(a) + \int_b^a \Phi(s)\dot{x}(s)ds, det\Psi \neq 0, \tag{10}$$

$$\lim_{\|x\|_D\to\infty} \frac{\|Fx\|_L}{\|x\|_D} = 0.$$

2. Preliminaries

A priori inequality (2) satisfied on the ball with radius r is said to satisfy the property Λ if it holds for all solutions $x_\lambda, |x_\lambda(a)| \le r$, of the equation

$$\dot{x} = \lambda F_0 x, \lambda \in [0,1]. \tag{11}$$

Let a canonical a priori inequality on the ball with radius r be satisfied for solutions of (1) and satisfies the property Λ. Then, by Leray Schauder theorem [3], the Cauchy problem

$$\dot{x} = Fx, \quad x(a) = \alpha, \quad |\alpha| \le r, \tag{12}$$

has at least one solution $x \in D$. Thus, the equation

$$x(t) = \alpha + \int_a^t z(s)ds \tag{13}$$

reduces (11) to the form

$$z = \lambda F_0(\alpha + \int_a^{(\cdot)} z(s)ds) \tag{14}$$

with continuous compact operator $\Omega : L \to L$,

$$\Omega z = \lambda F_0(\alpha + \int_a^{(\cdot)} Z(s)ds) \tag{15}$$

By a priori inequality, the a priori estimate

$$||z_\lambda||_L \le \int_a^b m(s, |\alpha|)ds \tag{16}$$

holds for any $\lambda \in [0,1]$.
Hence, by Leray-Schauder theorem [3], the equation $z = \Omega z$ has a solution z_1 and consequently

$$x(t) = \alpha + \int_a^t z_1(s)ds \tag{17}$$

is a solution of the Cauchy problem.

Definition 2.1.
[1] Consider the equation

$$\dot{x} = Fx, \tag{18}$$

a canonical a priori inequality is said to be satisfied in the ball with radius r if there exists a function $m : [a,b] \times [0,r] \to [0,\infty)$ such that $m(\cdot\,,\ s)$ is summable at each $s \in [0,r]$ and the inequality

$$|\dot{x}(t)| \le m(t,\ |x(a)|) \tag{19}$$

holds for any solution x of (1) with $|x(a)| \le r$.

Definition 2.2.
[1] The cauchy problem

$$\dot{y} = \Omega(t,y), \qquad y(a) = \beta \tag{20}$$

is said to have an increasing function $y(t,\beta)$on [a, b] if y is a solution such that for each $c \in (a,b]$, any solution y_c of the equation $\dot{y} = \Omega(t,y)$, defined on [a, c) and satisfying the initial condition $y_c(a) = \beta$, satisfies the inequality $y_c(t) \le y(t),\ t \in [a,c)$. The Cauchy problem of equation (20) is said to be the majorant Cauchy problem.

Definition 2.3.
[1] The majorant Cauchy problem (20) constructed according to operators , A_1, A_2, N and M is called the majorant Cauchy problem relevant to inequality $\left|\mathscr{H}(\theta x, \Sigma y)\right| \leq A_1 N M x + A_2 \left|y\right|$.

Definition 2.4.
[1] Consider the equation

$$\dot{y} = \Omega(t, y) \tag{21}$$

is said to be a majorant equation that corresponds to inequality
$|(Fx)(t)| \leq (A_1 N M x)(t) + (A_2 |\dot{x}|)(t),\, t \in [a,b],\ x \in \mathbf{D}$ if the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ is defined by

$$\Omega(t, y) = \omega(t, u(t)y)[A_1^*(I - A_2^*)^{-1}\upsilon](t). \tag{22}$$

Definition 2.5.
[1] The operator $F : \mathbf{D} \to \mathbf{L}$ is said to satisfy condition H if there exist the operators $\theta : \mathbf{D} \to \mathbf{Z}_1$, $\Sigma : \mathbf{L} \to \mathbf{Z}_2$, H : $\theta\mathbf{D} \times \Sigma\mathbf{L} \to \mathbf{L}$, H : $\theta\mathbf{D} \to \mathbf{L}$ such that the operator F may be represented in the form

$$Fx = \mathscr{H}(\theta x, \Sigma \dot{x}) \tag{23}$$

the product $H\theta : \mathbf{D} \to \mathbf{L}$ is continuous compact, and the function y = Hz is the unique solution to the equation

$$y = \mathscr{H}(z, \Sigma y) \tag{24}$$

for each $z \in \theta\mathbf{D}$.

3. Main results

Lemma 3.1 ([5]).
Let $A : \mathbf{L}^1 \to \mathbf{L}^1$ be a linear isotonic Volterra nilpotent operator and let the function $\upsilon \in \mathbf{L}^1{}_\infty$ be positive. Then for any positive $y \in \mathbf{L}^1$, the inequality

$$\int_a^t \upsilon(s)(Ay)(s)ds \geq \int_a^t k(s)y(s)ds,\, t \in [a,b]., \tag{25}$$

holds with

$$k(s) = \frac{d}{dt}\int_a^b \upsilon(s)(A_{\chi[a,t]})(s)ds, \tag{26}$$

$\chi[a,t]$is the characteristic function of the interval [a, t].

Proof. The inequality (25) represents by each $t \in [a,b]$ a linear functional on the space of functions summable on [a, t]. We set

$$\int_a^t \upsilon(s)(Ay)(s)ds = \int_a^t K(t,s)y(s)ds, \tag{27}$$

where the kernel K(t, s) is bounded by t.
Which implies the inequality

$$K(t,s) \geq K(b,s), \tag{28}$$

$$K(t,s) \leq K(b,s). \tag{29}$$

From equation (28) and (29), we get

$$K(t,s) = K(b,s). \tag{30}$$

Conversely: there exist t_0 and a set $\Delta \subset [a, t_0]$ of positive measure such that

$$K(t_0,s) < K(b,s),\, s \in \Delta \tag{31}$$

Let $\chi\Delta$ be the characteristic function of the set Δ. We have

$$l = \int_\Delta (K(b,s) - K(t_0,s))ds < 0. \tag{32}$$

Which leads to a contradiction (28). It can be verify that

$$\int_a^b K(b,s)ds = \int_a^b K(b,s)_{\chi[a,t]}(s)ds = \int_a^b \upsilon(s)(A_{\chi[a,t]})(s)ds. \tag{33}$$

The proof is complete. □

Remark 3.1.
We denote
$$k(t) = (A^*\upsilon)(t) \tag{34}$$
and
$$\int_a^b \upsilon(s)(Ay)(s)ds = \int_a^b (A^*\upsilon)(s)y(s)ds. \tag{35}$$
We consider the inequality
$$z \geq \mathscr{M}(\upsilon, z) := AN\mathscr{M}_1(\upsilon, z). \tag{36}$$
Hence the operator $\mathscr{M}_1$ acts from $\mathbb{R}^1 \times \mathbf{L}^1$ into a linear space Ξ of measurable functions $\xi : [a,b] \to \mathbb{R}^1$ and is denoted by
$$\mathscr{M}_1(\upsilon, z)(t) = q(t)\upsilon + u(t)\int_a^t \upsilon(s)z(s)ds \tag{37}$$
with positive $\upsilon \in \mathbf{L}^1_\infty q, u \in \Xi, q(t) \leq u(t)$ a.e. on [a, b], $N : \Xi \to \mathbf{L}^1$ is the operator of Nemytskii [6], $(N\xi)(t) = \omega((t)\xi(t)), \omega(t,\cdot)$ is continuous and increasing function. It is clear that the inequality (36) on [a, c] ⊂ [a, b] for $\upsilon \geq 0$. The solution is positive summable on [a, c] interval z such that
$$z(t) \geq M(\upsilon, z^c)(t) \tag{38}$$
a.e. on $[a,c]$. Hence z^c is a summable on [a, b] interval such that $z^c(t) = z(t)$ a.e.on interval [a, c]. In the majorant Cauchy problem, we define the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ by
$$\Omega(t,y) = \omega(t, u(t)y)(A^*\upsilon)(t). \tag{39}$$

Lemma 3.2.
The Cauchy problem (20) have an increasing function $y(t,\beta)$ on [a, b]. If $\upsilon \geq \beta$, then z be an interval on $[a,c] \subseteq [a,b]$ to inequality (36).

Proof. We consider the inequality
$$z(t) \geq (AN\xi_\upsilon)(t) \tag{40}$$
with $\xi_\upsilon(t) = u(t)y(t,\upsilon)$ holds a.e. on [a,c].
$$z^c(t) = \begin{cases} z(t) & \text{if } t \in [a,c], \\ 0 & \text{if } t \notin [a,c]. \end{cases} \tag{41}$$
It is clear that z^c is the solution to inequality (36) defined on [a, b]. The inequality
$$\eta(t) \geq u(t)\int_a^t \upsilon(s)(AN\eta)(s)ds + q(t)\upsilon \tag{42}$$
for $\eta(t) = \mathscr{M}_1(\upsilon, z^c)(t)$ holds a.e. on [a,b]. Here $\zeta(t) = \eta(t)/u(t)$.
We define
$$\zeta(t) \geq \int_a^t \upsilon(s)[AN(\zeta \cdot u)](s)ds + \upsilon, \tag{43}$$
and
$$\zeta(t) \geq \int_a^t (A^*\upsilon)(s)N(\zeta \cdot u)(s)ds + \upsilon. \tag{44}$$
Let w be the right hand side inequality. It means that $w \in \mathbf{D}^1$, $\dot{w} = \Omega(t,\zeta) \geq \Omega(t,w)$, $w(a) = \upsilon$. By Chaplygin theorem [8] on differential inequality we get $w(t) \geq y(t,\upsilon)$. Hence
$$\zeta(t) = \frac{\eta(t)}{u(t)} \geq y(t,\upsilon), \tag{45}$$
$$\therefore\ \eta(t) \geq u(t)y(t,\upsilon).$$
From the property of inequality (36), we get
$$z^c(t) \geq [AN(u(\cdot)y(\cdot,\upsilon))](t) \tag{46}$$
a.e. on [a, b]. The proof is complete. □

Theorem 3.1.
*Let F : **D** → **L** satisfy the conditions H and*

$$\left|\mathscr{H}(\theta x, \Sigma y)\right| \geq A_1 N M x + A_2 \left|y\right|. \quad (47)$$

The majorant Cauchy problem (20) have an increasing function $y(t, \beta)$ defined on [a, b] for $\beta \geq 0$. If a priori inequality (19) for the solutions to (18) holds, then the function m is summable.

Proof. Let us consider

$$|F x| = |\mathscr{H}(\theta x, \Sigma \dot{x})| \geq A_1 N M x + A_2 |\dot{x}|, \quad (48)$$

where

$$\dot{x} = \lambda H \theta x$$

The majorant Cauchy problem constructed according to the inequality $|x(a)| \leq \beta$ and $\lambda = 1$. Hence the inequality (19) is obvious for $\lambda = 0$ and it can be verify that $\lambda \in (0, 1)$. General solution of $\dot{x} = \lambda H \theta x$ is a solution to the equation

$$\dot{x} = \lambda \mathscr{H}(\theta x, \Sigma \frac{1}{\lambda} \dot{x}). \quad (49)$$

We set

$$|\dot{x}| \geq \lambda \left|\mathscr{H}(\theta x, \Sigma \frac{1}{\lambda} \dot{x})\right|. \quad (50)$$

Therefore, using (48)

$$|\dot{x}| \geq \lambda A_1 N M x + \lambda A_2 \left|\frac{1}{\lambda} \dot{x}\right| \geq A_1 N M x + A_2 |\dot{x}|. \quad (51)$$

Then, a.e. on [a, c],

$$|\dot{x}(t)| \leq m(t, |x(a)|), \quad (52)$$

with

$$m(t, \upsilon) = \left\{(I - A_2)^{-1} A_1 N z_\upsilon\right\}(t), z_\upsilon(t) = u(t) y(t, \upsilon). \quad (53)$$

We should be allowed that (18) is reducible with $F_0 = H\theta$ if F : **D** → **L** satisfies the condition H. We consider the reducibility to the canonical form if $\theta x = x(a)$.
Let F : **D** → **L** satisfy condition H, and

$$\left|\mathscr{H}(\theta x, \Sigma y\right| \leq A_1 N M x + A_2 \left|y\right| \quad (54)$$

$$\dot{x}(t) = (F x)(t), t \in [a, b]; \eta x = 0 \quad (55)$$

where $\eta : \mathbf{D} \to \mathbb{R}^n$.

$$\dot{x}(t) = \varphi(t, x(a)). \quad (56)$$

If $\dot{x} = F x$ is reducible to the form (56),then solvability of equation (55) is equivalent to equation

$$\eta[\alpha + \int_a^{(\cdot)} \varphi(s, \alpha) d s] = 0. \quad (57)$$

We denote $\alpha = A\alpha$ and $A : \mathbb{R}^n \to \mathbb{R}^n$,

$$A\alpha = \alpha - \eta[\alpha + \int_a^{(\cdot)} \varphi(s, \alpha) d s]. \quad (58)$$

The solution α_0 of (58) obtains to the solution x of problem (55), which contradicts with the solution of the Cauchy problem

$$\dot{x} = Fx, \quad x(a) = \alpha_0. \tag{59}$$

We convert the equation (55) to (58) the function defined by $\varphi(t,\alpha)$.

$$\left|\varphi(t,\alpha)\right| \le m(t,|\alpha|) \tag{60}$$

It gives a priori inequality (19).
Let the functional $\mu : \mathbf{L}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ increase, and

$$\left|x(a) - \eta x\right| \le \mu(|\dot{x}(\cdot)|, |x(a)|) \;\; \forall x \in \mathbf{D}. \tag{61}$$

Then the operator A has a fixed point on (55). We will assume:
(i) the set of positive solutions to a priori inequality

$$\delta \le \mu[m(\cdot,\delta),\delta] \tag{62}$$

is bounded;
(ii) the functions m(t, ·) and $(z,\cdot)$ increase and there exists $\delta > 0$ such that

$$\delta \ge \mu[m(\cdot,\delta),\delta] \tag{63}$$

Condition (62) is satisfied if

$$\overline{\lim_{\delta\to\infty}} \frac{1}{\delta}\mu[m(\cdot,\delta),\delta] < 1. \tag{64}$$

The Cauchy problem is solvable for all such that $|\alpha| \le \delta$ and a priori inequality (19) holds on the ball with radius δ.
Let $F : \mathbf{D} \to \mathbf{L}$ satisfy condition H. Then the equation $\dot{x} = Fx$ is equivalent to the equation

$$x(t) = x(a) + \int_a^t (H\theta x)(s)ds \tag{65}$$

with continuous compact $H\theta : \mathbf{D} \to \mathbf{L}$ and problem (55) is equivalent to the equation

$$x = \Pi x := x(a) - \eta x + \int_a^{(\cdot)} (H\theta x)(s)ds. \tag{66}$$

By Leray-Schauder theorem [3], (66) has a solution if there exists a general a priori estimate of all the solutions $x\lambda$ of the equation

$$x = \lambda\pi x. \tag{67}$$

$$||x_\lambda||_d \le d, \; \lambda \in [0,1], \; d > 0. \tag{68}$$

Any solution x of (66) is a solution to the equation

$$|\dot{x}(t)| \le m(t,|x(a)|.) \tag{69}$$

Let $\eta x = 0$. From equation (69) the inequality

$$|x(a)| \le \mu(|\dot{x}(\cdot)|, |x(a)|) \tag{70}$$

If x is a solution to problem, then the norm $|x(a)|$ satisfies the inequality

$$\delta \le \mu[m(\cdot,\delta),\delta]. \tag{71}$$

It obtained the existence of δ_0 such that $\delta_0 \ge \delta, \; \delta > 0$ that satisfies inequality (71). We denote $|x(a)| \le \delta_0$, and

$$||x||_D \le \delta_0 + sup\,||m(\cdot,\delta)||_{\mathbf{L}^1}. \tag{72}$$

$\therefore\, m(\,\cdot\,,\delta)$ is summable.
The proof is complete now. □

Remark 3.2.
The main difficulty of the priori estimate (68) is considered for $\lambda = 1$. In general case, any assumptions of the inequality (19) is impossible to get the priori estimate (68) for $\lambda \in (0,1)$.

4. An example

We consider the nonlinear boundary value problem

$$\sqrt{t}\,\ddot{x} = g(x) := \begin{cases} x^{5/2} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{73}$$

where

$$t \in [0,\tau],\ x(0) = \alpha,\ \ddot{x}(\tau) = \frac{x(\tau)+d}{\tau},\ \alpha, d \geq 0. \tag{74}$$

Since the Thomas-Fermi equation [2] contains in the statistical theory. Rewrite the equation in the form

$$\ddot{x} = y, \quad \dot{y} = \frac{1}{\sqrt{t}} g(x), \quad t \in [0,\tau], \tag{75}$$

$$x(0) = \alpha, \quad y(\tau) = \frac{x(\tau)+d}{\tau}. \tag{76}$$

From equation (75), we have

$$\int_0^\tau \sqrt{s}\,\dot{y}(s) y(s) ds = \int_0^\tau g(x(s)) y(s) ds. \tag{77}$$

$$\int_0^\tau \sqrt{s}\,\dot{y}(s) y(s) ds = \frac{1}{2}\sqrt{\tau} y^2(\tau) - \frac{1}{4}\int_0^\tau y^2(s) \frac{ds}{\sqrt{s}} \leq \sqrt{\tau} y^2(\tau) \tag{78}$$

Integration by parts,

$$\int_0^\tau g(x(s)) y(s) ds = \int_0^{x(\tau)} g(\xi) d\xi \leq \frac{2}{7} x^{7/2}(\tau) - \frac{2}{7}\alpha^{7/2}. \tag{79}$$

Therefore,

$$|x(\tau)| \leq \left\{ \alpha^{7/2} + \frac{7}{4}\sqrt{\tau} y^2(\tau) \right\}^{2/7}. \tag{80}$$

$$\left| y(\tau) \right| \leq \frac{\alpha^{7/2} + \frac{7}{4}\sqrt{\tau} y^2(\tau)^{2/7} + d}{\tau} \tag{81}$$

Therefore a priori inequality has the sublinear growth, there exists m > 0 such that $\left| y(\tau) \right| \leq$ m. Hence the general case every nonlinear boundary value problem demands to find a form of a priori inequality such that it allows us to obtain the required a priori estimate from the boundary condition.

References

[1] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, Introduction to the Theory of Functional Differential Equations: Methods and Applications, Hindawi Publishing Corporation, New York, USA, 2007.

[2] E. H. Lieb, Thomas-Fermi and related theories of atoms and molecules, Rev. Mod. Phys. 53 (1981) 603-641

[3] Jean Mawhin, Lery-Schauder Continuation Theorems in the absence of a Priori Bounds, Topological Methods in Nonlinear Analysis Journal of the Juliusz Schauder Center 9 (1997) 179-200.

[4] David W. Boyd, Inequalities for positive integral operators, Pacific Journal of Mathematics 38(1) (1971) 9-24.

[5] C. Corduneanu, Abstract Volterra Equations: A Survey, Mathematical and Computer Modelling 32 (2000) 1503-1528.

[6] Janusz MATKOWSKI, Functional Equations and Nemytskii Operators, Funkcialaj Ekvacioj 25 (1982) 127 –132.

[7] Jacek Szarski, Differential inequalities, Pan'stwowe Wydawn. Naukowe, 1967.

[8] B. N. Babkin, The theorem of S. A. Chaplygin on differential inequalities, Mat. Sb.(N.S.) 46 (1958) 389-398.