Int. J. Adv. Appl. Math. and Mech. 2(2) (2014) 150 - 158 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Similarity solution of natural convection boundary layer flow of non-Newtonian Sutterby fluids

Research Article

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Received 01 October 2014; accepted (in revised version) 11 November 2014

Abstract: In the present paper, using general group theoretic method, the similarity transformations are derived for steady, laminar, incompressible two dimensional natural convection boundary layer flow of non-Newtonian Sutterby fluid past a vertical isothermal plate. The application of a one-parameter deductive group transformation reduces the number of independent variables by one and consequently the system of governing non-linear partial differential equations with auxiliary conditions reduces to a non-linear ordinary differential equation with appropriate auxiliary conditions. Numerical solutions to the reduced non-linear similarity equations are then obtained by adopting Runge-Kutta and shooting method using the Nachtsheim- Swigert iteration technique with MATLAB. The numerical solution for the considered Sutterby fluid is derived for various values of flow consistency index A systematically in dimensionless form as an application of engineering aspects. Further the analysis is made of the velocity and temperature distribution in terms of dimensionless parameters and hence it represents behavior of all non-Newtonian Sutterby fluids.

MSC: 76M55 • 65L06

Keywords: Deductive group theoretic method • Similarity solutions • Sutterby fluids • MSABC • Skin friction and Nusselt number

1. Introduction

Polymer melts and polymer solutions possess the rheological phenomena which include non-Newtonian viscosity, normal stresses in steady shear flow and various time-dependent elastic effects [1]. The generalized Newtonian equation which describes non-Newtonian viscosity may be written as $\overline{\tau} = -\eta\overline{\Delta}$. For incompressible fluids, η may depend on the second and third invariants of $\overline{\Delta}$ (or of $\overline{\tau}$). A number of models have been proposed to describe viscosity data [2]. These models are merely mathematical functions of τ or Δ which conform to one or more of the characteristics of viscosity data.

Sutterby fluid is one of the most important non-Newtonian fluids representing constitutive equations for high polymer aqueous solutions. Sutterby model [3] describes the purely viscous non-Newtonian behavior in the shear rate range of interest as it has three constants. This paper deals with the laminar natural convection of a non-Newtonian fluid along a vertical isothermal surface and the analysis has been extended to solve the governing energy equation to obtain natural convection heat transfer characteristics. The boundary layer equations for a Sutterby fluid are solved numerically.

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Nomenclature

u, v	Velocity component in the boundary layer along x, y - axis respectively
$\tau_y x$	The non-vanishing shear tensor
x, y	Cartesian coordinates
θ	Heat transfer coefficient
ρ	Density of the fluid
α	Thermal conductivity
L	Characteristic length
U_0	Characteristic velocity
Pr	Prandtl number
Re_δ	Local Reynolds number
Gr	Grashoff number
$\mathscr{F}, \psi$	Mathematical function notations
A	Flow consistency index
B	Flow parameter
η	Similarity variable
F, G, H	Similarity functions
C_f	Skin friction
N_u	Nusselt number

The group theoretic method is of wide applicability and is a well accepted method to find the similarity solutions in many physical situations. It was first reported by Birkhoff [4] and later authors like Morgan [5], Hansen [6], Moran and Gaggioli [7] and Seshadri and Na [8] have contributed much to the development of the theory. The method has been applied intensively by Hansen and Na [9], Pakdemirli [10], Abd-el-Malek and Badran [11], Abd - el - Malek et al. [12]. In the present paper the deductive group method based on general group transformation is applied to derive similarity solutions. The similarity equations obtained are more general and systematic along with auxiliary conditions. Recently this method has been successfully applied to various non-linear problems by Hiral and Timol [12] and Darji and Timol [13].

2. Governing equations

The flow is assumed to be steady, incompressible, fully developed with uniform pressure across the cross-section of the annulus and with fluid properties independent of temperature. The basic equations of continuity, momentum and heat transfer of two dimensional steady incompressible, laminar natural convection flow over a vertical flat plate with a Cartesian coordinate system in usual notations are :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y} + g\beta\,\theta$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$$

where u and v are components of velocity in x and y directions. ρ is the fluid density, $\tau_y x$ is stress tensor and α is the thermal conductivity. Here the shearing stress $\tau_y x$ is related to the rate of stain by the arbitrary function:

$$\mathscr{F}\left(\tau_{yx}, \frac{\partial u}{\partial y}\right) = 0$$

The form of which differs for different fluid models.
Together with boundary conditions:

$$u = v = 0, \quad \theta = \theta_w \quad \text{for} \quad y = 0$$
$$u = 0, \quad \theta = 0 \quad \text{for} \quad y \to \infty$$

3. Formulation of the problem

Introducing the dimensionless quantities as:

$$\overline{x}=\frac{x\,Gr}{L}\qquad \overline{y}==\frac{y}{L}\left(\frac{\mathrm{Re}\;Gr}{3}\right)^{1/2}\qquad \overline{u}=\frac{u}{U}\qquad \overline{v}==\frac{v}{U}\left(\frac{\mathrm{Re}/3}{Gr}\right)^{1/2}$$

$$\overline{\tau}_{yx}=\frac{\tau_{yx}}{\rho U^2\left(\frac{\mathrm{Re}/3}{Gr}\right)^{1/2}}\qquad \overline{\theta}=\frac{\theta}{T'_w-T'_\infty}\qquad \overline{\theta_w}=\frac{\theta_w}{T'_w-T'_\infty}$$

$$\mathrm{Pr}=\frac{UL}{\alpha\,\mathrm{Re}}\qquad \mathrm{Re}=\frac{UL}{\nu}\qquad Gr=\frac{L}{U^2}g\beta\left(T'_w-T'_\infty\right)$$

and introducing the stream function ψ such that

$$u=\frac{\partial\psi}{\partial y}\quad\text{and}\quad v=-\frac{\partial\psi}{\partial x}$$

Continuity equation is satisfied identically and bars are dropped for simplicity. The above equation takes the following form:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial\tau_{yx}}{\partial y}+\theta \tag{1}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{1}{\mathrm{Pr}}\frac{\partial^2\theta}{\partial y^2} \tag{2}$$

$$\mathscr{F}\left(\tau_{yx},\frac{\partial^2\psi}{\partial y^2}\right)=0 \tag{3}$$

with the boundary conditions :

$$\frac{\partial\psi}{\partial y}(x,0)=\frac{\partial\psi}{\partial x}(x,0)=0,\qquad \theta(x,0)=\theta_w(x) \tag{4a}$$

$$\frac{\partial\psi}{\partial y}(x,\infty)=\theta(x,\infty)=0 \tag{4b}$$

4. Methodology and solution of the problem

Our method of solution depends on the application of a one-parameter deductive group of transformation to the partial differential Eqs. (1)-(3) along with auxiliary conditions (4). Under this transformation the two independent variables will be reduced by one and the differential equations will transforms into the ordinary differential equation.

4.1. The group systematic formulation

Introducing the group theoretic method

$$G:\quad \overline{Q}=\mathfrak{R}^Q(\varepsilon)Q+\mathfrak{D}^Q(\varepsilon) \tag{5}$$

where Q stands for $x, y, \tau_{yx}, \theta, \theta_w$. $\mathfrak{R}'s$ and $\mathfrak{D}'s$ are real-valued and are at least differentiable in the real argument ε.

4.2. The invariance analysis

For invariance invoking the group (5) in (1)-(4) and applying chain rule for transforming the derivatives we get

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}-\frac{\partial\overline{\tau}_{\overline{yx}}}{\partial\bar{y}}-\overline{\theta}=N_1(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau_{yx}}{\partial y}-\theta\right)$$

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{\theta}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{\theta}}{\partial\overline{y}}-\frac{1}{\Pr}\frac{\partial^2\overline{\theta}}{\partial\overline{y}^2}=N_2(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}\right)$$

$$\mathscr{F}\left(\overline{\tau}_{\overline{yx}},\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}\right)=N_3(\varepsilon)\mathscr{F}\left(\tau_{yx},\frac{\partial^2\psi}{\partial y^2}\right)$$

Applying chain rule for transforming the derivatives under the group (5) we get

$$\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{y}}\right)\frac{\partial\psi}{\partial y}\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{x}\mathfrak{R}^{y}}\right)\frac{\partial^2\psi}{\partial x\partial y}-\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{x}}\right)\frac{\partial\psi}{\partial x}\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{y2}}\right)\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\mathfrak{R}^{\tau_{yx}}}{\mathfrak{R}^{y}}\right)\frac{\partial\tau_{yx}}{\partial y}-\left(\mathfrak{R}^{\theta}\theta+\mathfrak{D}^{\theta}\right)$$
$$=N_1(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau_{yx}}{\partial y}-\theta\right)$$

$$\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{y}}\right)\frac{\partial\psi}{\partial y}\left(\frac{\mathfrak{R}^{\theta}}{\mathfrak{R}^{x}}\right)\frac{\partial\theta}{\partial x}-\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{x}}\right)\frac{\partial\psi}{\partial x}\left(\frac{\mathfrak{R}^{\theta}}{\mathfrak{R}^{y}}\right)\frac{\partial\theta}{\partial y}-\frac{1}{\Pr}\left(\frac{\mathfrak{R}^{\theta}}{\mathfrak{R}^{y2}}\right)\frac{\partial^2\theta}{\partial y^2}$$
$$=N_2(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{\Pr}\frac{\partial^2\theta}{\partial y^2}\right)$$

$$\mathscr{F}\left(\mathfrak{R}^{\tau_{yx}}\tau_{yx},\left(\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{y2}}\right)\frac{\partial^2\psi}{\partial y^2}\right)=N_3(\varepsilon)\mathscr{F}\left(\tau_{yx},\frac{\partial^2\psi}{\partial y^2}\right)$$

For the invariance of above equations

$$\frac{\mathfrak{R}^{\psi 2}}{\mathfrak{R}^{x}\mathfrak{R}^{y2}}=\frac{\mathfrak{R}^{\tau_{yx}}}{\mathfrak{R}^{y}}=\mathfrak{R}^{\theta}=N_1(\varepsilon)\quad\text{and}\quad\mathfrak{D}^{\theta}=0$$
$$\frac{\mathfrak{R}^{\psi}\mathfrak{R}^{\theta}}{\mathfrak{R}^{x}\mathfrak{R}^{y}}=\frac{\mathfrak{R}^{\theta}}{\mathfrak{R}^{y2}}=N_2(\varepsilon)$$
$$\mathfrak{R}^{\tau_{yx}}=1=\frac{\mathfrak{R}^{\psi}}{\mathfrak{R}^{y2}}=N_1(\varepsilon)\quad\text{and}\quad\mathfrak{D}^{\tau_{yx}}=0$$

The invariance of boundary conditions give:

$$\mathfrak{R}^{\theta}=\mathfrak{R}^{\theta_w}\quad\text{and}\quad\mathfrak{D}^{y}=\mathfrak{D}^{\theta}=\mathfrak{D}^{\theta_w}=0$$

On solving these we obtained

$$\mathfrak{R}^{x}=\mathfrak{R}^{y3},\qquad\mathfrak{R}^{\psi}=\mathfrak{R}^{y2},\qquad\mathfrak{R}^{\tau_{yx}}=1,\qquad\mathfrak{R}^{\theta}=\mathfrak{R}^{\theta_w}=1/\mathfrak{R}^{y}$$
$$N_1(\varepsilon)=\frac{1}{\mathfrak{R}^{y}},\qquad N_2(\varepsilon)=\frac{1}{\mathfrak{R}^{y^3}},\qquad N_3(\varepsilon)=1$$
$$\mathfrak{D}^{y}=\mathfrak{D}^{\tau_{yx}}=\mathfrak{D}^{\theta}=\mathfrak{D}^{\theta_w}=0$$

Finally, we get the one-parameter group $\overline{G}$, which transforms invariantly the differential equation (1)-(3) and the auxiliary conditions (4).
The group $\overline{G}$ is of the form :

$$\overline{G}\quad:\quad\begin{cases}\overline{x}=\mathfrak{R}^{y3}(\varepsilon)x+\mathfrak{D}^{x}(\varepsilon)\\ \overline{y}=\mathfrak{R}^{y}(\varepsilon)y\\ \overline{\psi}=\mathfrak{R}^{y2}(\varepsilon)\psi+\mathfrak{D}^{\psi}(\varepsilon)\\ \overline{\tau}_{\overline{yx}}=\tau_{yx}\\ \overline{\theta}=(\mathfrak{R}^{y})^{-1}(\varepsilon)\theta\\ \overline{\theta_w}=(\mathfrak{R}^{y})^{-1}(\varepsilon)\theta_w\end{cases}\tag{6}$$

4.3. The complete set of absolute invariants

Now, We proceed in our analysis to obtain a complete set of absolute invariants so that the original problem will transformed into an ordinary differential equation in a similarity variable via group theoretic method. We have applied HAMAD (2010) formulations for PDEs of 2- independent variables.
By considering $x_1 = x, x_2 = y, y_1 = \psi, y_2 = \theta, y_3 = \tau_{yx}, y_4 = \theta_w$ and the definitions of $\alpha_i, \beta_i; \quad i = 1,2,...6$. We get

$$\alpha_i = \left.\frac{\partial \mathfrak{R}^i}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon_o} \quad \text{and} \quad \beta_i = \left.\frac{\partial \mathfrak{D}^i}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon_o} \quad ; \quad i = 1,2,...6.$$

where ε_o denotes the value of $'\varepsilon'$ which yield the identity element of the group. The generator is given by

$$X = (\alpha_1 x_1 + \beta_1)\frac{\partial g}{\partial x_1} + (\alpha_2 x_2 + \beta_2)\frac{\partial g}{\partial x_2} + (\alpha_3 y_1 + \beta_3)\frac{\partial g}{\partial y_1} + (\alpha_4 y_2 + \beta_4)\frac{\partial g}{\partial y_2} + (\alpha_5 y_3 + \beta_5)\frac{\partial g}{\partial y_3} + (\alpha_6 y_4 + \beta_6)\frac{\partial g}{\partial y_4}$$

The above generator gives a rise characteristic equation which on solving and using the relations between α's and β's from Eq. (6) we obtained similarity variables as follow (See [7, 12, 13]) :

$$\left.\begin{aligned} &\eta = y(x+\beta)^{-\frac{1}{3}} \quad \text{where} \quad \beta = \frac{\beta_1}{\alpha_1} \\ &\psi = (x+\beta)^{\frac{2}{3}} F(\eta) - -\frac{\beta_3}{\alpha_3} \\ &\theta = (x+\beta)^{-\frac{1}{3}} H(\eta) \\ &\tau_{yx} = G(\eta) \\ &\theta_w = (x+\beta)^{-\frac{1}{3}} \end{aligned}\right\} \tag{7}$$

4.4. The reduction to an ordinary differential equation

The similarity transformations (7) maps equations (1)-(4) into the following non-linear ordinary differential equations:

$$F'^2 - 2FF'' - 3G' - 3H = 0 \tag{8}$$

$$H'' + \mathrm{Pr}(F'H + 2FH') = 0 \tag{9}$$

$$\mathscr{F}(G, F'') = 0 \tag{10}$$

$$F(0) = F'(0) = 0, \qquad F'(\infty) = 0 \tag{11a}$$

$$H(0) = 1, \qquad H(\infty) = 0 \tag{11b}$$

5. Numerical solution

For finding the numerical solution we have consider the Sutterby fluid model. Mathematically this model is given as

$$\tau_{yx} = \mu_0 \left[\frac{\sinh^{-1}\left(B\frac{\partial u}{\partial y}\right)}{\left(B\frac{\partial u}{\partial y}\right)}\right]^A \frac{\partial u}{\partial y}$$

Introducing the dimensionless quantities (defined in section 3) and applying the similarity variables:

$$G' = \frac{\alpha}{3} F'''(F'')^{-A}\left(\sinh^{-1}\lambda F''\right)^{A-1}\left[\frac{A\lambda F''}{\sqrt{1+\lambda^2 F''^2}} + (1-A)\sinh^{-1}\lambda F''\right] \tag{12}$$

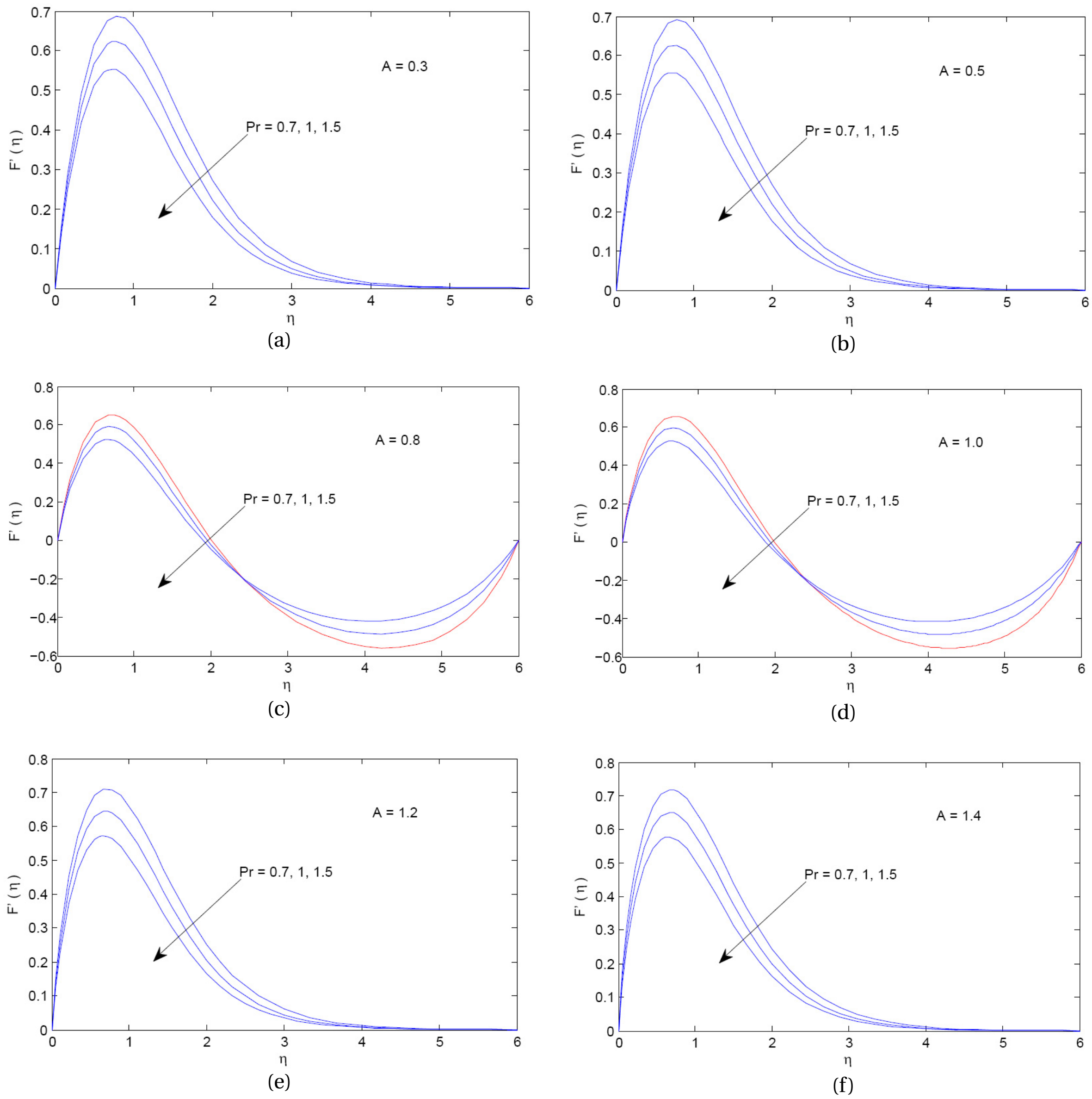

Fig. 1. Velocity profiles for various A

where $\alpha = \dfrac{\mu_0 \lambda^{-A}}{\mu}$ and $\lambda = \dfrac{B\,U}{L}\left(\dfrac{\mathrm{Re}\;Gr}{3}\right)^{1/2}$ are dimensionless numbers and can be referred as a flow parameters. Substituting (12) into (8) we get

$$F''' = \frac{1}{\alpha}\frac{\left(F'^2 - 2FF'' - 3H\right)\left(F''\right)^A \left(\sinh^{-1}\lambda F''\right)^{1-A}\left(1+\lambda^2 F''^2\right)^{1/2}}{\lambda\, AF'' + (1-A)\sinh^{-1}\lambda F''\sqrt{1+\lambda^2 F''^2}} \tag{13}$$

The numerical method applied to solve Eqs. (13) and (9) with the boundary conditions (11) is the Adams-Moulton procedure along with shooting method due to Nachtsheim and Swigert [15]. Integration is carried out using MATLAB ode solver with the step size 0.5. The numerical method applied to solve Eqs. (13) and (9) with the boundary conditions (11) is the Adams-Moulton procedure along with shooting method due to Nachtsheim and Swigert (1965). Integration is carried out using MATLAB ode solver with the step size 0.5. Starting from $\eta = 0, h = 0.5$ integration is performed until $\eta_{stop} = 10$. Starting from $\eta = 0, h = 0.5$ integration is performed until $\eta_{stop} = 10$. Fixing the non-dimensional numbers $\alpha = 10$, $\beta = 1.5 \times 10^{-2}$ and for different values of $Pr = 0.7, 1.0, 1.5$ the velocity profile is plotted (Fig. 1). Whereas by keeping the same α and β, temperature variation is generated for different Prandtl number (Fig. 2). From the graphs it is evident that K and Pr have great influence on the velocity and temperature distribution of the Sutterby fluids. All the figures are plotted in terms of dimensionless parameters. The physical quantity of interest are the coefficient of local skin friction C_f and Nusslet number N_u which are given by the equations:

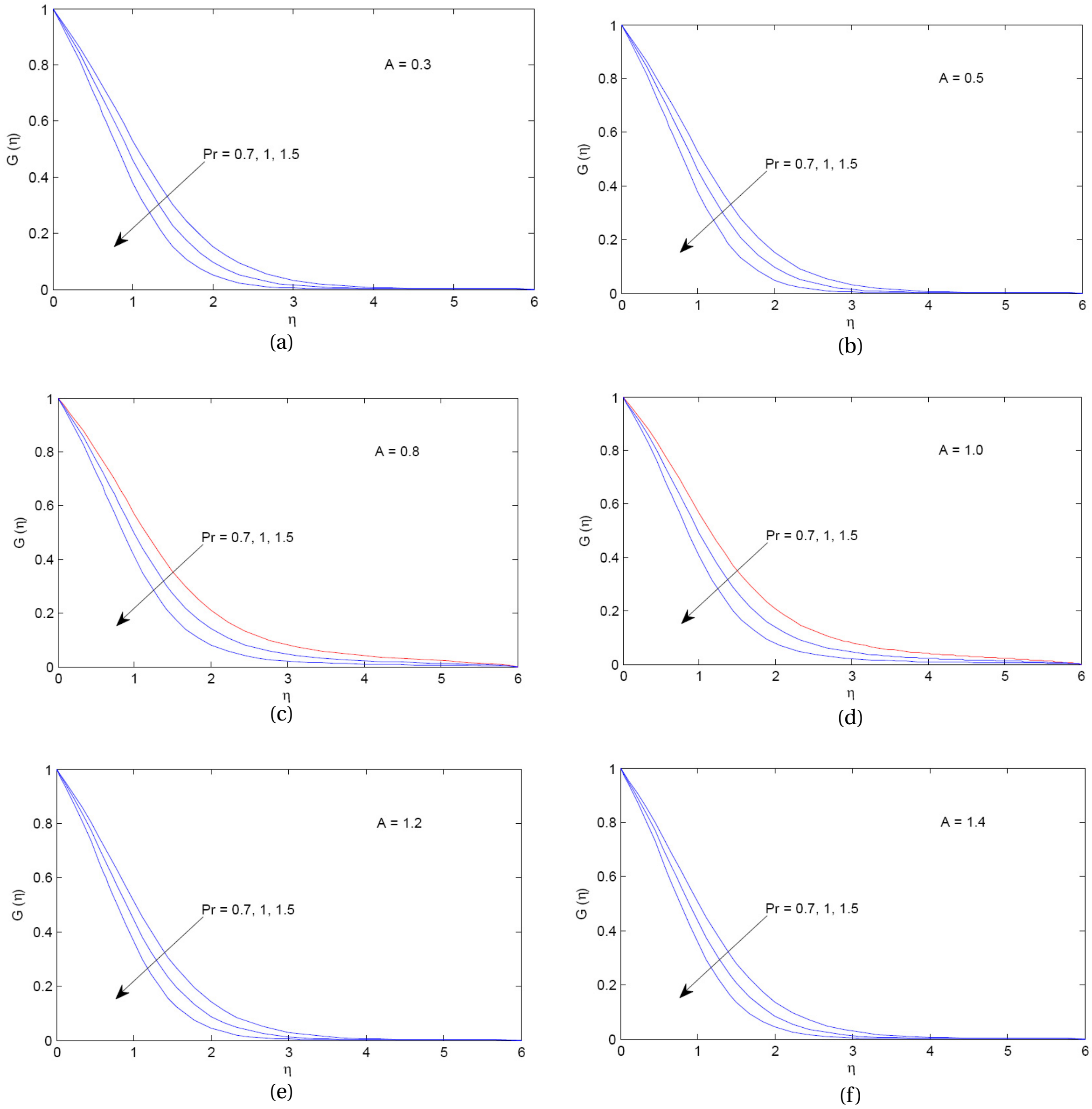

Fig. 2. Temperature profiles for various A

$$C_f = \frac{2\tau_w}{\sqrt{R_e}} \qquad \text{where} \quad \tau_w = \frac{\alpha}{3\sqrt{\beta}} \sin^{-1}\left(\sqrt{\beta} F''(0)\right)$$

$$N_u = -\frac{x}{T_w - T_\infty} \frac{\partial T}{\partial y}\bigg|_{y=0} = -\sqrt{R_e} H'(0)$$

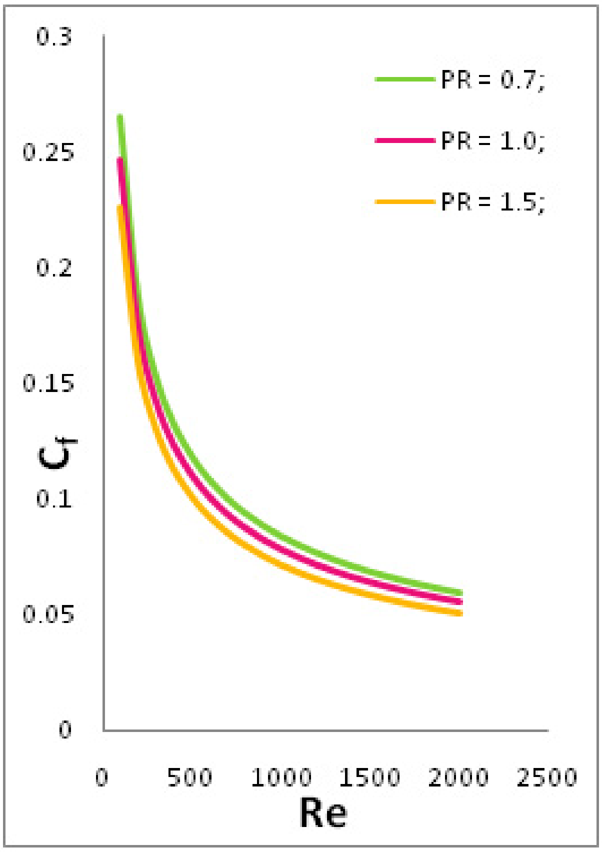

Fig. 3. Skin friction coefficent C_f

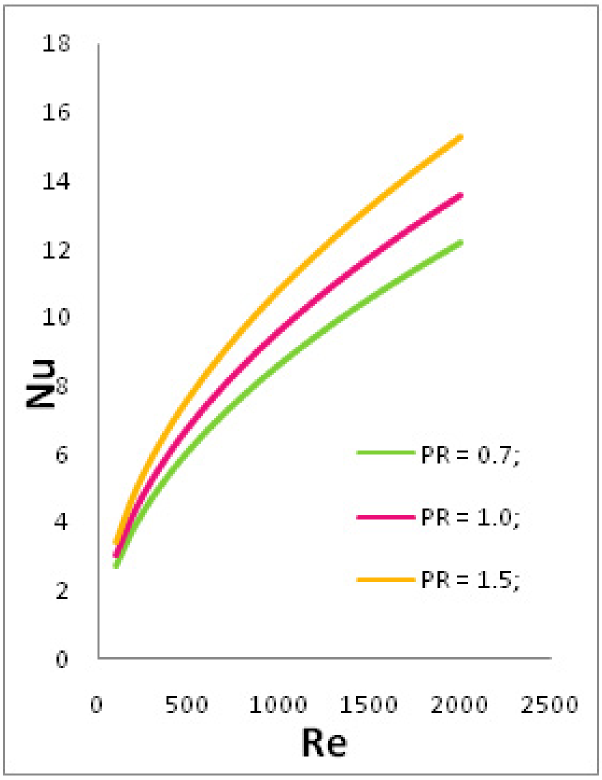

Fig. 4. Nusselt number N_u

6. conclusion

The numerical solution for the considered Sutterby fluid is derived for one value of flow consistency index $A = 0.3, 0.5, 0.8, 1.0, 1.2, 1.4$. Numerical results are presented in the form of graphs. The obtained results of velocity and temperature showed good agreement with the solution of Fujii et al [1]. The method appears to be insensitive to initial guesses and converges quickly to the solution. The important findings of the analysis of the results of the present problem are for fixed material constants as Prandtl number increases F' and G decrease rapidly. It is observed that Skin friction coefficient decreases when Reynold number increases but Nusselt number increases for the same.

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