Int. J. Adv. Appl. Math. and Mech. 2(2) (2014) 39 - 51 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Fixed point for generalized contractions on G- metric spaces

Research Article

Animesh Gupta *

H.No. 93/654, Ward No. 2 Gandhi Chowk, Pachmarhi-461881, Dist. Hoshangabad (M.P.), India.
Visiting at : Department of Mathematics & Computer Science, Govt. Model Science College, Jabalpur (M.P.) India.

Received 21 August 2014; accepted (in revised version) 19 December 2014

Abstract: We present fixed point results for generalized homotopy contractions on spaces with two G-metric. The focus is on continuation results for such type of mappings.

MSC: 54H25 • 46J10

Keywords: G- metric space • Non-self mappings • Generalized contraction • Fixed ooint

1. Introduction and preliminaries

The concept of G-metric space is given by Mustafa and Sims [1]. After the publication of their work there are many other works have appeared in this area some of them are [2–8] and their in.
The object of this paper is to establish some results for generalized contractions on spaces with two G-metric. The focus is on continuation results for such type of mappings. Our results generalize and improve many existing results in the literature.

Definition 1.1.

Let X be a non empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if x = y = z

(G_2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or more specially, a $G-$metric on X and the pair (X,G) is called a $G-$metric space.

Definition 1.2.

Let (X,G) be a $G-$metric space, and let $\{x_n\}$ be a sequence of points of X therefore, we say that $\{x_n\}$ is $G-$convergent to $x \in X$ if $\lim_{n,m\to+\infty} = G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists $n \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

* Corresponding author.
E-mail address: dranimeshgupta10@gmail.com

Lemma 1.1.
Let (X,G) be a G- metric space. The following statements are equivalent:

1. $\{x_n\}$ *is G- convergent to x,*
2. $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,
3. $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,
4. $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$,

Definition 1.3.
Let (X,G) be a G- metric space. A sequence $\{x_n\}$ is called a G- Cauchy sequence if, for any $\epsilon > 0$, there $n \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Lemma 1.2.
Let (X,G) be a G- metric space. The following statements are equivalent:

1. *The sequence* $\{x_n\}$ *is G- Cauchy,*
2. *for any* $\epsilon > 0$, *there exists* $n \in N$ *such that* $G(x_n, x_m, x_m) < \epsilon$ *for all* $m, n \geq N$.

Definition 1.4.
A G- metric space (X,G) is called G- complete if every G- Cauchy sequence is G- convergent in(X,G).

Every G- metric on X defines a metric d_G on X given by

$$d_G = G(x, y, y) + G(y, x, x) \quad for \ all \ \ x, y \in X. \tag{1}$$

Each G-metric G on X generates a topology τ_G on X which has as a base the family of open G-balls

$$\{B(x, r), x \in X, r > 0\},$$

where

$$B(x, r) = \{y \in X, G(x, y, y) < r\}$$

for all $x \in X$ and $r > 0$. Also, a nonempty subset A in the G-metric space (X, G) is G-closed if $\overline{A} = A$. Note that

$$x \in \overline{A} \Longleftrightarrow B(x, r) \cap A \neq 0,$$

for all $r > 0$.
By (1) it is easy to see that

Lemma 1.3.
Let (X, G) *be a* G*-metric space and* A *is a nonempty subset of* X. A *is said* G*-closed if for any sequence* $\{x_n\}$ *in* A *such that* $x_n \to x$ *as* $n \to \infty$, *then* $x \in A$.

Lemma 1.4.
If X is a G - metric space, then $G(x, y, y) = 2G(y, x, x)$ *for all* $x, y \in X$.

Lemma 1.5.
If X is a G- metric space, then $G(x, x, y) = G(x, x, z) + G(z, z, y)$ *for all* $x, y, z \in X$.

Throughout this article (X, G') will be a complete G-metric space and G another metric on X. If $x_0 \in X$ and $r > 0$ denote by

$$B(x_0, r) = \{x \in X : G(x_0, x, x) < r\}$$

and by $\overline{B(x_0, r)}^{G'}$ the G-closure of $B(x_0, r)$.

2. Fixed Point Results for Generalized Contraction on G - Metric Space

Theorem 2.1.
Let (X, G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0, r)^{G'}} \to X$. Suppose for any $x, y, z \in \overline{B(x_0, r)^{G'}}$ we have

$$G(Fx, Fy, Fz) \leq \alpha G(x, y, z), \tag{2}$$

where $0 \leq \alpha < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0, Fx_0, Fx_0) < (1-\alpha)r, \tag{3}$$

(2) if $G < G'$ then F is uniformly continuous form $(B(x_0, r), G)$ into (X, G'),

(3) if $G \neq G'$ then F is continuous from $(\overline{B(x_0, r)^{G'}}, G')$ into (X, G').

Then F has a fixed point, that is there exists $x \in \overline{B(x_0, r)^{G'}}$ with $F(x) = x$.

Proof. Let $x_1 = Fx_0$. From (3), since $\alpha < 1$, we have

$$G(x_0, x_1, x_1) < (1-\alpha)r < r$$

so $x_1 \in B(x_0, r)$.
Next let $x_2 = Fx_1$ and note that

$$\begin{aligned} G(x_1, x_2, x_2) &= G(Fx_0, Fx_1, Fx_1) \\ &\leq \alpha G(x_0, x_1, x_1) \\ G(x_1, x_2, x_2) &\leq \alpha(1-\alpha)r. \end{aligned}$$

Thus

$$\begin{aligned} G(x_0, x_2, x_2) &\leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\ &\leq (1-\alpha)r + \alpha(1-\alpha)r \\ &\leq (1-\alpha)r[1+\alpha] \\ &< (1-\alpha)r[1+\alpha+\alpha^2+\alpha^3+...] \\ &< (1-\alpha)r(1-\alpha)^{-1} \\ G(x_0, x_2, x_2) &< r. \end{aligned}$$

That is $x_2 \in B(x_0, r)$. Proceeding inductively we obtain

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq \alpha G(x_{n-1}, x_n, x_n) \\ &\leq ... \leq \alpha^n G(x_0, x_1, x_1) \\ &< \alpha^n (1-\alpha)r \end{aligned}$$

where $x_n = Fx_{n-1}$ for $n = 3, 4, 5....$ Since $\alpha \in [0, 1)$ it follows that $\alpha^n \in [0, 1)$ and thus

$$G(x_n, x_{n+1}, x_{n+1}) \leq (1-\alpha)r. \tag{4}$$

The lase inequality implies $x_{n+1} \in B(x_0, r)$ and the sequence $\{x_n\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.
If $G \geq G'$ this is trivial. Next suppose $G < G'$. Let $\epsilon > 0$ be given. Now from assumption-2 guarantees that there exists $\delta > 0$ such that

$$G'(Fx, Fy, Fy) < \epsilon \tag{5}$$

where $x, y \in B(x_0, r)$ and $G(x, y, y) < \delta$.
From above the sequence $\{x_n\}$ is a Cauchy sequence with respect to G, so we know that there exists N with

$$G(x_n, x_m, x_m) < \delta \tag{6}$$

for all $m, n \geq N$.
Now from (5) and (6) imply

$$G'(x_{n+1}, x_{m+1}, x_{m+1}) = G'(Fx_n, Fx_m, Fx_m) < \epsilon \tag{7}$$

whenever $m, n \geq N$ which proves that $\{x_n\}$ is a Cauchy sequence with respect to G'. Now since (X, G') is complete there exists $x \in \overline{B(x_0, r)^{G'}}$ with $G'(x_n, x, x) \to 0$ as $n \to \infty$. We claim that

$$x = Fx.$$

First consider the case $G \neq G'$. Notice

$$G'(x, Fx, Fx) \leq G'(x, x_n, x_n) + G'(x_n, Fx, Fx) \tag{8}$$

Let $n \to \infty$ and using assumption-3 we have $G(x, Fx, Fx) = 0$ and thus $x = Fx$ in this case. Next suppose that $G = G'$ that is assumption-2 and assumption-3 are not hold. Then

$$\begin{aligned} G(x, Fx, Fx) &\leq G(x, x_n, x_n) + G(x_n, Fx, Fx) \\ &\leq G(x, x_n, x_n) + G(Fx_{n-1}, Fx, Fx) \\ &\leq G(x, x_n, x_n) + \alpha G(x_{n-1}, x, x) \end{aligned}$$

as $n \to \infty$ we obtain $G(x, Fx, Fx) = 0$ and that $x = Fx$. Thus the proof of the theorem is complete. □

Next we present an homotopy result for this type of generalized contractions.

Theorem 2.2.
Let (X, G') be a complete G-metric space and G another metric on X. Let $Q \subset X$ be G'-closed and let $U \subset X$ be G-open and $U \subset Q$. Suppose $H : Q \times [0,1] \to X$ satisfies the following five properties:

(i) $x \neq H(x, \lambda)$ for $x \in Q \setminus U$ and $\lambda \in [0,1]$,

(ii) for any $\lambda \in [0,1]$ and $x, y \in Q$ we have

$$G(H(x,\lambda), H(y,\lambda), H(z,\lambda)) \leq \alpha G(x, y, z) \tag{9}$$

with $0 < \alpha < 1$,

(iii) $H(x, \lambda)$ is continuous in λ with respect to G, uniformly for $x \in Q$,

(iv) if $G < G'$ assume H is uniformly continuous form $U \times [0,1]$ endowed with the G-metric G on U into (X, G'),

(v) if $G \neq G'$ assume H is continuous from $Q \times [0,1]$ endowed with the G-metric G' on Q into (X, G').

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0,1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(., \lambda)$).

Proof. Let

$$A := \{\lambda \in [0,1];\ \text{there exists}\ x \in U\ \text{such that}\ H(x,\lambda) = x\}.$$

Since (H_0 has a fixed point and (i) holds we have $0 \in A$, and so the set A is homotopy. We will show A is open and closed in$[0,1]$ and so by the connectedness of $[0,1]$ we have $A = [0,1]$ and the proof is finished
First we show that A is closed in $[0,1]$.
Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of A for each k, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned} G(x_k, x_j, x_j) &= G(H(x_k,\lambda_k), H(x_j,\lambda_j), H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)) + G(H(x_k,\lambda), H(x_j,\lambda_j), H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)) + \alpha G(x_k, x_j, x_j) \\ &\leq (1-\alpha)^{-1} G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)). \end{aligned}$$

and (iii) guarantees that $\{x_k\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.
If $G < G'$ this is trivial. If $G < G'$ then

$$G'(x_k, x_j, x_j) = G'(H(x_k,\lambda_k), H(x_j,\lambda_j), H(x_j,\lambda_j))$$

and (iv) guarantees that $\{x_n\}$ is a Cauchy sequence with respect to G'. (note as well that $\{x_k\}$ is a Cauchy sequence with respect to G and $\{\lambda_k\}$ is Cauchy sequence in $[0,1]$). Now since (X, G') is complete there exists an $x \in Q$ such that $G'(x_k, x, x) \to 0$ as $n \to \infty$. Claim now that $x = H(x,\lambda)$.
We consider first case $G \neq G'$. Then

$$\begin{aligned} G'(x, H(x,\lambda), H(x,\lambda)) &\leq G'(x, x_k, x_k) + G'(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G'(x, x_k, x_k) + G'(H(x_k,\lambda), H(x,\lambda), H(x,\lambda)) \end{aligned}$$

together with (v), letting $k \to \infty$, we have

$$G'(x, H(x,\lambda), H(x,\lambda)) = 0$$

and so

$$x = H(x,\lambda).$$

We consider now the case $G = G'$. Then

$$\begin{aligned} G(x, H(x,\lambda), H(x,\lambda)) &\leq G(x, x_k, x_k) + G(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G(x, x_k, x_k) + G(H(x_k,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ &\leq G(x, x_k, x_k) + G(H(x_k,\lambda_k), H(x,\lambda_k), H(x,\lambda_k)) \\ &\quad + G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ &\leq G(x, x_k, x_k) + \alpha G(x_k, x, x) + G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)). \end{aligned}$$

Letting $k \to 0$ and using (iii) we have

$$G(x, H(x,\lambda), H(x,\lambda)) \leq 0.$$

So we have

$$G(x, H(x,\lambda), H(x,\lambda)) = 0,$$

that is $x = H(x,\lambda)$. Now form (i) we have $x \in U$. Consequently $\lambda \in A$ and so A is closed in [0,1].
We prove now A is open in [0,1].
Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0,\lambda_0)$. From U G-open there exists a G-ball

$$B(x_0\delta) = \{x \in X; G(x, x_0, x_0) < \delta\}, \delta > 0$$

and $B(x_0,\delta) \subset U$. From (iii) we have that H is uniformly continuous on $B(x_0,\delta)$.
Let

$$\epsilon = (1-\alpha)\delta > 0$$

and using the uniform continuity of H we have there exists $\eta = \eta(\delta) > 0$ such that for each $\delta \in [0,1], |\delta - \delta_0| \leq \eta$ with

$$G(H(x,\lambda), H(x,\lambda_0), H(x,\lambda_0) < \epsilon$$

for any $x \in B(x_0,\delta)$. So this property holds for $x = x_0$, and then we have

$$G(x_0, H(x_0,\lambda), H(x_0,\lambda)) = G(H(x_0,\lambda), H(x_0,\lambda), H(x_0,\lambda)) < (1-\alpha)\delta$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| \leq \eta$.
Using (ii), (iv) and (v) together with the Theorem 2.1 (in the case $r = \delta$ and $F = H_\lambda$) we get; there exists $x_\lambda \in \overline{B(x_0,\delta)}^{G'} \subset Q$ with $x_\lambda = H_\lambda(x\lambda)$ for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| \leq \eta$. But $x_\lambda \in U$ ((i) guarantees that) and so A contains all $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$. Consequently A is open in [0,1]. □

The following global result can be easy obtained from the above Theorem 2.1

Theorem 2.3.
Let (X, G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : X \to X$. Suppose for any $x, y, z \in X$ we have

$$G(Fx, Fy, Fz) \leq \alpha G(x, y, z), \tag{10}$$

where $0 \leq \alpha < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0, Fx_0, Fx_0) < (1-\alpha)r, \tag{11}$$

(2) *if $G < G'$ then F is uniformly continuous form (X, G') into (X, G'),*

(3) *if $G \neq G'$ then F is continuous from (X, G') into (X, G').*

Then F has a fixed point.

3. Fixed Point for Reich-Rus Type Generalized Contractions

Theorem 3.1.
Let (X, G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0, r)^{G'}} \to X$. Suppose for any $x, y, z \in \overline{B(x_0, r)^{G'}}$ we have

$$\begin{aligned} G(Fx, Fy, Fz) &\leq \alpha_1 G(x, y, z) + \alpha_2 G(x, Fx, Fx) \\ &\quad + \alpha_3 G(y, Fy, Fy) + \alpha_4 G(z, Fz, Fz), \end{aligned} \tag{12}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non negative numbers with $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0, Fx_0, Fx_0) < \left(1 - \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) r, \tag{13}$$

(2) *if $G < G'$ then F is uniformly continuous form $(B(x_0, r), G)$ into (X, G'),*

(3) *if $G \neq G'$ then F is continuous from $(\overline{B(x_0, r)^{G'}}, G')$ into (X, G').*

Then F has a fixed point, that is there exists $x \in \overline{B(x_0, r)^{G'}}$ with $F(x) = x$.

Proof. Let $x_1 = Fx_0$. From (13), since $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1$, we have

$$G(x_0, x_1, x_1) < \left(1 - \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) r < r$$

so $x_1 \in B(x_0, r)$.
Next let $x_2 = Fx_1$ and note that

$$\begin{aligned} G(x_1, x_2, x_2) &= G(Fx_0, Fx_1, Fx_1) \\ &\leq \alpha_1 G(x_0, x_1, x_1) + \alpha_2 G(x_0, Fx_0, Fx_0) \\ &\quad + \alpha_3 G(x_1, Fx_1, Fx_1) + \alpha_4 G(x_1, Fx_1, Fx_1) \\ &\leq \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) G(x_0, x_1, x_1) \\ G(x_1, x_2, x_2) &\leq \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right)\left(1 - \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) r. \end{aligned}$$

Let $0 \leq \beta = \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) < 1$.

Thus

$$\begin{aligned} G(x_0,x_2,x_2) &\leq G(x_0,x_1,x_1)+G(x_1,x_2,x_2) \\ &\leq (1-\beta)r+\beta(1-\beta)r \\ &\leq (1-\beta)r[1+\beta] \\ &< (1-\beta)r[1+\beta+\beta^2+\beta^3+...] \\ &< (1-\beta)r(1-\beta)^{-1} \\ G(x_0,x_2,x_2) &< r. \end{aligned}$$

That is $x_2 \in B(x_0,r)$. Proceeding inductively we obtain

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &= G(Fx_{n-1},Fx_n,Fx_n) \\ &\leq \beta G(x_{n-1},x_n,x_n) \\ &\leq ... \leq \beta^n G(x_0,x_1,x_1) \\ &< \beta^n(1-\beta)r \end{aligned}$$

where $x_n = Fx_{n-1}$ for $n = 3,4,5....$ Since $\beta \in [0,1)$ it follows that $\beta^n \in [0,1)$ and thus

$$G(x_n,x_{n+1},x_{n+1}) \leq (1-\beta)r. \tag{14}$$

The lase inequality implies $x_{n+1} \in B(x_0,r)$ and the sequence $\{x_n\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.
If $G \geq G'$ this is trivial. Next suppose $G < G'$. Let $\epsilon > 0$ be given. Now from assumption - 2 guarantees that there exists $\delta > 0$ such that

$$G'(Fx,Fy,Fy) < \epsilon \tag{15}$$

where $x,y \in B(x_0,r)$ and $G(x,y,y) < \delta$.
From above the sequence $\{x_n\}$ is a Cauchy sequence with respect to G, so we know that there exists N with

$$G(x_n,x_m,x_m) < \delta \tag{16}$$

for all $m,n \geq N$.
Now from (15) and (16) imply

$$G'(x_{n+1},x_{m+1},x_{m+1}) = G'(Fx_n,Fx_m,Fx_m) < \epsilon \tag{17}$$

whenever $m,n \geq N$ which proves that $\{x_n\}$ is a Cauchy sequence with respect to G'. Now since (X,G') is complete there exists $x \in \overline{B(x_0,r)}^{G'}$ with $G'(x_n,x,x) \to 0$ as $n \to \infty$. We claim tha

$$x = Fx.$$

First consider the case $G \neq G'$. Notice

$$G'(x,Fx,Fx) \leq G'(x,x_n,x_n)+G'(x_n,Fx,Fx) \tag{18}$$

Let $n \to \infty$ and using assumption-3 we have $G(x,Fx,Fx) = 0$, and thus $x = Fx$ in this case. Next suppose that $G = G'$ that is assumption-2 and assumption-3 are not hold. Then

$$\begin{aligned} G(x,Fx,Fx) &\leq G(x,x_n,x_n)+G(x_n,Fx,Fx) \\ &\leq G(x,x_n,x_n)+G(Fx_{n-1},Fx,Fx) \\ &\leq G(x,x_n,x_n)+\beta G(x_{n-1},x,x) \end{aligned}$$

as $n \to \infty$ we obtain $G(x,Fx,Fx) = 0$ and that $x = Fx$. Thus the proof of the theorem is complete. □

Now we present an homotopy result for this type of generalized contractions.

Theorem 3.2.
Let (X,G') be a complete G-metric space and G another metric on X. Let $Q \subset X$ be G'-closed and let $U \subset X$ be G-open and $U \subset Q$. Suppose $H : Q \times [0,1] \to X$ satisfies the following five properties:

(i) $x \neq H(x,\lambda)$ *for* $x \in Q \setminus U$ *and* $\lambda \in [0,1]$,

(ii) for any $\lambda \in [0,1]$ *and* $x, y \in Q$ *we have*

$$\begin{aligned} G(H(x,\lambda), H(y,\lambda), H(z,\lambda)) &\leq \alpha_1 G(x,y,z) \\ &+\alpha_2 G(x, H(x,\lambda), H(x,\lambda)) \\ &+\alpha_3 G(y, H(y,\lambda), H(y,\lambda)) \\ &+\alpha_4 G(z, H(z,\lambda), H(z,\lambda)) \end{aligned} \tag{19}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *are non negative numbers with* $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1$,

(iii) $H(x,\lambda)$ *is continuous in* λ *with respect to* G*, uniformly for* $x \in Q$,

(iv) if $G < G'$ *assume* H *is uniformly continuous form* $U \times [0,1]$ *endowed with the* G*-metric* G *on* U *into* (X, G'),

(v) if $G \neq G'$ *assume* H *is continuous from* $Q \times [0,1]$ *endowed with the* G*-metric* G' *on* Q *into* (X, G').

In addition assume H_0 *has a fixed point. Then for each* $\lambda \in [0,1]$ *we have that* H_λ *has a fixed point* $x_\lambda \in U$ *(here* $H_\lambda(.) = H(.,\lambda)$*).*

Proof. Let

$A := \{\lambda \in [0,1];$ *there exists* $x \in U$ *such that* $H(x,\lambda) = x\}$.

Since (H_0 has a fixed point and (i) holds we have $0 \in A$, and so the set A is homotopy. We will show A is open and closed in$[0,1]$ and so by the connectedness of $[0,1]$ we have $A = [0,1]$ and the proof is finished

First we show that A is closed in $[0,1]$.

Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of A for each k, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$\begin{aligned} G(x_k, x_j, x_j) &= G(H(x_k,\lambda_k), H(x_j,\lambda_j), H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)) \\ &\quad +G(H(x_k,\lambda), H(x_j,\lambda_j), H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)) \\ &\quad +\alpha G(x_k, x_j, x_j) \\ &\leq (1-\alpha)^{-1} G(H(x_k,\lambda_k), H(x_k,\lambda), H(x_k,\lambda)). \end{aligned}$$

and (iii) guarantees that $\{x_k\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.

If $G \geq G'$ this is trivial. If $G < G'$ then

$$G'(x_k, x_j, x_j) = G'(H(x_k,\lambda_k), H(x_j,\lambda_j), H(x_j,\lambda_j))$$

and (iv) guarantees that $\{x_n\}$ is a Cauchy sequence with respect to G'. (note as well that $\{x_k\}$ is a Cauchy sequence with respect to G and $\{\lambda_k\}$ is Cauchy sequence in$[0,1]$). Now since (X, G') is complete there exists an $x \in Q$ such that $G'(x_k, x, x) \to 0$ as $n \to \infty$. Claim now that $x = H(x,\lambda)$.

We consider first case $G \neq G'$. Then

$$\begin{aligned} G'(x, H(x,\lambda), H(x,\lambda)) &\leq G'(x, x_k, x_k) + G'(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G'(x, x_k, x_k) + G'(H(x_k,\lambda), H(x,\lambda), H(x,\lambda)) \end{aligned}$$

together with (v), letting $k \to \infty$, we have $G'(x, H(x,\lambda), H(x,\lambda)) = 0$ and so $x = H(x,\lambda)$.

We consider now the case $G = G'$. Then

$$\begin{aligned} G(x, H(x,\lambda), H(x,\lambda)) &\leq G(x, x_k, x_k) \\ &\quad +G(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G(x, x_k, x_k) \\ &\quad +G(H(x_k,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ &\leq G(x, x_k, x_k) \\ &\quad +G(H(x_k,\lambda_k), H(x,\lambda_k), H(x,\lambda_k)) \\ &\quad +G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ (1-\alpha_3-\alpha_4) G(x, H(x,\lambda), H(x,\lambda)) &\leq G(x, x_k, x_k) + \alpha_1 G(x_k, x, x) \\ &\quad +\alpha_2 G(x_k, H(x_k,\lambda_k), H(x_k,\lambda_k)) \\ &\quad +G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)). \end{aligned}$$

Letting $k \to 0$ and using (iii) we have

$$(1-\alpha_3-\alpha_4)G(x,H(x,\lambda),H(x,\lambda)) \le 0.$$

So we have $G(x,H(x,\lambda),H(x,\lambda)) = 0$, that is $x = H(x,\lambda)$. Now form (i) we have $x \in U$. Consequently $\lambda \in A$ and so A is closed in [0,1].
We prove now A is open in [0,1].
Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0,\lambda_0)$. From U G-open there exists a G-ball $B(x_0\delta) = \{x \in X; G(x,x_0,x_0) < \delta\}, \delta > 0$ and $B(x_0,\delta) \subset U$. From (iii) we have that H is uniformly continuous on $B(x_0,\delta)$.
Let $\epsilon = (1-\alpha)\delta > 0$ and using the uniform continuity of H we have there exists $\eta = \eta(\delta) > 0$ such that for each $\delta \in [0,1] \mid \delta - \delta_0 \mid \le \eta$ with $G(H(x,\lambda),H(x,\lambda_0),H(x,\lambda_0) < \epsilon$ for any $x \in B(x_0,\delta)$. So this property holds for $x = x_0$, and then we have

$$\begin{aligned} G(x_0,H(x_0,\lambda),H(x_0,\lambda)) &= G(H(x_0,\lambda),H(x_0,\lambda),H(x_0,\lambda)) \\ &< \left(1-\frac{\alpha_1+\alpha_2}{1-\alpha_3-\alpha_4}\right)\delta \end{aligned}$$

for $\lambda \in [0,1]$ and $\mid \lambda - \lambda_0 \mid \le \eta$.
Using (ii), (iv) and (v) together with the Theorem 3.1 (in the case $r = \delta$ and $F = H_\lambda$) we get; there exists $x_\lambda \in \overline{B(x_0,\delta)^{G'}} \subset Q$ with $x_\lambda = H_\lambda(x\lambda)$ for $\lambda \in [0,1]$ and $\mid \lambda - \lambda_0 \mid \le \eta$. But $x_\lambda \in U$ ((i) guarantees that) and so A contains all $\lambda \in [0,1]$ with $\mid \lambda - \lambda_0 \mid \le \eta$. Consequently A is open in [0,1]. □

Following result can be easily obtain from the Theorem 3.1

Theorem 3.3.
Let (X,G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : X \to X$. Suppose for any $x,y,z \in X$ we have

$$\begin{aligned} G(Fx,Fy,Fz) &\le \alpha_1 G(x,y,z) + \alpha_2 G(x,Fx,Fx) \\ &\quad + \alpha_3 G(y,Fy,Fy) + \alpha_4 G(z,Fz,Fz), \end{aligned} \tag{20}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are non negative numbers with $0 \le \alpha_1+\alpha_2+\alpha_3+\alpha_4 < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0,Fx_0,Fx_0) < \left(1-\frac{\alpha_1+\alpha_2}{1-\alpha_3-\alpha_4}\right)r, \tag{21}$$

(2) if $G < G'$ then F is uniformly continuous form (X,G') into (X,G'),

(3) if $G \ne G'$ then F is continuous from (X,G') into (X,G').

Then F has a fixed point.

4. Fixed Point Result for Ciri$\acute{c}$ Generalized Contractions

Theorem 4.1.
Let (X,G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0,r)^{G'}} \to X$. Suppose for any $x,y,z \in \overline{B(x_0,r)^{G'}}$ we have

$$\begin{aligned} G(Fx,Fy,Fz) &\le \alpha \max\{G(x,y,z),G(x,Fx,Fx), \\ &\qquad G(y,Fy,Fy),G(z,Fz,Fz)\}, \end{aligned} \tag{22}$$

where $0 \le \alpha < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0,Fx_0,Fx_0) < (1-\alpha)\,r, \tag{23}$$

(2) if $G < G'$ then F is uniformly continuous form $(B(x_0,r),G)$ into (X,G'),

(3) if $G \neq G'$ then F is continuous from $(\overline{B(x_0, r)^{G'}}, G')$ into (X, G').

Then F has a fixed point, that is there exists $x \in \overline{B(x_0, r)^{G'}}$ with $F(x) = x$.

Proof. Let $x_1 = F x_0$. From ((23)), since $\alpha < 1$, we have

$$G(x_0, x_1, x_1) < (1-\alpha) r < r$$

so $x_1 \in B(x_0, r)$.
Next let $x_2 = F x_1$ and note that

$$\begin{aligned} G(x_1, x_2, x_2) &= G(F x_0, F x_1, F x_1) \\ &\leq \alpha \max\{G(x_0, x_1, x_1), G(x_0, F x_0, F x_0) \\ &\quad G(x_1, F x_1, F x_1), G(x_1, F x_1, F x_1)\} \\ &\leq \alpha G(x_0, x_1, x_1) \\ G(x_1, x_2, x_2) &\leq \alpha(1-\alpha) r. \end{aligned}$$

Thus

$$\begin{aligned} G(x_0, x_2, x_2) &\leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\ &\leq (1-\alpha) r + \alpha(1-\alpha) r \\ &\leq (1-\alpha) r [1+\alpha] \\ &< (1-\alpha) r [1+\alpha+\alpha^2+\alpha^3+...] \\ &< (1-\alpha) r (1-\alpha)^{-1} \\ G(x_0, x_2, x_2) &< r. \end{aligned}$$

That is $x_2 \in B(x_0, r)$. Proceeding inductively we obtain

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(F x_{n-1}, F x_n, F x_n) \\ &\leq \alpha G(x_{n-1}, x_n, x_n) \\ &\ \ \vdots \\ &\leq \alpha^n G(x_0, x_1, x_1) \\ &< \alpha^n (1-\alpha) r \end{aligned}$$

where $x_n = F x_{n-1}$ for $n = 3, 4, 5....$ Since $\alpha \in [0, 1)$ it follows that $\alpha^n \in [0, 1)$ and thus

$$G(x_n, x_{n+1}, x_{n+1}) \leq (1-\alpha) r. \tag{24}$$

The lase inequality implies $x_{n+1} \in B(x_0, r)$ and the sequence $\{x_n\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.
If $G \geq G'$ this is trivial. Next suppose $G < G'$. Let $\epsilon > 0$ be given. Now from assumption - 2 guarantees that there exists $\delta > 0$ such that

$$G'(F x, F y, F y) < \epsilon \tag{25}$$

where $x, y \in B(x_0, r)$ and $G(x, y, y) < \delta$.
From above the sequence $\{x_n\}$ is a Cauchy sequence with respect to G, so we know that there exists N with

$$G(x_n, x_m, x_m) < \delta \tag{26}$$

for all $m, n \geq N$.
Now from (25) and (26) imply

$$G'(x_{n+1}, x_{m+1}, x_{m+1}) = G'(F x_n, F x_m, F x_m) < \epsilon \tag{27}$$

whenever $m,n \geq N$ which proves that $\{x_n\}$ is a Cauchy sequence with respect to G'. Now since (X,G') is complete there exists $x \in \overline{B(x_0,r)}^{G'}$ with $G'(x_n,x,x) \to 0$ as $n \to \infty$. We claim tha

$$x = Fx.$$

First consider the case $G \neq G'$. Notice

$$G'(x,Fx,Fx) \leq G'(x,x_n,x_n) + G'(x_n,Fx,Fx) \tag{28}$$

Let $n \to \infty$ and using assumption-3 we have $G(x,Fx,Fx) = 0$, and thus $x = Fx$ in this case. Next suppose that $G = G'$ that is assumption-2 and assumption-3 are not hold. Then

$$\begin{aligned} G(x,Fx,Fx) &\leq G(x,x_n,x_n) + G(x_n,Fx,Fx) \\ &\leq G(x,x_n,x_n) + G(Fx_{n-1},Fx,Fx) \\ &\leq G(x,x_n,x_n) + \alpha G(x_{n-1},x,x) \end{aligned}$$

as $n \to \infty$ we obtain $G(x,Fx,Fx) = 0$ and that $x = Fx$. Thus the proof of the theorem is complete. □

Now we present an homotopy result for this type of generalized contractions.

Theorem 4.2.
Let (X,G') be a complete G-metric space and G another metric on X. Let $Q \subset X$ be G'-closed and let $U \subset X$ be G-open and $U \subset Q$. Suppose $H : Q \times [0,1] \to X$ satisfies the following five properties:

(i) $x \neq H(x,\lambda)$ for $x \in Q \setminus U$ and $\lambda \in [0,1]$,

(ii) for any $\lambda \in [0,1]$ and $x, y \in Q$ we have

$$\begin{aligned} G(H(x,\lambda),H(y,\lambda),H(z,\lambda)) \leq\ & \alpha \max\{G(x,y,z), G(x,H(x,\lambda),H(x,\lambda)), \\ & G(y,H(y,\lambda),H(y,\lambda)), G(z,H(z,\lambda),H(z,\lambda))\} \end{aligned} \tag{29}$$

where $0 \leq \alpha < 1$,

(iii) $H(x,\lambda)$ is continuous in λ with respect to G, uniformly for $x \in Q$,

(iv) if $G < G'$ assume H is uniformly continuous form $U \times [0,1]$ endowed with the G-metric G on U into (X,G'),

(v) if $G \neq G'$ assume H is continuous from $Q \times [0,1]$ endowed with the G-metric G' on Q into (X,G').

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0,1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(.,\lambda)$).

Proof. Let

$A := \{\lambda \in [0,1]$; *there exists* $x \in U$ *such that* $H(x,\lambda) = x\}$.

Since (H_0 has a fixed point and (i) holds we have $0 \in A$, and so the set A is homotopy. We will show A is open and closed in$[0,1]$ and so by the connectedness of $[0,1]$ we have $A = [0,1]$ and the proof is finished
First we show that A is closed in $[0,1]$.
Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of A for each k, there exists $x_k \in U$ such that $x_k = H(x_k,\lambda_k)$. Now we have

$$\begin{aligned} G(x_k,x_j,x_j) &= G(H(x_k,\lambda_k),H(x_j,\lambda_j),H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k),H(x_k,\lambda),H(x_k,\lambda)) \\ &\quad + G(H(x_k,\lambda),H(x_j,\lambda_j),H(x_j,\lambda_j)) \\ &\leq G(H(x_k,\lambda_k),H(x_k,\lambda),H(x_k,\lambda)) + \alpha G(x_k,x_j,x_j) \\ &\leq (1-\alpha)^{-1} G(H(x_k,\lambda_k),H(x_k,\lambda),H(x_k,\lambda)). \end{aligned}$$

and (iii) guarantees that $\{x_k\}$ is a Cauchy sequence with respect to G. We claim that $\{x_n\}$ is a Cauchy sequence with respect to G'.
If $G \geq G'$ this is trivial. If $G < G'$ then

$$G'(x_k, x_j, x_j) \;=\; G'(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))$$

and (iv) guarantees that $\{x_n\}$ is a Cauchy sequence with respect to G'. (note as well that $\{x_k\}$ is a Cauchy sequence with respect to G and $\{\lambda_k\}$ is Cauchy sequence in$[0,1]$). Now since (X, G') is complete there exists an $x \in Q$ such that $G'(x_k, x, x) \longrightarrow 0$ as $n \longrightarrow \infty$. Claim now that $x = H(x, \lambda)$.
We consider first case $G \neq G'$. Then

$$\begin{aligned} G'(x, H(x,\lambda), H(x,\lambda)) &\leq G'(x, x_k, x_k) + G'(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G'(x, x_k, x_k) + G'(H(x_k,\lambda), H(x,\lambda), H(x,\lambda)) \end{aligned}$$

together with (v), letting $k \longrightarrow \infty$, we have $G'(x, H(x,\lambda), H(x,\lambda)) = 0$ and so $x = H(x,\lambda)$.
We consider now the case $G = G'$. Then

$$\begin{aligned} G(x, H(x,\lambda), H(x,\lambda)) &\leq G(x, x_k, x_k) + G(x_k, H(x,\lambda), H(x,\lambda)) \\ &= G(x, x_k, x_k) + G(H(x_k,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ &\leq G(x, x_k, x_k) + G(H(x_k,\lambda_k), H(x,\lambda_k), H(x,\lambda_k)) \\ &\quad + G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)) \\ G(x, H(x,\lambda), H(x,\lambda)) &\leq G(x, x_k, x_k) \\ &\quad + \alpha \max\{G(x_k, x, x), G(x_k, H(x_k,\lambda_k), H(x_k,\lambda_k)) \\ &\quad + G(x, H(x,\lambda), H(x,\lambda)), G(x, H(x,\lambda), H(x,\lambda))\} \\ &\quad + G(H(x,\lambda_k), H(x,\lambda), H(x,\lambda)). \end{aligned}$$

Letting $k \longrightarrow 0$ and using (iii) we have

$$G(x, H(x,\lambda), H(x,\lambda)) \leq 0.$$

So we have

$$G(x, H(x,\lambda), H(x,\lambda)) = 0,$$

that is $x = H(x,\lambda)$. Now form (i) we have $x \in U$. Consequently $\lambda \in A$ and so A is closed in [0,1].
We prove now A is open in [0,1].
Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. From U G-open there exists a G-ball

$$B(x_0 \delta) = \{x \in X; G(x, x_0, x_0) < \delta\}, \delta > 0$$

and

$$B(x_0, \delta) \subset U.$$

From (iii) we have that H is uniformly continuous on $B(x_0, \delta)$.
Let $\epsilon = (1-\alpha)\delta > 0$ and using the uniform continuity of H we have there exists $\eta = \eta(\delta) > 0$ such that for each $\delta \in [0,1], |\, \delta - \delta_0 \,| \leq \eta$ with

$$G(H(x,\lambda), H(x,\lambda_0), H(x,\lambda_0) < \epsilon$$

for any $x \in B(x_0, \delta)$. So this property holds for $x = x_0$, and then we have

$$G(x_0, H(x_0,\lambda), H(x_0,\lambda)) = G(H(x_0,\lambda), H(x_0,\lambda), H(x_0,\lambda)) < (1-\alpha)\delta$$

for $\lambda \in [0,1]$ and $|\, \lambda - \lambda_0 \,| \leq \eta$.
Using (ii), (iv) and (v) together with the Theorem 4.1 (in the case $r = \delta$ and $F = H_\lambda$) we get; there exists $x_\lambda \in \overline{B(x_0, \delta)}^{G'} \subset Q$ with $x_\lambda = H_\lambda(x\lambda)$ for $\lambda \in [0,1]$ and $|\, \lambda - \lambda_0 \,| \leq \eta$. But $x_\lambda \in U$ ((i) guarantees that) and so A contains all $\lambda \in [0,1]$ with $|\, \lambda - \lambda_0 \,| \leq \eta$. Consequently A is open in [0,1]. □

Following result can be easily obtained from Theorem 4.1

Theorem 4.3.
Let (X,G') be a complete G-metric space, G another G-metric on X, $x_0 \in X$, $r > 0$ and $F : X \to X$. Suppose for any $x,y,z \in X$ we have

$$G(Fx,Fy,Fz) \leq \alpha \max\{G(x,y,z), G(x,Fx,Fx), G(y,Fy,Fy), G(z,Fz,Fz)\}, \tag{30}$$

where $0 \leq \alpha < 1$.
In addition assume the following three properties hold:

(1)

$$G(x_0,Fx_0,Fx_0) < (1-\alpha)r, \tag{31}$$

(2) if $G < G'$ then F is uniformly continuous form (X,G') into (X,G'),

(3) if $G \neq G'$ then F is continuous from (X,G') into (X,G').

Then F has a fixed point.

References

[1] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2) (2006) 289-297.

[2] H. Aydi, A Fixed Point Result Involving A Generalized Weakly Contractive Condition in G -Metric spaces, Bull. of Math. Anal. and Appl. 3(4) (2011) 180-188.

[3] H. Aydi, Fixed point results for weakly contractive mappings in ordered partial metric spaces, J. Adv. Math. Stud., 4 (2) (2011) 1-12.

[4] H. Aydi, A common fixed point of integral type contraction in generalized metric spaces. J. Adv. Math. Stud. 5(1) (2012) 111-117.

[5] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point result in G-metric spaces, Int. J. Math. Math. Sci. 2009(2009) page 10, Article ID 283028.

[6] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete G-metric space, Fixed Point Theory Appl. 2009 (2009) page 10, Article ID 917175.

[7] Z. Mustafa, H. Obiedat and F. Awawdeh, Some of fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory Appl. 2008 (2008)page 12, Article ID 189870.

[8] W. Shatanawi, Fixed point theory for contractive mappings satisfying - maps in G-metric spaces, Fixed Point Theory Appl. 2010 (2010) 9 pages, Article ID 181650.