Int. J. Adv. Appl. Math. and Mech. 2(2) (2014) 21 - 38 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

An application of Meir-Keeler type tripled fixed point theorem

Research Article

Animesh Gupta *

H.No. 93/654, Ward No. 2 Gandhi Chowk, Pachmarhi-461881, Dist. Hoshangabad (M.P.), India
Visiting at : Department of Mathematics & Computer Science, Govt. Model Science College, Jabalpur (M.P.) India

Received 21 August 2014; accepted (in revised version) 13 November 2014

Abstract: Berinde and Borcut [1] introduced the concept of triple fixed point and proof some related fixed point theorem with some applications. The aim of this paper is to extend the result of Berinde and Borcut [1]. Indeed, we introduced the definition of generalized $g-$Meir-Keeler type contractions and prove some tripled fixed point theorems under a generalized $g-$Meir-Keeler type contractive condition. We also give an application of main results of this paper.

MSC: Primary 47H10 • Secondary 54H25

Keywords: Tripled coincidence point • Tripled common fixed point • Mixed strict $g-$monotone • Mixed strict monotone • Meir-Keeler type contraction

1. Introduction and preliminaries

The Banach contraction principle [2] is a classical and powerful tool in non linear analysis and has been generalized by many authors. Bhaskar and Lakshmikantham [3] introduced the concept of a coupled fixed point of mapping $F: X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered metric spaces. Later, various results in coupled fixed point have been obtained, see e.g. ([4]-[9]).
On the other hand, Berinde and Borcut [1] introduced the concept of triple fixed point and proof some related fixed point theorem. After this various results on tripled fixed point have been obtained. The following definitions are from [1].

Definition 1.1.

Let $(X, \preceq)$ be a partially ordered set, $F: X^3 \to X$ mapping. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$,

(i) $x_1, x_2 \in X, \ \ x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,

(ii) $y_1, y_2 \in X, \ \ y_1 \succeq y_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$,

(iii) $z_1, z_2 \in X, \ \ z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 1.2.

An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F: X^3 \to X$ if

$$F(x, y, z) = x, \ \ F(y, x, y) = y, \text{ and } F(z, y, x) = z.$$

* Corresponding author.
E-mail address: dranimeshgupta10@gmail.com

Definition 1.3.
Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$ two mappings. The mapping F is said to have the mixed $g-$ monotone property if for any $x, y, z \in X$.

i. $x_1, x_2 \in X,\ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,

ii. $y_1, y_2 \in X,\ g(y_1) \succeq g(y_2) \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$,

iii. $z_1, z_2 \in X,\ g(z_1) \preceq g(z_2) \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 1.4.
An element $(x, y, z) \in X^3$ is called a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if

$$F(x, y, z) = gx,\ \ F(y, x, y) = gy \ \text{ and } \ F(z, y, x) = gz.$$

Definition 1.5.
An element $(x, y, z) \in X^3$ is called a tripled common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if

$$F(x, y, z) = gx = x,\ \ F(y, x, y) = gy = y \ \text{ and } \ F(z, y, x) = gz = z.$$

Definition 1.6.
An element $x \in X$ is called a common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if

$$F(x, x, x) = gx = x.$$

Definition 1.7.
Let X be a non empty set. The mappings $F : X^3 \to X$ and $g : X \to X$ are commuting if for all $x, y, z \in X$,

$$g(F(x, y, z)) = F(g(x), g(y), g(z)).$$

Definition 1.8.
Let (X, d) be a metric space. The mappings F and g where $F : X^3 \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0$$
$$\lim_{n\to\infty} d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n))) = 0$$
$$and$$
$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} g(y_n) = y$ and $\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = z$ for some $x, y, z \in X$.

In [1] Berinde and Borcut proved the following theorem.

Theorem 1.1.
Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c \prec 1$ for which,

$$d(F(x, y, z), F(u, v, w)) \preceq a d(x, u) + b d(y, v) + c d(z, w) \tag{1}$$

For all $x \succeq u, y \preceq v, z \succeq w$. Assume either,

1. *F is continuous,*
2. *X has the following properties:*
   (a) *if non decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
   (b) *if non increasing sequence $y_n \to y$, then $y_n \succeq x$ for all n,*

If there exist $x_0, y_0, z_0 \in X$ *such that*

$$x_0 \preceq F(x_0, y_0, z_0), \quad y_0 \succeq F(y_0, x_0, y_0), \quad and \quad z_0 \preceq F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ *such that,*

$$F(x, y, z) = x, F(y, x, y) = y, and F(z, y, x) = z$$

Beside this in [10] Meir and Keeler generalized the well known Banach fixed point theorem [2] as follows;

Theorem 1.2.
Let $(X, \preceq)$ *be a complete metric space and* $T : X \times X \to X$ *be a given mapping. Suppose that, for any* $\epsilon \succ 0$ *then there exists* $\delta(\epsilon) \succ 0$ *such that*

$$\epsilon \preceq d(x, y) \prec \epsilon + \delta(\epsilon) \Longrightarrow d(Tx, Ty) \prec \epsilon \tag{2}$$

for all $x, y \in X$. *Then* T *admits a unique fixed point* $x_0 \in X$ *for all* $x \in X$, *the sequence* $\{T^n(x)\}$ *converges to* x_0.

Definition 1.9.
Let $(X, \preceq)$ be a partially ordered metric space and $F : X^3 \to X$ be a given mapping satisfying the following contraction condition,

$$\epsilon \preceq \frac{1}{3}(d(x, u) + d(y, v) + d(z, w)) \prec \epsilon + \delta(\epsilon) \Longrightarrow d(F(x, y, z), F(u, v, w)) \prec \epsilon \tag{3}$$

for all $x, y, z, u, v, w \in X$. Then F is a generalized Meir-Keeler type contraction.

Motivated by the results of Berinde and Borcut [1] and Meir-Keeler [10] we introduced the definition of $g-$ Meir-Keeler contractive mappings and prove some tripled common fixed point theorems under the generalized $g-$Meir-Keeler contractive condition.

2. Main Results

We introduced the following two definitions.

Definition 2.1.
Let $(X, \preceq)$ be a partially ordered set and $F : X^3 \to X$ and $g : X \to X$. We say that F has the mixed strict $g-$ monotone property if, for any $x, y, z \in X$,

$$\left.\begin{array}{lll} x_1, x_2 \in X, & g(x_1) \prec g(x_2) & \Rightarrow \quad F(x_1, y, z) \prec F(x_2, y, z) \\ y_1, y_2 \in X, & g(y_1) \prec g(y_2) & \Rightarrow \quad F(x, y_1, z) \succ F(x, y_2, z) \\ z_1, z_2 \in X, & g(z_1) \prec g(z_2) & \Rightarrow \quad F(x, y, z_1) \prec F(x, y, z_2). \end{array}\right\} \tag{4}$$

Definition 2.2.
Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$ be two mappings. We say that F is a generalized $g-$ Meir-Keeler type contraction if for all $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that for all $x, y, z, u, v, w \in X$ with $g(x) \preceq g(u)$, $g(y) \succeq g(v)$ and $g(z) \preceq g(w)$,

$$\begin{array}{rcl} \epsilon & \preceq & \frac{1}{3}[d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w))] \prec \epsilon + \delta(\epsilon) \\ & & \Rightarrow d(F(x, y, z), F(u, v, w)) \prec \epsilon. \end{array} \tag{5}$$

Lemma 2.1.
Let $(X, \preceq)$ *be a partially ordered set and* $F : X^3 \to X$ *and* $g : X \to X$ *be two mappings. We say that* F *is a generalized* $g-$ *Meir-Keeler type contraction then we have*

$$d(F(x, y, z), F(u, v, w)) \prec \frac{1}{3}[d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w))] \tag{6}$$

for all $x, y, z, u, v, w \in X$ *with* $g(x) \preceq g(u)$, $g(y) \succeq g(v)$ *and* $g(z) \preceq g(w)$.

Proof. Let $x, y, z, u, v, w \in X$ such that

$$g(x) \prec g(u),\ g(y) \succeq g(v),\ g(z) \preceq g(w)$$

or

$$g(x) \preceq g(u),\ g(y) \succ g(v),\ g(z) \preceq g(w)$$

or

$$g(x) \preceq g(u),\ g(y) \succeq g(v),\ g(z) \prec g(w)$$

then

$$d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w)) \succ 0.$$

Since F is a generalized $g-$ Meir-Keeler type contraction so

$$\epsilon = \frac{1}{3}\big(d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w))\big)$$

there exists $\delta(\epsilon) \succ 0$ such that, for all $x_0, y_0, z_0, u_0, v_0, w_0 \in X$ with $g(x_0) \preceq g(u_0)$, $g(y_0) \succeq g(v_0)$ and $g(z_0) \preceq g(w_0)$,

$$\begin{aligned} \epsilon &\preceq \frac{1}{3}\big(d(g(x_0), g(u_0)) + d(g(y_0), g(v_0)) + d(g(z_0), d(w_0))\big) \prec \epsilon + \delta(\epsilon) \\ &\Rightarrow\ d(F(x_0, y_0, z_0), F(u_0, v_0, w_0)) \prec \epsilon \end{aligned} \tag{7}$$

therefore putting $x_0 = x$, $y_0 = y$, $z_0 = z$, $u_0 = u$, $v_0 = v$ and $w_0 = w$, we have

$$\begin{aligned} \epsilon &\preceq \frac{1}{3}\big(d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w))\big) \prec \epsilon + \delta(\epsilon) \\ &\Rightarrow\ d(F(x, y, z), F(u, v, w)) \prec \epsilon \end{aligned} \tag{8}$$

this completes the proof. □

Theorem 2.1.
Let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(X)$ also g is continuous and commutative with F. Suppose that

(a) F has the mixed strict $g-$ monotone property;

(b) F is a generalized $g-$ Meir-Keeler type contraction;

(c) there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \prec F(x_0, y_0, z_0), \quad g(y_0) \succ F(y_0, x_0, y_0) \quad and \quad g(z_0) \prec F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x, y, z) = g(x),\ \ F(y, x, y) = g(y)\ \ and\ \ F(z, y, x) = g(z)$$

that is F and g have a tripled coincidence in X^3.

Proof. Let $x_0, y_0, z_0 \in X$ be such that,

$$\begin{aligned} &g(x_0) \prec F(x_0, y_0, z_0), \\ &g(y_0) \succ F(y_0, x_0, y_0), \\ &g(z_0) \prec F(z_0, y_0, x_0). \end{aligned}$$

Since $F(X^3) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that,

$$\begin{aligned} &g(x_1) = F(x_0, y_0, z_0), \\ &g(y_1) = F(y_0, x_0, y_0), \\ &g(z_1) = F(z_0, y_0, x_0). \end{aligned}$$

Again, from $F(X^3) \subseteq g(X)$ we can choose $x_2, y_2, z_2 \in X$ such that,

$$g(x_2) = F(x_1, y_1, z_1),$$
$$g(y_2) = F(y_1, x_1, y_1),$$
$$g(z_2) = F(z_1, y_1, x_1).$$

Continuing this process we can construct the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n, z_n), \\ g(y_{n+1}) &= F(y_n, x_n, y_n) \\ g(z_{n+1}) &= F(z_n, y_n, x_n) \end{aligned} \tag{9}$$

for all $n \succeq 0$.

Next we show that

$$\begin{aligned} g(x_n) &\prec g(x_{n+1}), \\ g(y_n) &\succ g(y_{n+1}), \\ g(z_n) &\prec g(z_{n+1}) \end{aligned} \tag{10}$$

for all $n \succeq 0$. If we take $n = 0$ in (10) then

$$\begin{aligned} g(x_0) &\prec g(x_1) = F(x_0, y_0, z_0), \\ g(y_0) &\succ g(y_1) = F(y_0, x_0, y_0), \\ g(z_0) &\prec g(z_1) = F(z_0, y_0, x_0). \end{aligned} \tag{11}$$

Since F has the mixed strict $g-$ monotone property, then we have

$$\begin{aligned} g(x_0) \prec g(x_1) &\Rightarrow F(x_0, y_1, z_1) \prec F(x_1, y_1, z_1), \\ g(y_0) \prec g(y_1) &\Rightarrow F(y_0, x_1, y_0) \succ F(y_1, x_1, y_1), \\ g(z_0) \prec g(z_1) &\Rightarrow F(z_0, y_1, x_1) \prec F(z_1, y_1, x_1). \end{aligned} \tag{12}$$

Continuing this process for each $n \succeq 1$, we have

$$\begin{aligned} g(x_{n+2}) &= F(x_{n+1}, y_{n+1}, z_{n+1}) \succ F(x_n, y_{n+1}, z_{n+1}) \succ F(x_n, y_n, z_n) = g(x_{n+1}), \\ g(y_{n+2}) &= F(y_{n+1}, x_{n+1}, y_{n+1}) \prec F(y_n, x_{n+1}, z_{n+1}) \prec F(y_n, x_n, y_n) = g(y_{n+1}), \\ g(z_{n+2}) &= F(z_{n+1}, y_{n+1}, x_{n+1}) \succ F(z_n, y_{n+1}, x_{n+1}) \succ F(z_n, y_n, x_n) = g(z_{n+1}), \end{aligned}$$

implies that,

$$\begin{aligned} g(x_0) \prec g(x_1) \prec g(x_2) \prec \cdots \prec g(x_n) \prec g(x_{n+1}) \prec \ldots, \\ g(y_0) \succ g(y_1) \succ g(y_2) \succ \cdots \succ g(y_n) \succ g(y_{n+1}) \succ \ldots, \\ g(z_0) \prec g(z_1) \prec g(z_2) \prec \cdots \prec g(z_n) \prec g(z_{n+1}) \prec \ldots. \end{aligned} \tag{13}$$

Denote that

$$\delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})). \tag{14}$$

Since $g(x_n) \succ g(x_{n-1}), g(y_n) \prec g(y_{n-1}), g(z_n) \succ g(z_{n-1})$.

It follows that from (9) and Lemma 2.1,

$$\begin{aligned} d(g(x_n), g(x_{n+1})) = & d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ & \preceq \frac{1}{3}\big[d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)) \\ & + d(g(z_{n-1}), d(z_n))\big]. \end{aligned} \tag{15}$$

In order to (15) we have

$$\begin{aligned} d(g(y_n), g(y_{n+1})) = & d(F(y_{n-1}, z_{n-1}, x_{n-1}), F(y_n, z_n, x_n)) \\ & \preceq \frac{1}{3}\big[d(g(y_{n-1}), g(y_n)) + d(g(z_{n-1}), g(z_n)) \\ & + d(g(x_{n-1}), d(x_n))\big] \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
d(g(z_n), g(z_{n+1})) &= d(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n)) \\
&\preceq \frac{1}{3}\big[d(g(z_{n-1}), g(z_n)) + d(g(x_{n-1}), d(x_n)) \\
&+ d(g(y_{n-1}), g(y_n))\big]. \qquad (17)
\end{aligned}
$$

Thus it follows from (14) - (17) that $\delta_n \prec \delta_{n-1}$. Its mean the sequence $\{\left(\frac{\delta_n}{3}\right)\}$ is monotone decreasing. So there exists $\delta^* \succeq 0$ such that $\lim_{n\to\infty}\left(\frac{\delta_n}{3}\right) = \delta^*$, that is ,

$$
\lim_{n\to\infty} \frac{1}{3}\big[d(g(z_{n-1}), g(z_n)) + d(g(x_{n-1}), d(x_n)) + d(g(y_{n-1}), g(y_n))\big] = \delta^*. \qquad (18)
$$

Next we show that $\delta^* = 0$. Suppose that $\delta^* \succ 0$ hold. Let $\delta^* = \epsilon$. Then there exists a positive integer m such that,

$$
\begin{aligned}
\epsilon &\preceq \frac{1}{3}\big(d(g(x_m), d(x_{m+1})) + d(g(y_m), g(y_{m+1})) + d(g(z_m), g(z_{m+1}))\big) \\
&= \epsilon + \delta(\epsilon). \qquad (19)
\end{aligned}
$$

Then, by using (10) also F is a generalized $g-$ Meir-Keeler type contraction, so

$$
d(F(x_m, y_n, z_m), F(x_{m+1}, y_{m+1}, z_{m+1})) \prec \epsilon, \qquad (20)
$$

from (9) we have

$$
d(g(x_{m+1}), g(x_{m+2})) \prec \epsilon. \qquad (21)
$$

On the other hand, by (15), we have

$$
\frac{1}{3}\big[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1})) + d(g(z_m), d(z_{m+1}))\big] \prec \epsilon \qquad (22)
$$

which contradiction of (19). Thus we have $\epsilon = \delta^* = 0$ so

$$
\lim_{n\to\infty} \frac{1}{3}\big[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), d(z_{n+1})))\big] = 0 \qquad (23)
$$

that is,

$$
\lim_{n\to\infty} \delta_n = 0 \qquad (24)
$$

Next we prove that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences in X.

Suppose that at least one of $\{g(x_n)\}$, $\{g(y_n)\}$ or $\{g(z_n)\}$ are not a Cauchy sequence. Then there exist $\epsilon \succ 0$ and two subsequences $\{l_k\}$ and $\{m_k\}$ of integers $m_k \succ l_k \succ k$ and

$$
\begin{aligned}
d(g(x_{l_k}), g(x_{m_k})) &\succeq \frac{\epsilon}{3}, \\
d(g(y_{l_k}), g(y_{m_k})) &\succeq \frac{\epsilon}{3}, \\
d(g(z_{l_k}), g(z_{m_k})) &\succeq \frac{\epsilon}{3} \qquad (25)
\end{aligned}
$$

for all $k \succeq 1$. Then we have

$$
r_k = d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) + d(g(z_{l_k}), g(z_{m_k})) \succeq \epsilon \qquad (26)
$$

for all $k \succeq 1$. Let m_k be the smallest number exceeding l_k such that (26) holds. Then we have

$$
d(g(x_{l_k}), g(x_{m_k-1})) + d(g(y_{l_k}), g(y_{m_k-1})) + d(g(z_{l_k}), g(z_{m_k-1})) \prec \epsilon \qquad (27)
$$

Thus from (14), (26), (27) and the triangular inequality, it follows that

$$\epsilon \preceq r_k$$

$$\begin{aligned}\epsilon \preceq & d(g(x_{l_k}), g(x_{m_k-1})) + d(g(x_{m_k-1}), g(x_{m_k})) \\ & + d(g(y_{l_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k})) \\ & + d(g(z_{l_k}), g(z_{m_k-1})) + d(g(z_{m_k-1}), g(z_{m_k})) \\ & \preceq \epsilon + \delta_{m_k-1} \end{aligned} \tag{28}$$

and so,

$$\epsilon \preceq \lim_{k\to\infty} r_k \preceq \lim_{k\to\infty} (\epsilon + \delta_{m_k-1}) \tag{29}$$

from (24) we have

$$\lim_{k\to\infty} r_k = \epsilon^+. \tag{30}$$

It follow from (9),(14) and the triangle inequality that

$$\begin{aligned} r_k &= d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) + d(g(z_{l_k}), g(z_{m_k})) \\ &\succeq \epsilon \\ &\preceq d(g(x_{l_k}), g(x_{l_k+1})) + d(g(x_{l_k+1}), g(x_{m_k+1})) + d(g(x_{m_k+1}), g(x_{m_k})) \\ &+ d(g(y_{l_k}), g(y_{l_k+1})) + d(g(y_{l_k+1}), g(y_{m_k+1})) + d(g(y_{m_k+1}), g(y_{m_k})) \\ &+ d(g(z_{l_k}), g(z_{l_k+1})) + d(g(z_{l_k+1}), g(z_{m_k+1})) + d(g(z_{m_k+1}), g(z_{m_k})) \\ &= \delta_{l_k} + \delta_{m_k} + d(g(x_{l_k+1}), g(x_{m_k+1})) \\ &+ d(g(y_{l_k+1}), g(y_{m_k+1})) + d(g(z_{l_k+1}), g(z_{m_k+1})) \\ &= \delta_{l_k} + \delta_{m_k} + d(F(x_{l_k}, y_{l_k}, z_{l_k}), F(x_{m_k}, y_{m_k}, z_{m_k})) \\ &+ d(F(y_{l_k}, x_{l_k}, y_{l_k}), F(y_{m_k}, x_{m_k}, y_{m_k})) \\ &+ d(F(z_{l_k}, y_{l_k}, x_{l_k}), F(z_{m_k}, y_{m_k}, x_{m_k})). \end{aligned} \tag{31}$$

From (13) we have $g(x_{l_k}) \prec g(x_{m_k})$, $g(y_{l_k}) \succ g(y_{m_k})$ and $g(z_{l_k}) \prec g(z_{m_k})$. It follows from Lemma 2.1 and (31) that

$$\begin{aligned} r_k &\prec \delta_{l_k} + \delta_{m_k} + d(g(x_{l_k}), g(x_{m_k})) \\ &+ d(g(y_{l_k}), g(y_{m_k})) + d(g(z_{l_k}), g(z_{m_k})) \end{aligned} \tag{32}$$

that is,

$$r_k \prec \delta_{l_k} + \delta_{m_k} + r_k \tag{33}$$

This is contradiction. Therefore, $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences in X. Since X is complete, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x, \ \lim_{n\to\infty} g(y_n) = y \ \ and \ \ \lim_{n\to\infty} g(z_n) = z. \tag{34}$$

Since $\{g(x_n)\}$,$\{g(z_n)\}$ are monotone increasing and $\{g(y_n)\}$ is monotone decreasing, so we have

$$g(x_n) \prec x, \ \ g(y_n) \succ y \ \ and \ \ g(z_n) \prec z \tag{35}$$

for all $n \succeq 1$. Thus it follows from (34) and the continuity of g that

$$\lim_{n\to\infty} g(x_n) = g(x), \ \lim_{n\to\infty} g(y_n) = g(y) \ \ and \ \ \lim_{n\to\infty} g(z_n) = g(z). \tag{36}$$

Thus, for all $m \succeq 1$, there exists a positive integer n_n such that, for all $n \succeq n_0$,

$$d(g(g(x_n)), g(x)) \prec \frac{1}{4m}, \ \ d(g(g(y_n)), g(y)) \prec \frac{1}{4m} \ \ and \ \ d(g(g(z_n)), g(z)) \prec \frac{1}{4m} \tag{37}$$

Hence from (9), the commutativity of F and g and the triangle inequality, we have

$$\begin{aligned} d(F(x,y,z), g(x)) &\preceq d(F(x,y,z), g(g(x_n))) \\ &+ d(g(g(x_n)), g(x)) \\ &= d(F(x,y,z), g(F(x_{n-1}, y_{n-1}, z_{n-1}))) \\ &+ d(g(g(x_n)), g(x)) \\ &= d(F(x,y,z), F(gx_{n-1}, gy_{n-1}, gz_{n-1})) \\ &+ d(g(g(x_n)), g(x)). \end{aligned} \tag{38}$$

Thus, it follows from (35), (37) and Lemma 2.1, that

$$\begin{aligned} d(F(x,y,z), g(x)) &\prec \frac{1}{3}[d(g(g(x_{n-1})), g(x)) + d(g(g(y_{n-1})), g(y)) \\ &+ d(g(g(z_{n-1})), g(z))] \\ &\prec \frac{1}{12m} + \frac{1}{12m} + \frac{1}{12m} + \frac{1}{4m} \\ &\frac{1}{2m} \to 0 \end{aligned} \tag{39}$$

as $m \to \infty$. Therefore, we have $F(x,y,z) = g(x)$. Similarly we can show that $F(y,z,y) = g(y)$ and $F(z,y,x) = g(z)$. This means that F and g have a coupled coincidence point in X^3. This completes the proof. □

Corollary 2.1.

Let $F : X^3 \to X$ be a mapping satisfying the following conditions:

(a) *F has the mixed strict monotone property;*

(b) *F is a generalized Meir-Keeler type contraction;*

(c) *there exist $x_0, y_0, z_0 \in X$ such that $x_0 \prec F(x_0, y_0, z_0)$, $y_0 \succ F(y_0, x_0, y_0)$ and $z_0 \succ F(z_0, y_0, x_0)$.*

Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = x, \ \ F(y,x,y) = y \ \ and \ \ F(z,y,x) = z$$

that is, F has a tripled fixed point in X^3.

Proof. The conclusion follows from Theorem 2.1, by taking $g = I$ (Identity mapping) on X. □

Now we introduced the product space X^3 with the following partial order; for all $(x,y,z),(u,v,w) \in X^3$,

$$(x,y,z) \succeq (u,v,w) \Longleftrightarrow u \prec x, \ \ v \succeq y, \ \ w \preceq z. \tag{40}$$

Theorem 2.2.

Suppose that all the hypothesis of Theorem 2.1 hold and further, for all $(x,y,z),(x^*,y^*,z^*) \in X^3$, there is $(u,v,w) \in X^3$ such that $(F(u,v,w), F(v,u,v), F(w,v,u))$ is comparable to $(F(x,y,z), F(y,x,y), F(z,y,x))$ and $(F(x^*,y^*,z^*), F(y^*,x^*,y^*), F(z^*,y^*,x^*))$. Then F and g have a unique tripled common fixed point in X^3 such that

$$F(x,y,z) = g(x) = x, \ \ F(y,x,y) = g(y) = y \ \ and \ \ F(z,y,x) = g(z) = z. \tag{41}$$

Proof. By Theorem 2.1, the set of tripled coincidence of the mappings F and g is non empty. First we show that if (x,y,z) and (x^*,y^*,z^*) are tripled coincidence points of F and g, that is, if

$$\begin{aligned} F(x,y,z) = g(x), \ \ F(y,x,y) = g(y) \ \ and \ \ F(z,y,x) = g(z). \\ F(x^*,y^*,z^*) = g(x^*), \ \ F(y^*,x^*,y^*) = g(y^*) \ \ and \ \ F(z^*,y^*,x^*) = g(z^*). \end{aligned} \tag{42}$$

then we have

$$g(x)=g(x^*)\ g(y)=g(y^*)\ and\ g(z)=g(z^*) \tag{43}$$

Put $u_0 = u$, $v_0 = v$, $w_0 = w$ and choose $u_1, v_1, w_1 \in X$ such that $g(u_1) = F(u_0, v_0, w_0)$, $g(v_1) = F(v_0, w_0, u_0)$ and $g(w_1) = F(w_0, u_0, v_0)$. Similarly as in the proof of Theorem 2.1, we can inductively define the sequences $\{g(u_n)\}$, $\{g(v_n)\}$ and $\{g(w_n)\}$ such that

$$g(u_{n+1})=F(u_n,v_n,w_n),\ g(v_{n+1})=F(v_n,u_n,v_n),\ and\ g(w_{n+1})=F(w_n,v_n,u_n) \tag{44}$$

for all $n \succeq 0$. Also, if we set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $x_0^* = x^*$, $y_0^* = y^*$, $z_0^* = z^*$ then we can defined the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(z_n)\}$ and $\{g(x_n^*)\}$, $\{g(y_n^*)\}$, $\{g(z_n^*)\}$ as follows,

$$\begin{aligned} g(x_{n+1})&=F(x_n,y_n,z_n),\ g(y_{n+1})=F(y_n,x_n,y_n),\ and\ g(z_{n+1})=F(z_n,y_n,x_n)\\ g(x_{n+1}^*)&=F(x_n^*,y_n^*,z_n^*),\ g(y_{n+1}^*)=F(y_n^*,x_n^*,y_n^*),\ and\ g(z_{n+1}^*)=F(z_n^*,y_n^*,x_n^*) \end{aligned} \tag{45}$$

for all $n \succeq 0$. Since

$$\begin{aligned} (F(x,y,z),F(y,x,y),F(z,y,x))&=(g(x_1),g(y_1),g(z_1))=(g(x),g(y),g(z))\\ (F(u,v,w),F(v,u,v),F(w,v,u))&=(g(u_1),g(v_1),g(w_1))=(g(u),g(v),g(w)) \end{aligned} \tag{46}$$

are comparable each other,so $g(x) \prec g(u_1)$, $g(y) \succeq g(v_1)$ and $g(z) \preceq g(w_1)$. It is easy to show that $(g(x), g(y), g(z))$ and $(g(u_n), g(v_n), g(w_n))$ are comparable each other, that is $g(x) \prec g(u_n)$, $g(y) \succeq g(v_n)$ and $g(z) \preceq g(w_n)$ for all $n \succeq 1$. Thus it follows from Lemma 2.1, that

$$\begin{aligned} &d(g(x),g(u_{n+1}))+d(g(y),g(v_{n+1}))+d(g(z),g(w_{n+1}))\\ &=d(F(x,y,z),F(u_n,v_n,w_n))+d(F(y,x,y),F(v_n,u_n,v_n))\\ &+d(F(z,y,x),F(w_n,v_n,u_n))\\ &\prec\frac{1}{3}[d(g(x),g(u_n))+d(g(y),g(v_n))+d(g(z),g(w_n))]\\ &+\frac{1}{3}[d(g(y),g(v_n))+d(g(x),g(u_n))+d(g(y),g(v_n))]\\ &+\frac{1}{3}[d(g(z),g(w_n))+d(g(x),g(u_n))+d(g(y),g(v_n))] \end{aligned} \tag{47}$$

and so

$$\begin{aligned} &\frac{1}{3}[d(g(x),g(u_{n+1}))+d(g(y),g(v_{n+1}))+d(g(z),g(w_{n+1}))]\\ &\prec\frac{1}{3^n}[d(g(x),g(u_1))+d(g(y),g(v_1))+d(g(z),g(w_1))]\to 0 \end{aligned} \tag{48}$$

as $n \to \infty$. Therefore, we have

$$\begin{aligned} &\lim_{n\to\infty} d(g(x),g(u_{n+1}))=0,\\ &\lim_{n\to\infty} d(g(y),g(v_{n+1}))=0,\\ &\lim_{n\to\infty} d(g(z),g(w_{n+1}))=0. \end{aligned} \tag{49}$$

Similarly, we can prove that

$$\begin{aligned} &\lim_{n\to\infty} d(g(x^*),g(u_{n+1}))=0,\\ &\lim_{n\to\infty} d(g(y^*),g(v_{n+1}))=0,\\ &\lim_{n\to\infty} d(g(z^*),g(w_{n+1}))=0. \end{aligned} \tag{50}$$

Thus by the triangle inequality, (49) and (50), we have

$$\begin{aligned} d(g(x),g(x^*))&\preceq d(g(x),g(u_{n+1}))+d(g(x^*),g(u_{n+1}))\to 0,\\ d(g(y),g(y^*))&\preceq d(g(y),g(v_{n+1}))+d(g(y^*),g(v_{n+1}))\to 0,\\ d(g(z),g(z^*))&\preceq d(g(z),g(w_{n+1}))+d(g(z^*),g(w_{n+1}))\to 0 \end{aligned} \tag{51}$$

as $n \to \infty$, which imply that $g(x) = g(x^*)$, $g(y) = g(y^*)$ and $g(z) = g(z^*)$.

Now, we prove that $g(x) = x$, $g(y) = y$ and $g(z) = z$.

Denote that $g(x) = \theta_1$, $g(y) = \theta_2$ and $g(z) = \theta_3$. Since $g(x) = F(x, y, z)$, $(g(y) = F(y, z, x)$ and $g(z) = F(z, x, y)$, by the commutativity of F and g, we have

$$g(\theta_1) = g(g(\theta_1)) = g(F(x, y, z)) = F(gx, gy, gz) = F(\theta_1, \theta_2, \theta_3) \tag{52}$$

$$g(\theta_2) = g(g(\theta_2)) = g(F(y, x, y)) = F(gy, gx, gy) = F(\theta_2, \theta_1, \theta_2) \tag{53}$$

$$g(\theta_3) = g(g(\theta_3)) = g(F(z, y, x)) = F(gz, gy, gx) = F(\theta_3, \theta_2, \theta_1) \tag{54}$$

Thus, $(\theta_1, \theta_2, \theta_3)$ is a tripled coincidence point of F and g.

Putting $x^* = \theta_1$, $y^* = \theta_2$ and $z^* = \theta_3$) in (53), it follows from (43) that

$$\begin{aligned} \theta_1 &= g(x) = g(x^*) = g(\theta_1) \\ \theta_2 &= g(y) = g(y^*) = g(\theta_2) \\ \theta_3 &= g(z) = g(z^*) = g(\theta_3) \end{aligned} \tag{55}$$

and so, from (52) and (53)

$$\begin{aligned} \theta_1 &= g(\theta_1) = F(\theta_1, \theta_2, \theta_3) \\ \theta_2 &= g(\theta_2) = F(\theta_2, \theta_1, \theta_2) \\ \theta_3 &= g(\theta_3) = F(\theta_3, \theta_2, \theta_1). \end{aligned} \tag{56}$$

Therefore $(\theta_1, \theta_2, \theta_3)$ is a tripled common fixed point of F and g.

Finally, to prove the uniqueness of the tripled common fixed point of F and g. Assume that (μ_1, μ_2, μ_3) is another tripled common fixed point of F and g. Then by (43), we have $\mu_1 = g(\mu_1) = g(\theta_1) = \theta_1$, $\mu_2 = g(\mu_2) = g(\theta_2) = \theta_2$ and $\mu_3 = g(\mu_3) = g(\theta_3) = \theta_3$. This complete the proof. □

Corollary 2.2.
Suppose that all the hypothesis of corollary (2.1) hold and further for all $(x, y, z), (x^*, y^*, z^*) \in X^3$ then there exists $(u, v, w) \in X^3$ that is comparable with (x, y, z) and ,(x^*, y^*, z^*). Then there a unique $x \in X$ such that $x = F(x, x, x)$.

3. Application

Now we give some applications of the main results in Section 2.

Theorem 3.1.
Let $F : X^3 \to X$ and $g : X \to X$ be two given mappings. Assume that there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\psi(0) = 0$ and $\psi(t) \succ 0$ for any $t \succ 0$,

(b) ψ is non decreasing and right continuous,

(c) for any $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that, for all $x, y, z, u, v, w \in X$ with $g(x) \preceq g(u)$, $g(y) \succeq g(v)$, and $g(z) \preceq g(w)$

$$\begin{aligned} &\epsilon \preceq \psi\left(\frac{1}{3}(d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w)))\right) \prec \epsilon + \delta(\epsilon) \\ &\quad \Rightarrow \psi[d(F(x, y, z), F(u, v, w))] \prec \epsilon \end{aligned}$$

Then F is a generalized g−Meir-Keeler type contraction.

Proof. for any $\epsilon \succ 0$, it follows from (a) $\psi(\epsilon) \succ 0$ and so there exists $\alpha \succ 0$ such that, for all $u, v, w, u^*, v^*, w^* \in X$ with $g(u) \preceq g(u^*)$, $g(v) \preceq g(v^*)$ and $g(w) \preceq g(w^*)$,

$$\begin{aligned} &\epsilon \preceq \psi\left(\frac{1}{3}(d(g(u), g(u^*)) + d(g(v), g(v^*)) + d(g(w), d(w^*)))\right) \prec \epsilon + \delta(\epsilon) \\ &\quad \Rightarrow \psi[d(F(u, v, w), F(u^*, v^*, w^*))] \prec \epsilon. \end{aligned} \tag{57}$$

From the right continuity of ψ, there exists $\delta \succ 0$ such that $\psi(\epsilon + \delta) \prec \psi(\epsilon)\alpha$. For any $x, y, z, u, v, w \in X$ such that $g(x) \preceq g(u)$, $g(y) \succeq g(v)$, $g(z) \preceq g(w)$ and

$$\epsilon \preceq \frac{1}{3}\left[d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))\right] \prec \epsilon + \delta \tag{58}$$

since ψ is non decreasing function, we get the following,

$$\begin{aligned} &\psi(\epsilon) \preceq \psi\left(\frac{1}{3}(d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), d(w)))\right) \prec \psi(\epsilon + \alpha) \\ &\qquad \prec \psi(\epsilon) + \alpha. \end{aligned} \tag{59}$$

by (58), we have

$$\psi[d(F(x, y, z), F(u, v, w))] \prec \psi(\epsilon)$$

and so

$$d(F(x, y, z), F(u, v, w)) \prec \epsilon$$

Therefore, it follows that F is a generalized $g-$Meir-Keeler type contraction. This completes the proof. □

Corollary 3.1.
Let $F : X^3 \to X$ be given mapping. Assume that there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\psi(0) = 0$ and $\psi(t) \succ 0$ for any $t \succ 0$,

(b) ψ is non decreasing and right continuous,

(c) for any $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that, for all $x, y, z, u, v, w \in X$ with $x \preceq u$, $y \succeq v$, and $z \preceq w$

$$\begin{aligned} &\epsilon \preceq \psi\left(\frac{1}{3}(d(x, u) + d(y, v) + d(z, w))\right) \prec \epsilon + \delta(\epsilon) \\ &\quad \Rightarrow \psi[d(F(x, y, z), F(u, v, w))] \prec \epsilon \end{aligned}$$

Then F is a generalized Meir-Keeler type contraction.

The following result is an immediate consequence of Theorems 2.1 and Theorem 3.1.

Corollary 3.2.
Let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(x)$, g is continuous and commutative with F. Also, suppose that

(a) F has the mixed strict $g-$monotone property;

(b) for any $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that, for all $x, y, z, u, v, w \in X$ with $g(x) \preceq g(u)$, $g(y) \succeq g(v)$, and $g(z) \preceq g(w)$,

$$\begin{aligned} &\epsilon \preceq \int_0^{\frac{1}{3}(d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w)))} \psi(t)dt \prec \epsilon + \delta(\epsilon) \\ &\quad \Rightarrow \int_0^{d(F(x,y,z), F(u,v,w))} \psi(t)dt \prec \epsilon \end{aligned} \tag{60}$$

where ψ is a locally integrable function from $[0,+\infty)$ into itself satisfying the following condition,

$$\int_0^s \psi(t)dt \succ 0, \tag{61}$$

for all $s \succ 0$

(c) *there exist $x_0, y_0, z_0 \in X$ such that*

$$g(x_0) \prec F(x_0, y_0, z_0)$$
$$g(y_0) \succ F(y_0, x_0, y_0)$$
$$g(z_0) \prec F(z_0, y_0, x_0)$$

Then there exists $(x, y, z) \in X \times X \times X$ such that

$$F(x, y, z) = g(x) = x,\ \ F(y, x, y) = g(y) = y\ \ and\ \ F(z, y, x) = g(z) = z,$$

that is, F and g have a tripled coincidence in X^3. Moreover, if $g(x_0)$, $g(y_0)$ and $g(z_0)$ are comparable to each other, then F and g have a unique tripled common fixed point in X^3.

Corollary 3.3.
Let $F : X^3 \to X$ be a mapping satisfying the following conditions,

(a) *F has the mixed strict monotone property;*

(b) *for any $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that for all $x, y, z, u, v, w \in X$ with $x \preceq u$, $y \succeq v$ and $z \preceq w$,*

$$\epsilon \preceq \int_0^{\frac{1}{3}(d(x,u)+d(y,v)+d(z,w))} \psi(t)dt \prec \epsilon + \delta(\epsilon)$$
$$\Rightarrow \int_0^{d(F(x,y,z),F(u,v,w))} \psi(t)dt \prec \epsilon \tag{62}$$

where ψ is a locally integrable function from $[0,+\infty)$ into itself satisfying the following condition,

$$\int_0^s \psi(t)dt \succ 0, \tag{63}$$

for all $s \succ 0$

(c) *there exist $x_0, y_0, z_0 \in X$ such that $x_0 \prec F(x_0, y_0, z_0)$, $y_0 \succ F(y_0, x_0, y_0)$ and $z_0 \succ F(z_0, y_0, x_0)$.*

Then there exists $(x, y, z) \in X \times X \times X$ such that

$$F(x, y, z) = x,\ \ F(y, x, y) = y\ \ and\ \ F(z, y, x) = z$$

that is F has a tripled coincidence in X^3. Moreover, if x_0, y_0 and z_0 are comparable to each other, then F has a unique tripled fixed point in X^3.

Corollary 3.4.
Let $F : X^3 \to X$ and $g : X \to X$ be two mappings such that $F(X^3) \subseteq g(x)$, g is continuous and commutative with F. Suppose that

(a) *F has the mixed strict $g-$monotone property;*

(b) *for any $\epsilon \succ 0$ there exists $\delta(\epsilon) \succ 0$ such that, for all $x, y, z, u, v, w \in X$ with $g(x) \preceq g(u)$, $g(y) \succeq g(v)$, and $g(z) \preceq g(w)$,*

$$\int_0^{d(F(x,y,z),F(u,v,w))} \psi(t)dt \preceq k \int_0^{\frac{1}{3}(d(g(x),g(u))+d(g(y),g(v))+d(g(z),g(w)))} \psi(t)dt \tag{64}$$

where $k \in (0,1)$ and ψ is a locally integrable function from $[0,+\infty)$ into itself satisfying the following condition,

$$\int_0^s \psi(t)dt \succ 0, \tag{65}$$

for all $s \succ 0$,

(c) *there exist* $x_0, y_0, z_0 \in X$ *such that* $g(x_0) \prec F(x_0, y_0, z_0)$, $g(y_0) \succ F(y_0, z_0, x_0)$ *and* $g(z_0) \succ F(z_0, x_0, y_0)$.

Then there exists $(x, y, z) \in X^3$ *such that,*

$$F(x,y,z) = g(x) = x,\ \ F(y,x,y) = g(y) = y\ \ and\ \ F(z,y,x) = g(z) = z$$

that is, F *and* g *have a tripled coincidence in* X^3. *Moreover, if* $g(x_0)$, $g(y_0)$ *and* $g(z_0)$ *are comparable to each other, then* F *and* g *have a unique tripled common fixed point in* X^3.

Corollary 3.5.
Let $F : X^3 \to X$ *be a mapping satisfying the following conditions,*

(a) F *has the mixed strict monotone property;*

(b) *for any* $\epsilon \succ 0$ *there exists* $\delta(\epsilon) \succ 0$ *such that for all* $x, y, z, u, v, w \in X$ *with* $x \preceq u$, $y \succeq v$, *and* $z \preceq w$,

$$\int_0^{d(F(x,y,z),F(u,v,w))} \psi(t)dt \preceq k \int_0^{\frac{1}{3}(d(x,u)+d(y,v)+d(z,w))} \psi(t)dt \tag{66}$$

where $k \in (0,1)$ *and* ψ *is a locally integrable function from* $[0,+\infty)$ *into itself satisfying the following condition,*

$$\int_0^s \psi(t)dt \succ 0 \tag{67}$$

for all $s \succ 0$,

(c) *there exist* $x_0, y_0, z_0 \in X$ *such that* $x_0 \prec F(x_0, y_0, z_0)$, $y_0 \succ F(y_0, x_0, y_0)$ *and* $z_0 \succ F(z_0, y_0, x_0)$.

Then there exists $(x, y, z) \in X^3$ *such that*

$$F(x,y,z) = x,\ \ F(y,x,y) = y\ \ and\ \ F(z,y,x) = z$$

that is, F *has a tripled coincidence in* X^3. *Moreover if* x_0, y_0 *and* z_0 *are comparable to each other, then* F *has a unique tripled fixed point in* X^3.

Finally by using the above results, we show the existence of solutions for the following integral equation:

$$\begin{aligned}(x(t), y(t), z(t)) = \Bigg(&\int_0^T G(t,s)[f(s,x(s)+\lambda x(s)-(f(s,y(s))+\lambda y(s))]ds,\\ &\int_0^T G(t,s)[f(s,y(s)+\lambda y(s)-(f(s,z(s))+\lambda z(s))]ds,\\ &\int_0^T G(t,s)[f(s,z(s)+\lambda z(s)-(f(s,x(s))+\lambda x(s))]ds\Bigg)\end{aligned} \tag{68}$$

where $x, y, z \in C(I,R)$ where $C(I,R)$ is the set of continuous functions from I into R, $T \succ 0$, $f : I \times R \to R$ is continuous function and

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T-1}} & if\ \ 0 \preceq s \prec t \preceq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T-1}} & if\ \ 0 \preceq t \prec s \preceq T \end{cases}$$

Definition 3.1.
A lower solution for the integral type Eq. (68) is an element $(\alpha, \beta, \gamma) \in \left(C^1(I,R)\right)^3$ such that

$$\begin{aligned}\alpha'(t) + \lambda\beta(t) + \lambda\gamma(t) \preceq f(t,\alpha(t)) - f(t,\beta(t)) - f(t,\gamma(t)),\ \ \alpha(0) \prec \alpha(T),\\ \beta'(t) + \lambda\gamma(t) + \lambda\alpha(t) \preceq f(t,\beta(t)) - f(t,\gamma(t)) - f(t,\alpha(t)),\ \ \beta(0) \succeq \beta(T),\\ \gamma'(t) + \lambda\alpha(t) + \lambda\beta(t) \preceq f(t,\gamma(t)) - f(t,\alpha(t)) - f(t,\beta(t)),\ \ \gamma(0) \preceq \gamma(T),\end{aligned} \tag{69}$$

where $C^1(I,R)$ denotes the set of differentiable functions from I to R.

Next we prove the existence of solution for the integral Eq. (68).

Theorem 3.2.
Let Ψ be the class of the functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) ψ is increasing,

(b) for any $x \succeq 0$, there exists $k \in [0,1)$ such that $\psi(x) \prec (k/3)x$.

In the integral Eq. (68) suppose that there exists $\lambda \succ 0$ such that for all $x, y \in R$ with $y \succ x$

$$0 \prec f(t,y) + \lambda y - [f(t,x) + \lambda(x)] \preceq \lambda\psi(y-x), \tag{70}$$

where $\psi \in \Psi$. If a lower solution of the integral Eq. (68) exists then the solution of integral Eq. (68) exists.

Proof. Define a mapping $F : (C(I,R))^3 \to C(I,R)$ by

$$F(x(t), y(t), z(t)) = \int_0^T G(t,s)[f(s, x(s) + \lambda x(s) \\ -(f(s,y(s)) + \lambda y(s)) - (f(s,z(s)) + \lambda z(s))]ds, \tag{71}$$

Note that, if $(x(t), y(t), z(t)) \in (C(I,R))^3$ is tripled fixed point of F, then $(x(t), y(t), z(t))$ is the solution of integral Eq. (68).

Now, we check the hypothesis in Corollary 3.4 as follows:

1. $X^3 = (C(I,R))^3$ is a partially ordered set if we define the order relation in X^3 as follows;

$$(u(t), v(t), w(t)) \preceq (x(t), y(t), z(t)) \tag{72}$$

iff $u(t) \prec x(t)$, $v(t) \succeq y(t)$ $w(t) \prec z(t)$, for all $(u(t), v(t), w(t)), (x(t), y(t), z(t)) \in X^3$ and $t \in I$.

2. (X,d) is a complete metric space if we define a metric d as follows;

$$d(x(t), y(t)) = \sup_{t\in I}\{|x(t) - y(t)| : x(t), y(t) \in X\}. \tag{73}$$

3. The mapping F has the mixed strict monotone property. In fact by hypothesis, if $x_2 \succ x_1$, then we have

$$f(t,x_2) + \lambda x_2 \succ f(t,x_1) + \lambda x_1 \tag{74}$$

which implies that for any $t \in I$,

$$F(x_2(t), y(t), z(t)) = \int_0^T G(t,s)[f(s, x_2(s)) + \lambda x_2(s) \\ -(f(s,y(s)) + \lambda y(s)) - (f(s,z(s)) + \lambda z(s))]ds$$

and

$$F(x_1(t), y(t), z(t)) = \int_0^T G(t,s)[f(s, x_1(s)) + \lambda x_1(s) \\ -(f(s,y(s)) + \lambda y(s)) - (f(s,z(s)) + \lambda z(s))]ds,$$

that is,

$$F(x_2(t), y(t), z(t)) \succ F(x_1(t), y(t), z(t)). \tag{75}$$

Similarly if $y_1 \succ y_2$, then we have

$$f(t,y_2) + \lambda y_2 \succ f(t,y_1) + \lambda y_1 \tag{76}$$

which implies that for any $t \in I$,

$$F(x(t), y_2(t), z(t)) = \int_0^T G(t,s)[f(s, x(s)) + \lambda x(s) \\ -(f(s,y_2(s)) + \lambda y_2(s)) - (f(s,z(s)) + \lambda z(s))]ds$$

and

$$F(x(t), y_1(t), z(t)) = \int_0^T G(t,s)[f(s,x(s)) + \lambda x(s) \\ -(f(s,y_1(s)) + \lambda y_1(s)) - (f(s,z(s)) + \lambda z(s))]ds.$$

that is

$$F(x(t), y_2(t), z(t)) \prec F(x(t), y_1(t), z(t)) \tag{77}$$

for any $t \in I$.

Also if $z_1 \prec z_2$, then we have

$$f(t,z_2) + \lambda z_2 \succ f(t,z_1) + \lambda z_1 \tag{78}$$

$$F(x(t), y(t), z_2(t)) = \int_0^T G(t,s)[f(s,x(s)) + \lambda x(s) \\ -(f(s,y(s)) + \lambda y(s)) - (f(s,z_2(s)) + \lambda z_2(s))]ds$$

and

$$F(x(t), y(t), z_1(t)) = \int_0^T G(t,s)[f(s,x(s)) + \lambda x(s) \\ -(f(s,y(s)) + \lambda y(s)) - (f(s,z_1(s)) + \lambda z_1(s))]ds$$

that is

$$F(x(t), y(t), z_2(t)) \succ F(x(t), y(t), z_1(t)) \tag{79}$$

In fact, let $(x, y, z) \preceq (u, v, w)$ and $t \in I$ then we have

$$\begin{aligned}
& d(F(x(t), y(t), z(t)), F(u(t), v(t), u(t))) \\
& = \sup\{| F(x(t), y(t), z(t)) - F(u(t), v(t), u(t)) |: t \in I\} \\
& = \sup_{t\in I}\{| \int_0^T G(t,s)[(f(s,x(s)) + \lambda x(s)) - (f(s,y(s)) + \lambda y(s)) \\
& -(f(s,z(s)) + \lambda z(s))]ds \\
& - \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s) - (f(s,v(s)) + \lambda v(s)) \\
& -(f(s,w(s)) + \lambda w(s))]ds |: t \in I\} \\
& \preceq \sup_{t\in I}\{| \int_0^T G(t,s)\big[(f(s,x(s)) + \lambda x(s)) - (f(s,y(s)) + \lambda y(s)) \\
& -(f(s,z(s)) + \lambda z(s)) - (f(s,u(s)) + \lambda u(s)) \\
& -(f(s,v(s)) + \lambda v(s)) - (f(s,w(s)) + \lambda w(s))\big]ds.
\end{aligned} \tag{80}$$

Since the function $\psi(x)$ is non decreasing and $(x, y, z) \preceq (u, v, w)$, we have

$$\begin{aligned}
\psi(x(s) - u(s)) \preceq \psi(d(x(s), u(s))) \\
\psi(y(s) - v(s)) \preceq \psi(d(y(s), v(s))) \\
\psi(z(s) - w(s)) \preceq \psi(d(z(s), w(s)))
\end{aligned} \tag{81}$$

$$
\begin{aligned}
& d(F(x(t),y(t),z(t)),F(u(t),v(t),u(t))) \\
& \preceq \sup_{t\in I}\int_0^T G(t,s)\,|\,\lambda\psi(x(s)-u(s))+\lambda\psi(y(s)-v(s))+\lambda\psi(z(s)-w(s))\,|\,ds \\
& \preceq \lambda\sup_{t\in I}\int_0^T G(t,s)\,|\,\psi(d(x(s),u(s)))+\psi(d(y(s),v(s)))+\psi(d(z(s),w(s)))\,|\,ds \\
& = \lambda(\psi(d(x(s),u(s)))+\psi(d(y(s),v(s)))+\psi(d(z(s),w(s))))\sup_{t\in I}\int_0^T G(t,s)ds \\
& = \lambda(\psi(d(x(s),u(s)))+\psi(d(y(s),v(s))) \\
& +\psi(d(z(s),w(s))))\sup_{t\in I}\frac{1}{e^{\lambda T}-1}\left[\frac{1}{\lambda}e^{\lambda(T+s-t)}\right]_0^t+\left[\frac{1}{\lambda}e^{\lambda(s-t)}\right]_t^T \\
& = \lambda(\psi(d(x(s),u(s)))+\psi(d(y(s),v(s)))+\psi(d(z(s),w(s)))) \\
& \prec \frac{k}{3}(d(x(s),u(s))+d(y(s),v(s))+d(z(s),w(s))) \\
& \preceq \frac{k}{3}\sup\{|\,x(t)-u(t)\,|: t\in I\}\ +\ \frac{k}{3}\sup\{|\,y(t)-v(t)\,|: t\in I\} \\
& +\frac{k}{3}\sup\{|\,z(t)-w(t)\,|: t\in I\} \\
& = \frac{k}{3}(d(x(t),u(t))+d(y(t),v(t))+d(z(t),w(t)))
\end{aligned} \tag{82}
$$

Then by the Lemma 6, F is generalized Meir-Keeler type contraction.
Finally, let $(\alpha(t),\beta(t),\gamma(t))\in(C^1(I,R))^3$ be a lower solution for the integral Eq. (68) then we show that

$$
\alpha\prec F(\alpha,\beta,\gamma),\ \ \beta\succeq F(\beta,\alpha,\beta),\ \ \gamma\preceq F(\gamma,\beta,\alpha) \tag{83}
$$

Indeed, we have

$$
\alpha'(t)+\lambda\beta(t)+\lambda\gamma(t)\preceq f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t))
$$

for any $t\in I$ and so

$$
\alpha'(t)+\lambda\alpha(t)\preceq f(t,\alpha(t))-f(t,\beta(t))-f(t,\gamma(t))+\lambda\alpha(t)-\lambda\beta(t)-\lambda\gamma(t) \tag{84}
$$

for any $t\in I$.
Multiplying by $e^{\lambda t}$ in (84), we get the following:

$$
\begin{aligned}
\big((\alpha(t)+\lambda\alpha(t))e^{\lambda t}\big)' &\preceq [(f(t,\alpha(t))+\lambda\alpha(t)) \\
&\quad -(f(t,\beta(t))-\lambda\beta(t))-(f(t,\gamma(t))-\lambda\gamma(t))]e^{\lambda t}
\end{aligned} \tag{85}
$$

for any $t\in I$, which implies that

$$
\begin{aligned}
\alpha(t)e^{\lambda t} &\preceq \alpha(0)+\int_0^t[(f(s,\alpha(s))+\lambda\alpha(s)) \\
&\quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]e^{\lambda s}ds
\end{aligned} \tag{86}
$$

for any $t\in I$, this implies that

$$
\begin{aligned}
& \alpha(0)e^{\lambda t}\prec\alpha(T)e^{\lambda T} \\
& \quad\preceq \alpha(0)+\int_0^T[(f(s,\alpha(s))+\lambda\alpha(s))-(f(s,\beta(s))-\lambda\beta(s)) \\
& \quad -(f(s,\gamma(s))-\lambda\gamma(s))]e^{\lambda s}ds
\end{aligned} \tag{87}
$$

and so

$$
\begin{aligned}
& \alpha(0)\prec\int_0^T\frac{e^{\lambda s}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\
& \quad -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds
\end{aligned} \tag{88}
$$

Thus it follows from (86) and (88) that

$$\alpha(t)e^{\lambda t} \prec \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds \\ +\int_0^t \frac{e^{\lambda(T-s)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds \tag{89}$$

and so

$$\alpha(t) \prec \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds \\ +\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[(f(s,\alpha(s))+\lambda\alpha(s)) \\ -(f(s,\beta(s))-\lambda\beta(s))-(f(s,\gamma(s))-\lambda\gamma(s))]ds \tag{90}$$

$$\alpha(t) \prec \int_0^T G(t,s)[f(s,\alpha(s)+\lambda\alpha(s) \\ -(f(s,\beta(s))+\lambda\beta(s))-(f(s,\gamma(s))+\lambda\gamma(s))]ds \\ = F(\alpha(t),\beta(t),\gamma(t)) \tag{91}$$

for any $t \in I$.
Similarly, we have $\beta(t) \succeq F(\beta(t),\alpha(t),\beta(t))$ and $\gamma(t) \preceq F(\gamma(t),\beta(t),\alpha(t))$. Therefore by Corollary 2.1, F has a tripled fixed point. □

4. Acknowledgement

The authors would like to express their sincere thanks to the editor and the anonymous referees for their valuable comments and useful suggestions in improving the article.

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