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# On a new sequence of $q$-Baskakov-Szasz-Stancu operators

**Research Article**

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**Abstract:** This paper deals with Stancu type generalization of $q$-analogue of Baskakov-Szasz operators introducing a new sequence of positive $q$-integral operators. We show that it is a weighted approximation process in the polynomial space of continuous functions defined on $[0,\infty)$. An estimate for the rate of convergence and weighted approximation properties are also obtained for these operators.

**MSC:** 41A25 • 41A35

**Keywords:** Stancu type generalization • Continuity of functions • $q$-integrals • $q$-Baskakov-Szasz operators • Rate of Convergence • Weighted approximation.



## 1. Introduction

In approximation theory, $q$-calculus makes our research very interesting. In the year 1987, firstly $q$-analogue of classical Bernstein polynomials was given by A. Lupas [11]. In 1997, the most important $q$-analogue of the Bernstein polynomials was introduced by Phillips [13]. After that many researchers worked in this direction and proposed many $q$-operators and motivated their different properties related to some special functions, number theory and convergence behaviour. Gupta et al. [6] established the generating functions of some $q$-basis functions. In the theory of approximation, the convergence is important. In this context we mention some results for the convergence of $q$-discrete operators due to [2], [7] etc.
In the year 2012, Gupta-Kim-Lee [6] proposed the $q$-analogue of a new sequence of linear positive operators and modified the well known Baskakov-Szasz operators by Agrawal-Mohammad [1], which is given as

$$D_n^q(f;x)=[n]_q\sum_{v=1}^{\infty}p_{n,v}^q(x)\int_0^{q/(1-q^n)}q^{-v}s_{n,v-1}^q(t)f(tq^{-v})d_qt+p_{n,0}^q(x)f(0), \tag{1}$$

where

$$p_{n,v}^q(x)=\begin{pmatrix} n+v-1\\ v\end{pmatrix}q^{v(v-1)/2}\frac{x^v}{(1+x)_q^{-(n+v)}}$$
$$s_{n,v}^q(t)=E_q(-[n]_qt)\frac{([n]_qt)^v}{[v]_q!}$$

In case $q=1$, we get the original Baskakov-Szasz operators. It is observed that the above operators reproduce constant as well as linear functions.
Now we discuss Stancu type generalization [15] of the above $q$-Baskakov-Szasz operators. Very recently some results on Szasz-Mirakyan- Stancu operators are obtained by Maheshwari-Garg [12]. The Stancu variant is based on two

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parameters $\alpha,\beta$ satisfying $0\le\alpha\le\beta$. So for $0<q<1$ and $x\in[0,\infty)$, we propose $q$-Baskakov-Szasz-Stancu operators as

$$M^q_{n,\alpha,\beta}(f;x)=[n]_q\sum_{v=1}^{\infty}p^q_{n,v}(x)\int_0^{q/(1-q^n)}q^{-v}s^q_{n,v-1}(t)f\left(q^{-v}\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)\right)d_qt+p^q_{n,0}(x)f\left(\frac{\alpha}{[n]_q+\beta}\right), \quad (2)$$

where Baskakov operators $p^q_{n,v}$ and Szasz operators $s^q_{n,v}$ are defined as above.

The literature of $q$-calculus is described in different branches of Mathematics and Physics. Recently the $q$-analogues of the Baskakov operators and their Kantorovich and Durrmeyer variants have been studied by Aral-Gupta [2], [3] and Gupta-Radu [8] respectively. The history of $q$-calculus and its use to obtain some results can be seen in Ernst [4] and Kac-Cheung [9]. Some of the notations and concepts in $q$-calculus are recalled here. Hence for a real number $q\in(0,1)$ and $n\in N$

$$\begin{aligned}(x;q)_n=(1+x)^n_q &=: (1+x)(1+qx)...(1+q^{n-1}x), \qquad n=1,2,...\\ &=: 1 \qquad\qquad\qquad\qquad\qquad\qquad n=0.\end{aligned}$$

The $q$-binomial coefficients are given by

$$\left[\begin{array}{c} n \\ v \end{array}\right]_q=\frac{[n]_q!}{[v]_q![n-v]_q!}, \qquad 0\le v\le n.$$

The $q$-derivative of a function $f$ that is $D_qf$ is given by

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x}, \qquad x\neq 0$$

and $(D_qf)(0)=f'(0)$ provided $f'(0)$ exists.

The $q$-Gamma integral is defined by Koornwinder [10] as

$$\Gamma_q(t)=\int_0^{\frac{1}{1-q}}x^{l-1}E_q(-qx)d_qx, \qquad l>0,$$

where

$$E_q(x)=\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_q!}; \qquad \Gamma_q(l+1)=[l]_q\Gamma_q(l), \qquad \Gamma_q(1)=1.$$

Furthermore, $q$-Beta function defined by Solo-Kac [14] is

$$B_q(l,m)=K(A,l)\int_0^{\infty/A}\frac{x^{l-1}}{(1+x)^{l+m}_q}d_qx,$$

where $K(x,l)=\frac{1}{x+1}x^l\left(1+\frac{1}{x}\right)^l_q(1+x)^{1-l}_q$. Also $K(x,l)$ is a $q$-constant, that is, $K(qx,l)=K(x,l)$ and in case of $l$ to be an integer, it is independent of $x$. In particular for any integer $n\in N$, we have

$$K(x,n)=q^{\frac{n(n-1)}{2}}, \qquad K(x,0)=1, \qquad B_q(l,m)=\frac{\Gamma_q(l)\Gamma_q(m)}{\Gamma_q(l+m)}.$$

For details on $q$-Beta functions, we refer the readers to Sole-Kac [14].

Now for $f\in C[0,\infty)$, $q>0$ and $n\in N$ Aral-Gupta [2] introduced $q$-Baskakov operators are defined as

$$\begin{aligned}B_{n,q}(f;x)&=\sum_{v=0}^{\infty}\left[\begin{array}{c} n+v-1 \\ v \end{array}\right]q^{\frac{v(v-1)}{2}}\frac{x^v}{(1+x)^{n+v}_q}f\left(\frac{[v]_q}{q^{v-1}[n]_q}\right)\\ &=: \sum_{v=0}^{\infty}p^q_{n,v}(x)f\left(\frac{[v]_q}{q^{v-1}[n]_q}\right). \qquad (3)\end{aligned}$$

Taking $0<\alpha<\beta$, the operators (2) reduce to the operators (1). Hence in this paper, we estimate local approximation theorem and the rate of convergence of these new operators as well as their weighted approximation properties.

## 2. Moment estimations

In this section, we give moments and higher order moments for operators (3) and (1). Also we estimate the corresponding moments for new operators (2). Also we give higher order moments.

**Lemma 2.1.**
*For the operators (3), the following equalities hold:*

$$B_{n,q}(1;x)=1, \qquad B_{n,q}(t;x)=x,$$
$$B_{n,q}(t^2;x)=x^2+\frac{x}{[n]_q}\left(1+\frac{x}{q}\right).$$

*Proves are along the lines of Aral-Gupta [2].*

**Lemma 2.2.**
*For the operators (1), the following equalities hold:*

$$D_n^q(1;x)=1, \qquad D_n^q(t;x)=x,$$
$$D_n^q(t^2;x)=x^2+\frac{x}{[n]_q}\left(1+q+\frac{x}{q}\right).$$

*Proves can be seen in Gupta et al. [6].*

**Lemma 2.3.**
*For $q\in(0,1)$ and $x\in[0,\infty)$, we have*

$$D_n^q((t-x)^2;x)=\frac{x(x+q[2]_q)}{q[n]_q}.$$

**Lemma 2.4.**
*For $q\in(0,\infty)$, $n\in N$ and $0<\alpha<\beta$, the equalities hold*

$$M_{n,\alpha,\beta}^q(1,x)=1, \qquad M_{n,\alpha,\beta}^q(t,x)=\frac{[n]_q x+\alpha}{[n]_q+\beta},$$
$$M_{n,\alpha,\beta}^q(t^2,x)=\frac{[n]_q+q[n]_q^2}{q([n]_q+\beta)^2}x^2+\frac{(1+q+2\alpha)[n]_q}{([n]_q+\beta)^2}x+\left(\frac{\alpha}{[n]_q+\beta}\right)^2.$$

*Proof.* Obviously $M_{n,\alpha,\beta}^q(1,x)=1$. Now we proceed as

$$\begin{aligned}M_{n,\alpha,\beta}^q(t,x)&=[n]_q\sum_{v=1}^{\infty}p_{n,v}^q(x)\int_0^{q/(1-q^n)}q^{-v}s_{n,v-1}^q(t)q^{-v}\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)d_q t+p_{n,0}^q(x)\left(\frac{\alpha}{n+\beta}\right)\\&=\left(\frac{[n]_q}{[n]_q+\beta}\right)D_n^q(t;x)+\frac{\alpha}{[n]_q+\beta}D_n^q(1;x)\\&=\frac{[n]_q x+\alpha}{[n]_q+\beta},\end{aligned}$$

using Lemma 2.2. In the similar way, we have

$$\begin{aligned}M_{n,\alpha,\beta}^q(t^2,x)&=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2D_n^q(t^2;x)+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}D_n^q(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\&=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left[x^2+\frac{x}{[n]_q}(1+q+\frac{x}{q})\right]+\frac{2\alpha[n]_q x}{([n]_q+\beta)^2}+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\&=\frac{[n]_q+q[n]_q^2}{q([n]_q+\beta)^2}x^2+\frac{(1+q+2\alpha)[n]_q}{([n]_q+\beta)^2}x+\left(\frac{\alpha}{[n]_q+\beta}\right)^2.\end{aligned}$$

□

**Lemma 2.5.**
*For $0<\alpha<\beta$ and $0<q<1$, the central moments for our operators (2) are*

$$T^q_{n,m,\alpha,\beta}(x)=M^q_{n,\alpha,\beta}((t-x)^m,x)$$
$$=[n]_q\sum_{v=1}^{\infty}p^q_{n,v}(x)\int_0^{q/(1-q^n)}q^{-v}s^q_{n,v-1}(t)q^{-mv}\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}-x\right)^m d_qt+p^q_{n,0}(x)\left(\frac{\alpha}{n+\beta}-x\right)^m.$$

*Then we have the first three moments as*

$$T^q_{n,0,\alpha,\beta}(x)=1 \tag{4}$$

$$T^q_{n,1,\alpha,\beta}(x)=\frac{\alpha-\beta x}{[n]_q+\beta} \tag{5}$$

$$T^q_{n,2,\alpha,\beta}(x)=\left(\frac{\alpha}{[n]_q+\beta}\right)^2+\left[(1+q+2\alpha)\frac{[n]_q}{([n]_q+\beta)^2}-\frac{2\alpha}{[n]_q+\beta}\right]x+\left[\frac{q[n]_q^2+[n]_q}{([n]_q+\beta)^2}-\frac{2[n]_q}{[n]_q+\beta}+1\right]x^2. \tag{6}$$

*Proof.* From Lemma 2.2 it is obvious that $T^q_{n,0,\alpha,\beta}(x)=1$. Further we get

$$T^q_{n,1,\alpha,\beta}(x)=M^q_{n,\alpha,\beta}((t-x),x)=M^q_{n,\alpha,\beta}(t,x)-xM^q_{n,\alpha,\beta}(1,x)$$
$$=\frac{\alpha-\beta x}{[n]_q+\beta},$$

$$T^q_{n,2,\alpha,\beta}(x)=M^q_{n,\alpha,\beta}((t-x)^2,x)$$
$$=M^q_{n,\alpha,\beta}(t^2,x)-2xM^q_{n,\alpha,\beta}(t,x)+x^2M^q_{n,\alpha,\beta}(1,x)$$
$$=\left(\frac{\alpha}{[n]_q+\beta}\right)^2+\left[(1+q+2\alpha)\frac{[n]_q}{([n]_q+\beta)^2}-\frac{2\alpha}{[n]_q+\beta}\right]x+\left[\frac{q[n]_q^2+[n]_q}{([n]_q+\beta)^2}-\frac{2[n]_q}{[n]_q+\beta}+1\right]x^2.$$

□

**Remark 2.1.**
For $0<\alpha<\beta$ and $0<q<1$, it can be proved easily from above lemma that

$$M^q_{n,\alpha,\beta}((t-x)^2,x)\le\frac{x(x+q[2]_q)}{q([n]_q+\beta)}.$$

**Higher Order Moments** Now we consider for higher order moments for the operators (2).

**Lemma 2.6 ([5]).**
*For $0<q<1$, we have for operators (3)*

$$B_{n,q}(t^3;x)=\frac{1}{[n]_q}x+\frac{1+2q}{q^2}\frac{[n+1]_q}{[n]_q^2}x^2+\frac{1}{q^3}\frac{[n+1]_q[n+2]_q}{[n]_q^2}x^3$$
$$B_{n,q}(t^4;x)=\frac{1}{[n]_q^3}x+\frac{1+3q+3q^2}{q^3}\frac{[n+1]_q}{[n]_q^3}x^2+\frac{(1+3q+5q^2+3q^3)}{q^5[2]_q}\frac{[n+1]_q[n+2]_q}{[n]_q^3}x^3$$
$$+\frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{q^6[2]_q[3]_q[4]_q}\frac{[n+1]_q[n+2]_q[n+3]_q}{[n]_q^3}x^4.$$

**Lemma 2.7 ([6]).**
*Let $0<q<1$, then for operators (1) we have*

$$D_n^q(t^3;x)=\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^3+\left(\frac{1+2q}{q^2}\frac{[n+1]_q}{[n]_q^2}+\frac{2+3q+q^2}{[n]_q}\right)x^2+\frac{1+2q+2q^2+q^3}{[n]_q^2}x$$

$$\begin{aligned}D_n^q(t^4;x)=&\frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{q^6[2]_q[3]_q[4]_q}\frac{[n+1]_q[n+2]_q[n+3]_q}{[n]_q^3}x^4\\&+\left(\frac{1+3q+5q^2+3q^3}{q^5[2]_q}+q(3+2q+q^2)\right)\frac{[n+1]_q[n+2]_q}{[n]_q^3}x^3\\&+\left(\begin{array}{c}\frac{1+3q+3q^2}{q^3}\frac{[n+1]_q}{[n]_q}+\frac{(1+2q)(3+2q+q^2)}{q}\frac{[n+1]_q}{[n]_q^3}\\+\frac{q^2(3+4q+3q^2+q^3)}{[n]_q^2}+\frac{q(3+4q+3q^2+q^3)}{[n]_q^3}\end{array}\right)x^2\\&+\left(\frac{1}{[n]_q}+\frac{q(3+5q+6q^2+5q^3+3q^4+q^5)}{[n]_q^3}\right)x.\end{aligned}$$

**Lemma 2.8.**
*For $q\in(0,1)$ and $0<\alpha<\beta$, we have*

$$\begin{aligned}M_{n,\alpha,\beta}^q(t^3;x)=&\left(\frac{\alpha}{[n]_q+\beta}\right)^3+\left(\frac{[n]_q}{[n]_q+\beta}\right)^3\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^3+\left(\frac{[n]_q}{[n]_q+\beta}\right)^3\\&\times\left\{\frac{2+3q+q^2}{[n]_q}+\frac{1+2q}{q^2}\frac{[n+1]_q}{[n]_q}+\frac{(q[n]_q+1)3\alpha}{q[n]_q^2}\right\}x^2\\&+\left(\frac{[n]_q}{[n]_q+\beta}\right)^3\left\{(1+2q+2q^2+q^3)+\frac{3(1+q)\alpha}{[n]_q^2}+\frac{3\alpha^2}{[n]_q^2}\right\}x\end{aligned}$$

*and*

$$\begin{aligned}M_{n,\alpha,\beta}^q(t^4;x)=&\left[\frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{q^6[2]_q[3]_q[4]_q}\frac{[n]_q[n+1]_q[n+2]_q[n+3]_q}{([n]_q+\beta)^4}\right]x^4+\\&\left[\frac{1+3q+q^2(5+4\alpha[2]_q)+3q^3+q^6(3+2q+q^2)[2]_q}{q^5[2]_q}\frac{[n]_q[n+1]_q[n+2]_q}{([n]_q+\beta)^4}\right]x^3\\&+\{\frac{1+3q+3q^2}{q^3}\frac{[n+1]_q[n]_q^3}{([n]_q+\beta)^4}+\frac{(1+2q)(4\alpha+3q+2q^2+q^3)}{q^2}\frac{[n+1]_q[n]_q}{([n]_q+\beta)^4}\\&+\frac{[n]_q^2}{([n]_q+\beta)^4}+\frac{[n]_q}{([n]_q+\beta)^4}\}x^2+\left[\frac{[n]_q^3}{([n]_q+\beta)^4}+2\alpha\{2+3\alpha+2\alpha^2+(4+3\alpha)q\right.\\&\left.+4q^2+2q^3\}\frac{[n]_q^2}{([n]_q+\beta)^4}+q(3+5q+6q^2+5q^3+3q^4+q^5)\frac{[n]_q}{([n]_q+\beta)^4}\right]x+\left(\frac{\alpha}{[n]_q+\beta}\right)^4.\end{aligned}$$

*Proof.*

$$\begin{aligned}M_{n,\alpha,\beta}^q(t^3;x)=&[n]_q\sum_{v=1}^{\infty}p_{n,v}^q(x)\int_0^{q/(1-q^n)}q^{-v}s_{n,v-1}^q(t)q^{-3v}\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)^3d_qt+p_{n,0}^q(x)\left(\frac{\alpha}{[n]_q+\beta}\right)^3\\=&\left(\frac{[n]_q}{[n]_q+\beta}\right)^3D_n^q(t^3;x)+\frac{3\alpha[n]_q^2}{([n]_q+\beta)^3}D_n^q(t^2;x)+\frac{3\alpha^2[n]_q}{([n]_q+\beta)^3}D_n^q(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^3\\=&\left(\frac{[n]_q}{[n]_q+\beta}\right)^3(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^3+\left(\frac{1+2q}{q^2}\frac{[n+1]_q}{[n]_q^2}+\frac{2+3q+q^2}{[n]_q}\right)x^2\\&+\frac{1+2q+2q^2+q^3}{[n]_q^2}x)+\frac{3\alpha[n]_q^2}{([n]_q+\beta)^3}(x^2+\frac{x}{[n]_q}\left(1+q+\frac{x}{q}\right)+\frac{3\alpha^2[n]_q}{([n]_q+\beta)^3}x\\&+\left(\frac{\alpha}{[n]_q+\beta}\right)^3\end{aligned}$$

using Lemma 2.2 and Lemma 2.6. Collecting the coefficients of $x, x^2, x^3$ we have the required result. Similarly we find that

$$M^q_{n,\alpha,\beta}(t^4;x)=[n]_q\sum_{\nu=1}^{\infty}p^q_{n,\nu}(x)\int_0^{q/(1-q^n)}q^{-\nu}s^q_{n,\nu-1}(t)q^{-4\nu}\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)^4 d_q t+p^q_{n,0}(x)\left(\frac{\alpha}{[n]_q+\beta}\right)^4$$
$$=\left(\frac{[n]_q}{[n]_q+\beta}\right)^4 D^q_n(t^4;x)+\frac{4\alpha[n]^3_q}{([n]_q+\beta)^4}D^q_n(t^3;x)+\frac{6\alpha^2[n]^2_q}{([n]_q+\beta)^4}D^q_n(t^2;x)$$
$$+\frac{4\alpha^3[n]_q}{([n]_q+\beta)^4}D^q_n(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^4$$
$$=\left(\frac{[n]_q}{[n]_q+\beta}\right)^4\left(\frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{q^6[2]_q[3]_q[4]_q}\frac{[n+1]_q[n+2]_q[n+3]_q}{[n]^3_q}x^4\right.$$
$$+\left(\frac{1+3q+5q^2+3q^3}{q^5[2]_q}+q(3+2q+q^2)\right)\frac{[n+1]_q[n+2]_q}{[n]^3_q}x^3+$$
$$\left[\frac{1+3q+3q^2}{q^3}\frac{[n+1]_q}{[n]_q}+(3+4q+3q^2+q^3)\left(\frac{q^2}{[n]^2_q}+\frac{q}{[n]^3_q}\right)+\frac{[n+1]_q}{[n]^3_q}\right.$$
$$\left.\times\frac{3+8q+5q^2+2q^3}{q}\right]x^2+\left.\left(\frac{1}{[n]_q}+\frac{q(3+5q+6q^2+5q^3+3q^4+q^5)}{[n]^3_q}\right)x\right)$$
$$+\frac{4\alpha[n]^3_q}{([n]_q+\beta)^4}\left[\frac{[n+1]_q[n+2]_q}{q^3[n]^2_q}x^3+\left(\frac{1+2q}{q^2}\frac{[n+1]_q}{[n]^2_q}+\frac{2+3q+q^2}{[n]_q}\right)x^2\right.$$
$$\left.+\frac{1+2q+2q^2+q^3}{[n]^2_q}x\right]+\frac{6\alpha^2[n]^2_q}{([n]_q+\beta)^4}\left(x^2+\frac{x}{[n]_q}\left(1+q+\frac{x}{q}\right)\right)+\frac{4\alpha^3[n]_q}{([n]_q+\beta)^4}x$$
$$+\left(\frac{\alpha}{[n]_q+\beta}\right)^4$$

using Lemmas 2.2 and Lemma 2.6. Rearranging the coefficients of $x, x^2, x^3, x^4$ we get the required lemma. □

### Definition 2.1 (Peetre's $K$-functional).

Let us consider the space $C_B[0,\infty)$ of all the continuous and bounded functions $f$ that is $f\in C_B[0,\infty)$ and endowed with the norm $\|f\|=\{|f(x)|: x\in[0,\infty)\}$, then the $K$-functional

$$K_2(f,\delta)=\inf_{g\in W^2_\infty}\{\|f-g\|+\delta\|g''\|\},$$

where $\delta>0$ and $W^2_\infty=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. Also there exists an absolute constant $C>0$ such that $K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta})$, where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|$$

is the second order modulus of smoothness of $f\in C_B[0,\infty)$.

Also, for $f\in C_B[0,\infty)$ a usual modulus of continuity is given by

$$\omega(f,\delta)=\sup_{0<h<\delta}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|.$$

### Definition 2.2 (Rate of convergence).

Let $B_{x^2}[0,\infty)$ be the set of all functions $f\in[0,\infty)$ satisfying the condition $|f(x)|\le M_f(1+x^2)$, $M_f$ is a constant depending on $f$. We denote the subspace of all continuos functions by $C_{x^2}[0,\infty)$ belonging to $B_{x^2}[0,\infty)$. Again, we suppose $C^*_{x^2}[0,\infty)$ be the subspace of all the functions $f\in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is defined as $\|f\|_{x^2}=\sup_{x\to\infty}\frac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of $f$ on the closed interval $[0,a]$ for $a>0$, by

$$\omega_a(f,\delta)=\sup_{|t-x|\le\delta}\sup_{x,t\in[0,a]}|f(t)-f(x)|.$$

We know that for a function $f\in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)\to 0$.

**Definition 2.3.**
Here we define some classes of functions
If there exists some constant $M_f > 0$ corresponding to function $f$, then we have

$$C_m[0,\infty) =: \left\{ f \in C[0,\infty) : |f(x)| < M_f(1+x^m); \|f\|_m := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m} \right\},$$

$$C_m^*[0,\infty) =: \left\{ f \in C_m[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty, \forall m \in N \right\},$$

## 3. Direct estimates

In this section we give some direct theorems and asymptotic formula using our operators (2).

**Theorem 3.1.**
*Let $f \in C_B[0,\infty)$ and $0<q<1$. Then for all $x \in [0,\infty)$ and $n \in N$, there exists an absolute constant $C>0$ such that*

$$|M_{n,\alpha,\beta}^q(f;x) - f(x)| \le C\omega_2\left(f, \sqrt{\frac{x(x+q[2]_q}{q([n]_q+\beta)}}\right).$$

*Proof.* Let $g \in W_\infty^2$ and $x, t \in\in [0,\infty)$, then by Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-w)g''(w)dw.$$

From Lemma 2.5 and Remark 2.1

$$M_{n,\alpha,\beta}^q(g(t);x) - g(x) = g'(x)M_{n,\alpha,\beta}^q(t-x;x) + M_{n,\alpha,\beta}^q\left(\int_x^t (t-w)g''(w)dw; x\right).$$

Using $\int_x^t (t-w)g''(w)dw \le (t-x)^2\|g''\|$, we have

$$\begin{aligned}|M_{n,\alpha,\beta}^q(g(t);x) - g(x)| &= M_{n,\alpha,\beta}^q(t-x;x)g'(x) + M_{n,\alpha,\beta}^q((t-x)^2;x)\|g''\| \\ &\le \frac{x(x+q[2]_q)}{q([n]_q+\beta)}\|g''\|.\end{aligned}$$

Also from (2)

$$\begin{aligned}|M_{n,\alpha,\beta}^q(f;x)| = [n]_q \sum_{v=1}^{\infty} p_{n,v}^q(x) \int_0^{q/(1-q^n)} q^{-v} s_{n,v-1}^q(t) \left| f\left(q^{-v}\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)\right)\right| d_q t + p_{n,0}^q(x)\left| f\left(\frac{\alpha}{[n]_q+\beta}\right)\right| \\ \le \|f\|.\end{aligned}$$

Hence we can have

$$\begin{aligned}|M_{n,\alpha,\beta}^q(f(t);x) - f(x)| &\le |M_{n,\alpha,\beta}^q(f-g;x) - (f-g)(x)| + |M_{n,\alpha,\beta}^q(g;x) - g(x)| \\ &\le \|f-g\| + \frac{x(x+q[2]_q)}{q([n]_q+\beta)}\|g''\|.\end{aligned}$$

Taking infimum overall $g \in W_\infty^2$ and then from Peetre's $K$-function, we get

$$|M_{n,\alpha,\beta}^q(f(t);x) - f(x)| \le C\omega_2\left(f, \sqrt{\frac{x(x+q[2]_q}{q([n]_q+\beta)}}\right).$$

Hence the required theorem. □

**Theorem 3.2.**
*Let $f \in C_{x^2}$, $q \in (0,1)$ and $\omega_{(a+1(f,\delta))}$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, $\forall a > 0$. Then for every $n > 2$, we have*

$$\|M_{n,\alpha,\beta}^q(f) - f\| \le \frac{6M_f a(1+a^2)(a+2)}{q([n]_q+\beta)} + 2\omega(f, \sqrt{\frac{a(a+q[2]_q)}{q([n]_q+\beta)}}.$$

*Proof.* For $x \in [0,a]$ and $t > a+1$, as $t-x > 1$, we have

$$\begin{aligned}|f(t)-f(x)| &\leq M_f(2+x^2+t^2)\\ &\leq M_f(2+3x^2+(t-x)^2)\\ &\leq 6M_f(1+a^2)(t-x)^2. \end{aligned} \tag{7}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t)-f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \qquad \delta > 0 \tag{8}$$

From (7) and (8), for $x \in [0,a]$ and $t \geq 0$, we have

$$|f(t)-f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta). \tag{9}$$

Hence

$$\begin{aligned}|M^q_{n,\alpha,\beta}(f;x)-f(x)| &\leq M^q_{n,\alpha,\beta}(|f(t)-f(x)|;x)\\ &\leq 6M_f(1+a^2)M^q_{n,\alpha,\beta}((t-x)^2;x)+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}M^q_{n,\alpha,\beta}((t-x)^2;x)^{1/2}\right).\end{aligned}$$

Therefor by using Schwarz inequality and Remark 1,

$$\begin{aligned}|M^q_{n,\alpha,\beta}(f;x)-f(x)| &\leq \frac{6M_f(1+a^2)(x(x+q[2]_q)}{q([n]_q+\beta)}+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x(x+q[2]_q)}{q([n]_q+\beta)}}\right)\\ &\leq \frac{6M_f a(1+a^2)(a+2)}{q([n]_q+\beta)}+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{a(a+q[2]_q)}{q([n]_q+\beta)}}\right).\end{aligned}$$

Taking $\delta = \sqrt{\frac{a(a+q[2]_q)}{q([n]_q+\beta)}}$, we get the required assertion. □

**Theorem 3.3.**
*Let $q_n \in (0,1)$, then the sequence $M^{q_n}_{n,\alpha,\beta}(f)$ converges to $f$ uniformly on $[0,A]$, for each $f \in C^*_2[0,\infty)$ if and only if* $\lim_{n\to\infty} q_n = 1$.
*Its proof is obvious as $q, q_n \in (0,1)$.*

**Theorem 3.4.**
*If $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to 1$ as $n \to \infty$ for any $f \in C^*_2[0,\infty)$ such that $f', f'' \in C^*_2[0,\infty)$, the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}(M^{q_n}_{n,\alpha,\beta}(f;x)-f(x)) = (\alpha-\beta x)f'(x)+\frac{1}{2}x(x+2)f''(x).$$

*Proof.* Assume that $f, f', f'' \in C^*_2[0,\infty)$ for all $x \in [0,\infty)$. Therefore by Taylor's formula

$$f(t) = f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{10}$$

where $r(t,x)$ is the Peano form of the remainder and $r(t,x) \in C^*_2[0,\infty)$. Also $\lim_{t\to x} r(t,x) = 0$. Now applying the operators $M^{q_n}_{n,\alpha,\beta}$ in (10), we get

$$\begin{aligned}[n]_{q_n}(M^{q_n}_{n,\alpha,\beta}(f;x)-f(x)) = f'(x)[n]_{q_n}M^{q_n}_{n,\alpha,\beta}(t-x;x)+\frac{1}{2}f''(x)[n]_{q_n}M^{q_n}_{n,\alpha,\beta}((t-x)^2;x)\\ +[n]_{q_n}M^{q_n}_{n,\alpha,\beta}(r(t;x)(t-x)^2;x).\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$M^{q_n}_{n,\alpha,\beta}(r(t;x)(t-x)^2;x) \leq \sqrt{M^{q_n}_{n,\alpha,\beta}(r^2(t;x);x)}\sqrt{M^{q_n}_{n,\alpha,\beta}(((t-x)^2;x))} \tag{11}$$

As $r^2(x;x)=0$ and $r^2(.;x) \in C^*_2[0,\infty)$, it follows that $\forall x \in [0,A]$

$$lim_{n\to\infty} M^{q_n}_{n,\alpha,\beta}(r^2(t;x);x) = r^2(x;x) = 0 \tag{12}$$

uniformly and so RHS of (11) becomes zero. Therefore we get

$$\begin{aligned}lim_{n\to\infty}[n]_{q_n}(M^{q_n}_{n,\alpha,\beta}(f;x)-f(x)) = lim_{n\to\infty} f'(x)[n]_{q_n}M^{q_n}_{n,\alpha,\beta}(t-x;x)+\frac{1}{2}f''(x)[n]_{q_n}M^{q_n}_{n,\alpha,\beta}((t-x)^2;x)\\ +[n]_{q_n}M^{q_n}_{n,\alpha,\beta}(r(t;x)(t-x)^2;x)\\ = (\alpha-\beta x)f'(x)+\frac{1}{2}x(x+2)f''(x)\end{aligned}$$

Hence the theorem. □

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