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A model of a fishery resource in the presence of water hyacinth, the case of lake Victoria

Research Article

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Abstract: In this paper, we propose and analyse a mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones: water hyacinth zone and water hyacinth free zone, fishing was allowed in both zones. The developed model describes the quantitative relationship of water hyacinth and fish biomass densities. The conditions for the validity of the model were established. The biological equilibria of the model was determined, and proved to be not locally stable. Small increments on the parameters of the model associated with water hyacinth shifted the equilibria of the model significantly, moreover in each shift of an equilibrium point, the fish subpopulation in water hyacinth zone decreased more rapidly than the fish subpopulation in water hyacinth free zone. Optimal harvesting policy in the two zones was discussed using Pontryagin's maximum principle where by mathematical equations governing the optimal fish biomass densities and optimal harvestings at any time t were determined.
The model best fits the Lake Victoria fishery which has greatly hampered by water hyacinth infestation. The study provides a mathematical proof that infestation of water hyacinth have negative effects on fish stocks and fish production.

MSC: 65P99 • 92B05 • 37N25

Keywords: fishery • water hyacinth • equilibria • mathematical model • optimal harvesting • stability analysis

1. Introduction

The presence of water hyacinth in Lake Victoria has been cited as far as 1989 [1, 4, 5]. The water hyacinth effect has been greatly felt in the fish industry. Available literature [3, 6, 7, 12, 13] links water hyacinth infestation to fish production. Water hyacinth infestation appears at different intensities ranging from low to moderate and high. In areas with high infestation the impact of the hyacinth has been severe on fishing activities due to increase time to access the fishing ground and or landing site.
Water hyacinth mats generates large amount of organic matter. As the organic matter decomposes, biological oxygen demand increases and water quality deteriorates. The oxygen can be reduced to such low levels that it leads to massive fish kills due to oxygen depletion in the water column [11]. This results in loss of aquatic biodiversity [9]. Fish production has been affected by water hyacinth as its mats blocks the path for fishermen, and thus not only increases the time used to access the fishing ground but also leads to the destruction of fishing boats and gears. This leads to increases in the cost of fishing and by so doing leading to diminished fish catch and profitability.
The idea of this study is developed on the work of [5] where their model focused on the effect of water hyacinth on the catchability of fish and ignored the biological effects of hyacinths on the fish stock function, moreover their

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model did not assume equilibrium. The model in this study is built on the assumptions that the catch per unit effort is directly proportional to the density of fish, the density of fish is directly proportional to the abundance of fish at time t and the catchability coefficient decreases as the abundance level of water hyacinth increases (the water hyacinth coverage of the fishing grounds reduces the ease with which fish is harvested). Further it was assumed that water hyacinth abundance leads to biological effects such as fish death and fish migration.The main objective of this study was to analyse the effect of water hyacinth on the equilibrium of fish biomass densities. We model this phenomenon in aquatic habitat that consists of two zones: zone I, water hyacinth zone and zone II, water hyacinth free zone.The conditions for the existence of biological equilibria were derived and their stability were analysed.

2. Mathematical model

Consider two equal zones (interms of area and volume of water) in Lake Victoria. First zone being a zone with water hyacinth and the second is the water hyacinth free zone, and fishing is taking place in both zones. It should be noted that we are considering the same fish species in both zones.

ZONE I	ZONE II
Water hyacinth zone	Water hyacinth free zone

Table 1. Parameters and variables of the model

Variables	Descriptions
η	rate at which water hyacinth reduces fish catch
σ	fish migration rate from water hyacinth zone to water hyacinth free zone
α	fish death rate due to oxygen depletion caused by water hyacinth
β	fish death rate due to water pollution caused by water hyacinth
E	fishing effort applied equally to both zone
q	catchability coefficient of the fishing vessel
$x(t)$	biomass density of fish subpopulation in water hyacinth zone
$y(t)$	biomass density of fish subpopulation in water hyacinth free zone
r	intrinsic growth rate of fish subpopulation in water hyacinth zone
s	intrinsic growth rate of fish subpopulation in water hyacinth free zone
K	carrying capacity of fish subpopulation in water hyacinth zone
L	carrying capacity of fish subpopulation in water hyacinth free zone

In this study it is assumed that the rate at which water hyacinth reduces fish catch (η) increases as the water hyacinth abundance increases, since water hyacinth are living plants (weeds) which grow with time.

Therefore,

$$\text{Fish catch in water hyacinth zone} = qEx - \eta qEx = (1-\eta)qEx \tag{1}$$

When $\eta = 1$, then no fish catch in water hyacinth zone.

The dynamics of fish sub populations in zone I and zone II may be governed by the following autonomous system of differential equations:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - (1-\eta)qEx - (\sigma+\alpha+\beta)x, \qquad (2)\text{a}$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) + \sigma x - qEy, \qquad (2)\text{b}$$

$$x(0) > 0, y(0) > 0.$$

The parameters $r, s, q, \sigma, \eta, \alpha, \beta, K$ and L are assumed to be positive constants, and $0 < q < 1$, and $0 < \eta < 1$. Rearrangement of equation (2), a technique applied by [2], gives

$$\frac{dx}{dt} = -x\left[\frac{r}{K}x + \sigma + \alpha + \beta + qE - (r + \eta qE)\right], \qquad (3)\text{a}$$

$$\frac{dy}{dt} = -y\left[\frac{s}{L}y + qE - s\right] + \sigma x. \qquad (3)\text{b}$$

We noted that in (3)a if $\sigma+\alpha+\beta+qE-(r+\eta qE)>0$ then $\frac{dx}{dt}<0$, similarly in (3)b we observe that if there is no fish migration from water hyacinth zone to water hyacinth free zone (i.e, $\sigma=0$) and $qE-s>0$ then $\frac{dy}{dt}<0$.

Therefore, inorder for $\frac{dx}{dt}>0$ and $\frac{dy}{dt}>0$ we require:

$$\sigma+\alpha+\beta+qE<r+\eta qE, \quad (4)a$$
$$qE<s. \quad (4)b$$

Hence throughout our analysis in this work we assume equation (4) to hold.

If $\eta=1$ (No fish catch in water hyacinth zone) then (4) becomes

$$\sigma+\alpha+\beta<r, \quad (5)a$$
$$qE<s. \quad (5)b$$

3. Existence of equilibria

The equilibria $P(x^\star,y^\star)$ of the model (2) is obtained by solving $\frac{dx}{dt}=\frac{dy}{dt}=0$. We obtain

$$x^\star=\frac{K}{r}\left[r+\eta qE-(qE+\sigma+\alpha+\beta)\right] \quad (6)$$

knowing the value of $x^\star$, the value of $y^\star$ can be computed using (7),

$$y^{\star 2}+ay^\star+b=0 \quad (7)$$

where,

$$a=\frac{-L}{s}(s-qE), \quad (8)a$$
$$b=\frac{-\sigma KL}{sr}\left[r+\eta qE-(qE+\sigma+\alpha+\beta)\right]. \quad (8)b$$

4. Dynamical behaviour of equilibria

4.1. Local stability

The variational matrix for the system (2) at $P(x^\star,y^\star)$ is given by

$$J(x^\star,y^\star)=\begin{bmatrix} r-\frac{2rx^\star}{K}-(1-\eta)qE-(\sigma+\alpha+\beta) & 0 \\ \sigma & s-\frac{2sy^\star}{L}-qE \end{bmatrix} \quad (9)$$

The characteristic equation of (9) is

$$\xi^2+M_1\xi+M_2=0 \quad (10)$$

where,

$$M_1=\frac{2rx^\star}{K}+\frac{2sy^\star}{L}+\sigma+\alpha+\beta+2qE-(s+r+\eta qE) \quad (11)a$$
$$M_2=\left[\frac{2rx^\star}{K}+\sigma+\alpha+\beta+qE-(r+\eta qE)\right]\left[\frac{2sy^\star}{L}+qE-s\right] \quad (11)b$$

Applying the Routh-Hurwitz criteria [10], $P(x^\star,y^\star)$ is LAS if and only if $M_1>0$ and $M_2>0$.
Analysing (11) using (4) above we see that

$$[\sigma+\alpha+\beta+qE-(r+\eta qE)]<0 \quad \text{and} \quad qE-s<0.$$

Hence,

- M_1 is **NOT** always positive for all $(x^\star,y^\star)$
- M_2 is **NOT** always positive for all $(x^\star,y^\star)$.

Therefore the equilibria $P(x^\star,y^\star)$ is **NOT** locally stable.

Table 2. Variations of parameters associated with the effect of water hyacinth

Parameters	Fig. 1	Fig. 2	Fig. 3	Fig. 4
η	0.20	0.21	0.22	0.23
σ	0.10	0.11	0.12	0.13
α	0.40	0.41	0.42	0.43
β	0.30	0.31	0.32	0.33

4.2. Numerical analysis of the model

Consider an example with the following parameters of the model

$K = 1900, L = \frac{8000}{43}, E = 10, q = 0.00625, r = 0.95, s = 0.50$ and the following varied parameters in Table 2,

- It is assumed that as η increases also σ, α and β increases.
- In all four cases equation (4) holds.

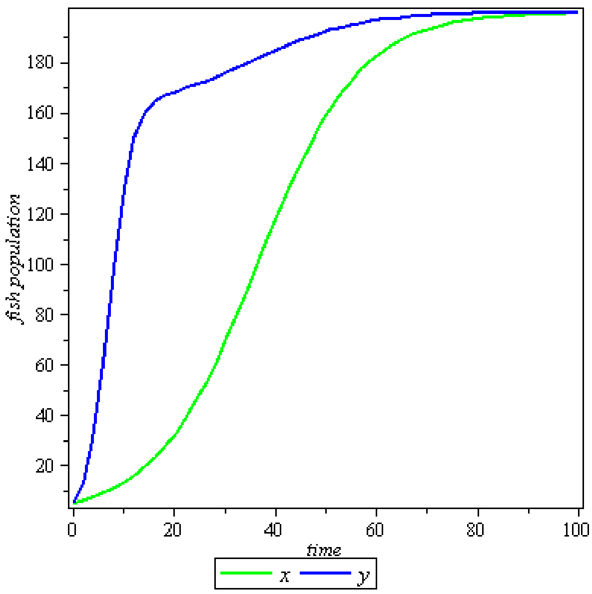

Fig. 1. Equilibrium state of x and y when $\eta = 0.20, \sigma = 0.10, \alpha = 0.40, \beta = 0.30$

5. Optimal harvesting policy

In this section we employ the Pontryagins's Maximum Principle to obtain the optimal value of η for the optimal harvesting in both zones. In this case η is considered as a control variable.
Consider c as the the total costs of harvesting in both zones, and p as the selling price per unit biomass of harvested fish. The present value J of a continous time-stream of revenues is given by

$$J = \int_0^{\infty} e^{-\delta t} \left[pqE(x+y) - cE \right] dt \tag{12}$$

where δ is the instantaneous rate of annual discount. If we consider harvesting effort as a dynamic variable which change with the change of revenues then the dynamics becomes

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - (1-\eta)qEx - (\sigma + \alpha + \beta)x, \quad \text{(13)a}$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma x - qEy, \quad \text{(13)b}$$

$$\frac{dE}{dt} = \theta E\left[pq(1-\eta)x + pqy - c\right]. \quad \text{(13)c}$$

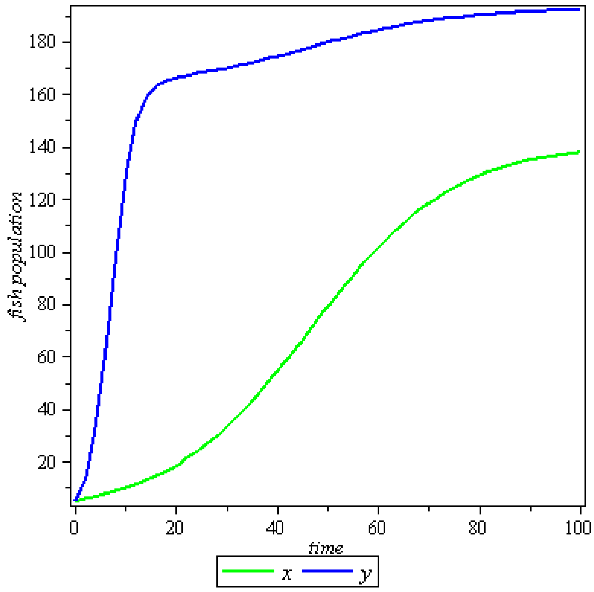

Fig. 2. Equilibrium state of x and y when $\eta = 0.21, \sigma = 0.11, \alpha = 0.41, \beta = 0.31$

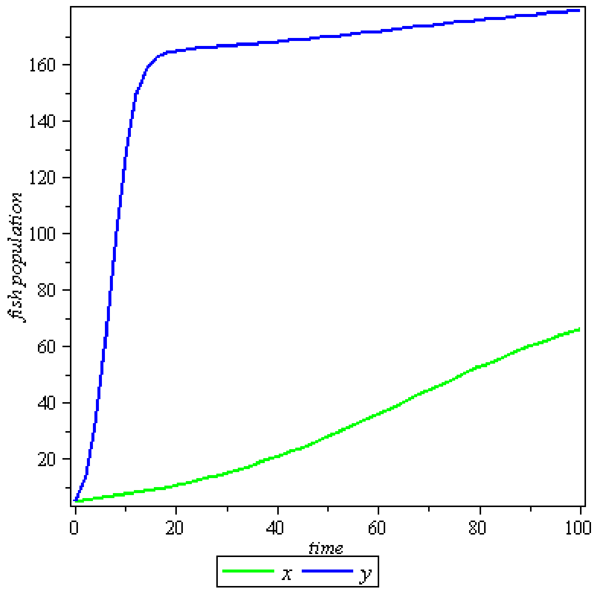

Fig. 3. Equilibrium state of x and y when $\eta = 0.22, \sigma = 0.12, \alpha = 0.42, \beta = 0.32$

where θ is the stiffness parameter.
When the system (13) is at equilibrium $P(x^\star, y^\star)$ the total fishing costs in both zones is given by

$$c = pq[(1-\eta)x^\star + y^\star] \tag{14}$$

Thus, our objective is to maximize J subject to the state equations (13) and to the control constraints $0 \le E \le E_{max}$ and $0 < \eta < \eta_{max}$
The corresponding Hamiltonian function is

$$\begin{aligned} H = {} & e^{-\delta t}[pqE(x+y) - cE] + \lambda_1\left[rx\left(1-\frac{x}{K}\right) - (1-\eta)qEx - (\sigma+\alpha+\beta)x\right] \\ & + \lambda_2\left[sy\left(1-\frac{y}{L}\right) + \sigma x - qEy\right] + \lambda_3[\theta E\left(pq(1-\eta)x + pqy - c\right)] \end{aligned} \tag{15}$$

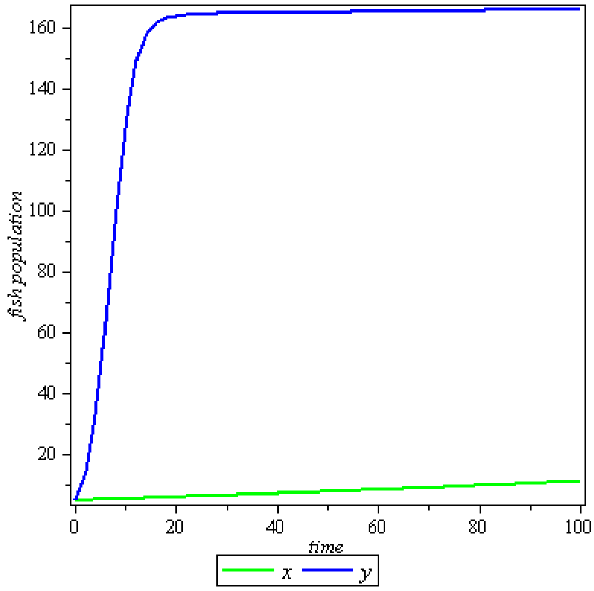

Fig. 4. Equilibrium state of x and y when $\eta = 0.23, \sigma = 0.13, \alpha = 0.43, \beta = 0.33$

Therefore, singular control are given by

$$\frac{\partial H}{\partial \eta} = 0 \tag{16)a}$$

$$\frac{\partial H}{\partial E} = 0 \tag{16)b}$$

Applying (16)a we obtain $\lambda_3 = 0$, and using (16)b we obtain:

$$e^{-\delta t}[pqx + pqy - c] = \lambda_1(1-\eta)qx + \lambda_2 qx \tag{17}$$

Equating coefficients of x and y in equation (17) resulted into

$$\lambda_1 = \frac{pe^{-\delta t}}{(1-\eta)} \tag{18)a}$$

$$\lambda_2 = pe^{-\delta t} \tag{18)b}$$

Again, by Pontryagin's maximum principle we have:

$$\frac{\mathrm{d}\lambda_1}{\mathrm{dt}} = -\frac{\partial \mathrm{H}}{\partial x} \tag{19)a}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{dt}} = -\frac{\partial \mathrm{H}}{\partial y} \tag{19)b}$$

Using, (19)a and (18)b resulted into the following equation

$$\frac{d\lambda_1}{dt} - A_1\lambda_1 = -A_2 e^{-\delta t} \tag{20}$$

where,

$$A_1 = -\left[r - \frac{2xr}{K} - (1-\eta)qE - (\sigma + \alpha + \beta)\right]$$

$$A_2 = p(qE + \sigma)$$

Solving (20) using the method employed by [8] resulted into the following

$$\lambda_1 = \left(\frac{A_2}{A_1 + \delta}\right) e^{-\delta t} \tag{22}$$

Equating (18)a with (22) resulted into (23),

$$A_2(1-\eta)=p(A_1+\delta) \tag{23}$$

Applying (19)b with litle simplifications resulted into the following

$$\frac{d\lambda_2}{dt}-B_1\lambda_2=-B_2e^{-\delta t} \tag{24}$$

where,

$$B_1=-\left(s-\frac{2sy}{L}-qE\right)$$
$$B_2=pqE$$

solving (24) using the method employed by [8] resulted into the following

$$\lambda_2=\left(\frac{B_2}{B_1+\delta}\right)e^{-\delta t} \tag{26}$$

Equating (18)b with (26) resulted into the following

$$\delta=\frac{B_2-pB_1}{p} \tag{27}$$

Substituting (27) into (23)gives the following

$$A_2(1-\eta)=p(A_1-B_1)+B_2 \tag{28}$$

Substituting expressions for A_1, A_2, B_1 and B_2 into (28) and making necessary simplifications we obtain the expression for optimal η corresponding with the optimal harvesting as

$$\eta_\delta=\frac{r+\frac{2sy}{L}-(\frac{2rx}{K}+s+\alpha+\beta)}{\sigma} \tag{29}$$

for which $\quad 0<\eta_\delta<1.$

Therefore, any harvesting of x (fish subpopulation in water hyacinth zone) and y (fish subpopulation in water hyacinth free zone) such that (29) is satisfied will be optimal.

Using equation (23) with the substitutions of A_1 and A_2 results into (30)a, while using (27) with the substitutions of B_1 and B_2 results into (30)b

$$\delta=s-\frac{2s}{L}y \tag{30)a}$$
$$\delta=r-\frac{2r}{K}x-\sigma(1-\eta)-(\sigma+\alpha+\beta) \tag{30)b}$$

Equating (30)a and (30)b results into(31), a governing equation for optimal fish biomass in both zones at any time t

$$\frac{2s}{L}y_\delta(t)-\frac{2r}{K}x_\delta(t)+r-s-\sigma(1-\eta)-(\sigma+\alpha+\beta)=0 \tag{31}$$

where,

$x_\delta(t)=$ optimal fish biomass in water hyacinth zone at any time t

$x_\delta(t)=$ optimal fish biomass in water hyacinth free zone at any time t.

For instance the optimal fish biomass equations for the above example is summarized in Table 3,

Optimal harvestings in the water hyacinth zone are computed using linear functions as shown in Table 4,

The optimal harvesting at any time t in water hyacinth free zone is governed by the following equations in all cases

$$h_2(t)=qEy_\delta(t)=0.0625y_\delta(t)$$

Table 3. Optimal fish biomass equations

Figure name	Optimal fish biomass equation
Fig. 1	$-0.001x_\delta(t)+0.00537y_\delta(t)-0.4300=0$
Fig. 2	$-0.001x_\delta(t)+0.00537y_\delta(t)-0.4669=0$
Fig. 3	$-0.001x_\delta(t)+0.00537y_\delta(t)-0.5036=0$
Fig. 4	$-0.001x_\delta(t)+0.00537y_\delta(t)-0.5401=0$

Table 4. Optimal harvesting equations in water hyacinth zone

Figure name	Optimal harvesting in water hyacinths zone $=(1-\eta)qEx_\delta(t)$
Fig. 1	$h(t)=0.050x_\delta(t)$
Fig. 2	$h(t)=0.049375x_\delta(t)$
Fig. 3	$h(t)=0.04875x_\delta(t)$
Fig. 4	$h(t)=0.048125x_\delta(t)$

6. Conclusion

We have presented Lotka-Volterra model (2) for a fishery resource in the presence of water hyacinth. We investigated the effect of water hyacinth on the equilibria of fish biomass densities. The equilibria of the model was analysed using Routh-Hurwitz criteria and proved to be not locally stable. The study has proved that water hyacinth have severe impact on fish stock, slight increments on the model parameters associated with water hyacinth completely changed the equilibria of the model, where fish subpopulation in the water hyacinth zone were extremly decreasing in the equilibria. Water hyacinth may lead to fish extinction, therefore efforts should be done to eradicate it with whatever means which are environmentally friendly.

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