Thermal stress analysis due to surface heat source

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Abstract: The present manuscript deals with the heat transfer and thermal stress analysis of cylinder due to internal heat generation under steady temperature conditions. The internal heat generation is taken as cylindrical surface heat source in annular region of linear length of cylinder and is situated concentrically inside the cylinder. The upper surface of a cylinder is subjected to arbitrary temperature whereas the lower surface and circular boundary surface are thermally insulated. The integral transform methods are used for the solution of non-homogeneous boundary value problem. The theory of linearized thermoelasticity based on solution of Navier's equation in terms of Goodier's thermoelastic displacement potential, Michell's function and the Boussinesq's function for cylindrical co-ordinate system have been used for discussion and analysis of thermal stresses.

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1. Introduction

Thermoelasticity is based on temperature changes induced by expansion and compression of the test part. Although this coupling between mechanical deformation and thermal energy has been known for over a century. After World War second, there was very rapid development of thermoelasticity, stimulated by various engineering sciences. Thermoelasticity contains the generalized theory of heat conduction, the generalized theory of the thermal stresses. A considerable progress in the field of air-craft and machine structures, mainly with gas and steam turbines and the emergence of new topics in chemical engineering have given rise to numerous problems in which thermal stresses play an important role and frequently even a primary role.

Sipailov et al. [1] studied the effect of heat transfer in the impulse method of measurement on the value of the maximum temperature and on the time for achieving its half-value. Taking this heat transfer effect into account increases the accuracy of measuring the thermophysical properties. Lohle et al. [2], analyze inverse heat conduction problems by the analysis of the system impulse response by the application of the non-integer system identification (NISI) method. Stakhanova et al. [3] carried out heat transfer on model fragments of fuel rod claddings during impulse supply of power. The time taken for reaching steady state for different parameters of impulse and the values of heat-transfer coefficient are determined. Lee [4] obtained three dimensional series solution for elastic thick plate subjected to general temperature distribution. Hata [5] concerned with a method for calculating the thermal-stress distribution in a nonhomogeneous thick elastic plate under steady distribution of the surface temperature whose shear modulus and coefficient of thermal expansion are assumed to be functions of z. Kulkarni et al. [6] determined the temperature changes and thermal stresses due to conduction of heat in the thick circular plate under transient temperature conditions and analyzed his analytical results for heat treatment given in the annular region which is described by Dirac-delta function.

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2. Heat transfer analysis

2.1. Formulation of the Problem

The steady state temperature of the cylinder satisfies the heat conduction equation,

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r)}{k} = 0
\]  
(1)

with the boundary conditions

\[
\frac{\partial T}{\partial r} = 0 \text{ at } r = a, \ -h \leq z \leq h
\]  
(2)

\[
T = f(r) \text{ on } z = h, \ a \leq r \leq b
\]  
(3)

\[
\frac{\partial T}{\partial z} = 0 \text{ at } z = -h, \ a \leq r \leq b
\]  
(4)

The set of Eqs. (1)-(4) constitute mathematical formulation for temperature change within cylinder due to internal heat source \( g(r) \) which is cylindrical surface heat source of radius \( r = r_1, \ 0 < r < a \) and of strength \( g_s \), of linear length of cylinder and is situated concentrically inside the cylinder at \( r = r_1 \) given by

\[
g(r) = \frac{g_s \delta(r - r_1)}{2\pi r k}
\]  
(5)

2.2. The solution for temperature change

To obtain the expression for temperature \( T(r, z) \), introduce the finite Hankel transform over the variable \( r \) and its inverse transform defined as in [7]

\[
\mathcal{T}(a_n, z) = \int_a^b r K_0(a_n, r) T(r, z) dr
\]  
(6)

\[
T(r, z) = \sum_{n=1}^{\infty} \mathcal{T}(a_n, z) K_0(a_n, r)
\]  
(7)

where

\[
K_0(a_n, r) = \frac{\sqrt{2} J_0(a_n r)}{a_n J_0(a_n a)}
\]  
(8)

and \( a_1, a_2 \) are roots of the transcendental equation

\[
J_1(aa) = 0
\]  
(9)

\( J_n(x) \) is Bessel function of the first kind of order \( n \). This transform satisfies the relation

\[
H \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = -a_n^2 \mathcal{T}(a_n, z)
\]  
(10)

and

\[
H \left[ \frac{\partial^2 T}{\partial z^2} \right] = \frac{d^2 \mathcal{T}}{dz^2}
\]  
(11)

On applying the finite Hankel transform defined in the Eq. (6) to the Eq. (1), one obtain

\[
\frac{d^2 \mathcal{T}}{dz^2} - a_n^2 \mathcal{T} + \frac{\mathcal{G}(r)}{k} = 0
\]  
(12)

where \( \mathcal{T} \) is the Hankel transform of \( T \) and \( g(r) \) is the Hankel transform of \( g(r) \). On solving (12) under condition given in Eqs. (3) and (4), one obtain

\[
\mathcal{T}(a_n, z) = \left\{ \left[ \tilde{f}(a_n) \left( \frac{\mathcal{G}(r)}{a_n^2 k} \right) \left[ \cosh(a_n z + h) \right] + \frac{\tilde{g}(r)}{a_n^2 k} \right] \right \}
\]  
(13)

On applying inverse Hankel transform defined in (7), one obtain the expression for temperature as

\[
T = \sum_{n=1}^{\infty} \left\{ \left[ \tilde{f}(a_n) \frac{\sqrt{2} J_0(a_n r)}{a_n J_0(a_n a)} \right] \left( \mathcal{T}(a_n) - \frac{\mathcal{G}(r)}{a_n^2 k} \left[ \cosh(a_n z + h) \right] + \frac{\tilde{g}(r)}{a_n^2 k} \right) \right\}
\]  
(13)

where \( \tilde{f}(a_n) \) is Hankel transform of \( f(r) \) and \( \mathcal{G}(r) \) is Hankel transform of surface heat source which is given by

\[
\mathcal{G}(r) = \frac{\sqrt{2} g_s}{a_n J_0(a_n a)}
\]  
(14)

\( \tau_i = 0 \), the temperature change

\[
\tau = T
\]  
(15)
3. Thermal stress analysis

3.1. Development of thermoelastic equations

Following Noda et al. [8], The Naviers equations for axisymmetric thermoelastic problems can be expressed as

\[ \nabla^2 u_r - \frac{u_r}{r^2} + \left( \frac{1}{1-2\nu} \right) \frac{\partial e}{\partial r} - 2a(1+\nu) \frac{\partial \tau}{\partial r} + \frac{2(1+\nu)}{E} F_r = 0 \]  
(16)

\[ \nabla^2 u_z + \left( \frac{1}{1-2\nu} \right) \frac{\partial e}{\partial z} - 2a(1+\nu) \frac{\partial \tau}{\partial z} + \frac{2(1+\nu)}{E} F_z = 0 \]  
(17)

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \)

\( e \) - dilatation, \( E \) - Young\'s modulus \( \nu \) - coefficient of linear thermal expansion \( \lambda \) - Poisson ratio.

The solution of Naviers Eqs. (16) and (17) without body forces can be expressed by Goodiers thermoelastic displacement potential \( f \) and Boussinesq harmonic functions \( j \) and \( y \) under the axisymmetric conditions.

The Goodiers thermoelastic displacement potential \( f \) must satisfy the governing equations \( \nabla^2 f = K\tau \),

That is

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K\tau \]  
(19)

where \( K \) is Restraint coefficient as

\[ K = \frac{\beta}{\lambda + 2\mu} = \frac{1+\nu}{1-\nu} \alpha \]

where \( \beta \) - thermoelastic constant, \( \lambda \) and \( \mu \) - Lames elastic constants.

Boussinesq harmonic functions \( \phi \) and \( \psi \) must satisfy the governing equations

\[ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]  
(20)

and

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \]  
(21)

when deformation in the cylindrical coordinate system are discussed, Michells function \( M \) instead of Boussinesq harmonic functions \( \phi \) and \( \psi \) is often used.

Taking

\[ M = -\int (\phi + z\psi) dz \]  
(22)

The Michell\'s function \( M \) must satisfy

\[ \nabla^2 \nabla^2 M = 0 \]  
(23)

The component of the displacement and stresses are represented by the thermoelastic displacement potential \( \phi \) and Michell\'s function \( M \) as

\[ u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \]  
(24)

\[ u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \]  
(25)

\[ \sigma_{rr} = 2G[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2})] \]  
(26)

\[ \sigma_{\theta z} = 2G[\frac{1}{r} \frac{\partial \phi}{\partial z} - K\tau + \frac{\partial}{\partial r}(\nu \nabla^2 M - \frac{1}{r} \frac{\partial M}{\partial r})] \]  
(27)

\[ \sigma_{zz} = 2G[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}(2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}] \]  
(28)

\[ \sigma_{rr} = 2G[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}] \]  
(29)

For traction free surface the stress functions

\[ \sigma_{rr} = \sigma_{zz} = 0 \text{ at } r = 0 \]  
(30)

The set of Eqs. (16) to (30) constitute mathematical formulation for displacement and thermal stresses developed within solid due to temperature change.
3.2. The solution for displacement and thermal stresses

Assuming displacement function $\phi(r, z)$ as

$$
\phi(r, z) = \sum_{n=1}^{\infty} D_n \left( \frac{J_0(a_n r)}{J_0(a_n a)} \right) \left( \sum_{n=1}^{\infty} \left( \frac{J_0(a_n r)}{a_n^2 k} \right) \frac{(z + h) \sinh[a_n(z + h)]}{\cosh(2a_n h)} \right)
$$

(31)

Using $\phi$ in (19), one has

$$
D_n = \frac{K \bar{f}(a_n)}{a \sqrt{2a_n}}
$$

. Thus Eq. (31) becomes

$$
\phi(r, z) = \sum_{n=1}^{\infty} \frac{K \bar{f}(a_n)}{a \sqrt{2a_n}} \frac{J_0(a_n r)}{J_0(a_n a)} \left( \frac{g(r)}{a_n^2 k} \right) \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{\cosh(2a_n h)} \right)
$$

(32)

Now suitable form of Michell function $M$ satisfying (23) is given by

$$
M = \left( \frac{\sqrt{2} \bar{f}(a_n)}{a} \right) \frac{J_0(a_n r)}{J_0(a_n a)} \left( \frac{g(r)}{a_n^2 k} \right) \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{\cosh(2a_n h)} \right) \times \{ B_n \sinh[a_n(z + h)] + C_n a_n [z + h] \cosh[a_n(z + h)] \}
$$

(33)

where $B_n$ and $C_n$ are arbitrary functions which can be determined by boundary condition on the traction free surface given in the Eq. (30). Now using Eqs. (13), (15), (32), and (33) in (24) to (29), one obtains the expressions for displacements and stresses respectively as

$$
u_r = \left( \frac{K}{\sqrt{2}a} \right) \sum_{n=1}^{\infty} \left( \frac{\bar{f}(a_n)}{a} \right) \frac{\bar{g}(r)}{a_n^2 k} \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{2 \cosh(2a_n h)} \right) + B_n a_n^2 \cosh[a_n(z + h)] + C_n a_n^2 \cosh[a_n(z + h)] + a_n [z + h] \sinh[a_n(z + h)]
$$

(34)

$$
u_z = \left( \frac{K}{\sqrt{2}a} \right) \sum_{n=1}^{\infty} \left( \frac{\bar{f}(a_n)}{a} \right) \frac{\bar{g}(r)}{a_n^2 k} \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{2 \cosh(2a_n h)} \right) + B_n a_n^2 \cosh[a_n(z + h)] + C_n a_n^2 [2(1-2\nu) \sinh[a_n(z + h)] - a_n [z + h] \cosh[a_n(z + h)]
$$

(35)

$$
\sigma_{rr} = \left( \frac{\sqrt{2}G K}{a} \right) \sum_{n=1}^{\infty} \left( \frac{\bar{f}(a_n)}{a} \right) \frac{\bar{g}(r)}{a_n^2 k} \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{2 \cosh(2a_n h)} \right)
$$

(36)

$$
\sigma_{\theta\theta} = \left( \frac{\sqrt{2}G K}{a} \right) \sum_{n=1}^{\infty} \left( \frac{\bar{f}(a_n)}{a} \right) \frac{\bar{g}(r)}{a_n^2 k} \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{2 \cosh(2a_n h)} \right)
$$

(37)

$$
\sigma_{zz} = \left( \frac{\sqrt{2}G K}{a} \right) \sum_{n=1}^{\infty} \left( \frac{\bar{f}(a_n)}{a} \right) \frac{\bar{g}(r)}{a_n^2 k} \left[ \frac{J_1(a_n r)}{J_0(a_n r)} \right] \left( \frac{(z + h) \sinh[a_n(z + h)]}{2 \cosh(2a_n h)} \right)
$$

(38)
\[ \sigma_{rz} = \left( \frac{\sqrt{2}GK}{a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \]
\[ \left\{ - \sinh[a_n(z + h)] + a_n(z + h) \cosh[a_n(z + h)] \right\} \cdot \frac{2 \cosh(2a_n h)}{2a_n^2 \sinh[a_n(z + h)]} + B_n a_n^3 \sinh[a_n(z + h)] \right\} \right) \]

Now in order to satisfy Eq. (30) solving Eqs. (38) and (39) for \( B_n \) and \( C_n \) one obtain,

\[ B_n = \frac{(1 - 2\nu)}{2a_n^3 \cosh(2a_n h)} \]

and

\[ C_n = \frac{1}{2a_n^3 \cosh(2a_n h)} \]

Using these values of \( B_n \) and \( C_n \) in Eqs. (34) to (39) one obtain the expressions for displacements and stresses as

\[ u_r = \left( \frac{K}{\sqrt{2}a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \left[ \frac{(1 - \nu) \cosh[a_n(z + h)]}{a_n \cosh(2a_n h)} \right] \]

\[ u_z = \left( \frac{K}{\sqrt{2}a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \left[ \frac{(1 - \nu) \sinh[a_n(z + h)]}{a_n \cosh(2a_n h)} \right] \]

\[ \sigma_{rr} = \left( \frac{\sqrt{2}GK}{a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \left[ \frac{(1 - \nu) \cosh[a_n(z + h)]}{a_n \cosh(2a_n h)} \right] \]

\[ \sigma_{\theta \theta} = \left( \frac{\sqrt{2}GK}{a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \left[ \frac{(1 - \nu) \sinh[a_n(z + h)]}{a_n \cosh(2a_n h)} \right] \]

\[ \sigma_{zz} = \left( \frac{\sqrt{2}GK}{a} \right) \sum_{n=1}^{\infty} \left( \tilde{f}(a_n) - \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right) \left[ J_1(a_n r) - \frac{J_0(a_n r)}{a_n} \right] \left[ \frac{\tilde{g}(a_n) - \bar{g}(a_n)}{a_n^2} \right] \]

and

\[ \sigma_{rz} = 0 \]

### 3.3. Numerical conclusions

#### 3.3.1. Special case

Setting

\[ f(r) = (r^2 - a^2)^2 \]

Applying finite Hankel transform as defined in Eq. (6) to the Eq. (48), one obtain

\[ \tilde{f}(a_n) = \int_0^a \sqrt{2} J_0(a_n r) (r^2 - a^2)^2 J_1(a_n r) \, dr \]

\[ \tilde{f}(a_n) = \frac{8 \sqrt{2} ((8 - a^2 a_n^2) J_1(a_n a) - 4 a a_n J_0(a_n a))}{a_n^2 J_0(a_n a)} \]

#### 3.3.2. Dimensions

Radius of plate \( a = 1 \text{ m} \). Thickness (Height) of plate \( h = 2 \text{ m} \).
3.3.3. Thermoelastic constants

Poisson ration \( \nu = 0.35 \). Lame constant \( \mu = 26.67 \). Young’s modulus \( E = 130\, GPa \). Strength of Internal heat source \( g_s = 10^0\, C \).

3.3.4. Roots of transcendental equation

Let \( \alpha_1 = 3.8317, \alpha_2 = 7.0156, \alpha_3 = 10.1735, \alpha_4 = 13.3237, \alpha_5 = 16.470, \alpha_6 = 19.6159, \alpha_7 = 22.7601, \alpha_8 = 25.9037, \alpha_9 = 29.0468, \alpha_{10} = 32.18 \) are the roots of transcendental equation \( J_1(\alpha a) = 0 \).

3.3.5. Material properties

The numerical calculations has been performed for iron (pure) materials, having properties, Thermal diffusivity \( \alpha = 20.3410^{6}(m^2/s) \). Thermal conductivity \( k = 72.7(W/mk) \). Density \( \rho = 7897\, kg/m^3 \). Specific heat \( C_p = 452(J/kgK) \). Coefficient of linear thermal expansion \( \alpha_t = 11.810^{-6}(1/K) \). For convenience setting \( A = \frac{16}{a \times 10^3}, B = \frac{8K(1+V)a}{a(1-V)10^3}, C = \frac{16GK(1+V)a}{a(1-V)10^3} \) in the expressions (13), (42) to (46).

![Fig. 1. The temperature function \( T/A \) in radial direction](image1)

![Fig. 2. The temperature function \( T/A \) in axial direction](image2)

3.4. Concluding remarks

The attempt made for the analysis of the heat transfer and thermal stresses within cylinder due to internal heat generation under steady temperature conditions. The internal heat generation is taken as cylindrical surface heat source in annular region of linear length of cylinder and is situated concentrically inside the cylinder. As a special...
case mathematical model is constructed for Iron (Pure) thick circular plate with the material properties specified as above. From Figs. 1 and 2, as a result of cylindrical surface heat source the heat flow can be observed from centre to outer circular boundary in radial direction whereas due to temperature prescribed on upper surface heat flows from upper surface to lower surface in axial direction. From Figs. 3 and 4, radial displacement function \( u_r \) shows sinusoidal fluctuations from centre to outer circular surface in radial direction whereas in axial direction it is zero at centre and on circular boundary of cylinder. The displacement takes place at annular region \( r = r_1 = 0.5 \) where heat is generated. From Figs. 5 and 6, axial displacement function \( u_z \) is maximum on upper surface and decreases towards lower surface in both radial and axial direction. From Figs. 7 and 8, the radial stress function \( \sigma_{rr} \) develops tensile stresses in radial direction within circular region \( 0.2 \leq r \leq 1 \) and it develops at annular region \( r = r_1 = 0.5 \) where heat is generated and decreases from upper surface to lower surface in axial direction. Also it can be observed that the radial stress function \( \sigma_{rr} \) satisfies the equilibrium conditions i.e. \( \sigma_{rr} = 0 \) at outer traction free surface \( r = 1 \). From Figs. 9 and 10, the angular stress function \( \sigma_{\theta\theta} \) develops compressive stresses near centre in radial and axial direction it develops tensile stresses on the upper surface where as it is almost zero at the lower surface. From Figs. 11 and 12, the axial stress function \( \sigma_{zz} \) develops on the upper surface, it is almost zero on the lower surface, also it develops compressive stresses near centre in radial and axial direction it develops tensile stresses on the upper surface where as it is almost zero at the lower surface. Due to applying the cylindrical surface heat source in annular region, the radial and axial displacements occurs near the centre and plate expands towards the lower direction. It means we may find out that displacement and stress components occurs near heat source. The results presented here will be more useful in engineering problem particularly in the engineering systems subjected
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Fig. 5. The axial displacement function $u_z/B$ in radial direction

Fig. 6. The axial displacement function $u_z/B$ in axial direction

to cylindrical surface heat source.
Fig. 7. The radial stress function $\sigma_{rr}/C$ in radial direction

Fig. 8. The radial stress function $\sigma_{rr}/C$ in axial direction

Fig. 9. The stress function $\sigma_{\theta\theta}/C$ in radial direction
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Fig. 10. The stress function $\sigma_{\theta\theta}/C$ in axial direction

Fig. 11. The stress function $\sigma_{zz}/C$ in radial direction

Fig. 12. The stress function $\sigma_{zz}/C$ in axial direction
References