Fixed point for generalized contractions on G-metric spaces

Animesh Gupta *
H.No. 93/654, Ward No. 2 Gandhi Chowk, Pachmarhi-461881, Dist. Hoshangabad (M.P .), India. Visiting at : Department of Mathematics & Computer Science, Govt. Model Science College, Jabalpur (M.P .) India.

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Abstract: We present fixed point results for generalized homotopy contractions on spaces with two G-metric. The focus is on continuation results for such type of mappings.

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1. Introduction and preliminaries

The concept of G-metric space is given by Mustafa and Sims [1]. After the publication of their work there are many other works have appeared in this area some of them are [2–8] and their in. The object of this paper is to establish some results for generalized contractions on spaces with two G-metric. The focus is on continuation results for such type of mappings. Our results generalize and improve many existing results in the literature.

Definition 1.1.
Let X be a non empty set and let \( G : X \times X \times X \to R^+ \) be a function satisfying the following properties:

(G1) \( G(x, y, z) = 0 \) if \( x = y = z \)

(G2) \( G(x, x, y) > 0 \) for all \( x, y \in X \) with \( x \neq y \),

(G3) \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( x \neq y \),

(G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots \) (symmetry in all three variables),

(G5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \) (rectangle inequality).

Then the function \( G \) is called a generalized metric or more specially, a G–metric on \( X \) and the pair \((X, G)\) is called a G–metric space.

Definition 1.2.
Let \((X, G)\) be a G–metric space, and let \( \{x_n\} \) be a sequence of points of \( X \) therefore, we say that \( \{x_n\} \) is G–convergent to \( x \in X \) if \( \lim_{n,m \to +\infty} G(x, x_n, x_m) = 0 \). that is, for any \( \epsilon > 0 \), there exists \( n \in N \) such that \( G(x, x_n, x_m) < \epsilon \) for all \( n, m \geq N \). We call \( x \) the limit of the sequence and write \( x_n \to x \) or \( \lim_{n \to +\infty} x_n = x \).

* Corresponding author.
E-mail address: dranimeshgupta10@gmail.com
Lemma 1.1.
Let \((X,G)\) be a G- metric space. The following statements are equivalent:

1. \(\{x_n\}\) is G- convergent to \(x\),
2. \(G(x_n, x, x)\to 0\) as \(n\to +\infty\),
3. \(G(x_n, x)\to 0\) as \(n\to +\infty\),
4. \(G(x_n, x_m, x)\to 0\) as \(n, m\to +\infty\),

Definition 1.3.
Let \((X,G)\) be a G- metric space. A sequence \(\{x_n\}\) is called a G- Cauchy sequence if, for any \(\varepsilon > 0\), there exists \(N\in\mathbb{N}\) such that \(G(x_n, x_m, x_l) < \varepsilon\) for all \(n, m, l\geq N\), that is, \(G(x_n, x_m, x)\to 0\) as \(n, m, l\to +\infty\).

Lemma 1.2.
Let \((X,G)\) be a G- metric space. The following statements are equivalent:

1. The sequence \(\{x_n\}\) is G- Cauchy,
2. for any \(\varepsilon > 0\), there exists \(N\in\mathbb{N}\) such that \(G(x_n, x_m, x) < \varepsilon\) for all \(n, m\geq N\).

Definition 1.4.
A G- metric space \((X,G)\) is called G- complete if every G- Cauchy sequence is G- convergent in \((X,G)\).

Every G- metric on \(X\) defines a metric \(d_G\) on \(X\) given by

\[
d_G = G(x, y, y) + G(y, x, x) \quad \text{for all } x, y \in X.
\] (1)

Each G-metric \(G\) on \(X\) generates a topology \(\tau_G\) on \(X\) which has as a base the family of open G-balls

\[
\{B(x, r), x \in X, r > 0\},
\]

where

\[
B(x, r) = \{y \in X : G(x, y, y) < r\}
\]

for all \(x \in X\) and \(r > 0\). Also, a nonempty subset \(A\) in the G-metric space \((X,G)\) is G-closed if \(\overline{A} = A\). Note that

\[
x \in \overline{A} \iff B(x, r) \cap A \neq 0,
\]

for all \(r > 0\).

By (1) it is easy to see that

Lemma 1.3.
Let \((X,G)\) be a G-metric space and \(A\) is a nonempty subset of \(X\). \(A\) is said G-closed if for any sequence \(\{x_n\}\) in \(A\) such that \(x_n \to x\) as \(n \to \infty\), then \(x \in A\).

Lemma 1.4.
If \(X\) is a G- metric space, then \(G(x, y, y) = 2G(y, x, x)\) for all \(x, y \in X\).

Lemma 1.5.
If \(X\) is a G- metric space, then \(G(x, x, y) = G(x, x, z) + G(z, z, y)\) for all \(x, y, z \in X\).

Throughout this article \((X, G')\) will be a complete G-metric space and \(G\) another metric on \(X\). If \(x_0 \in X\) and \(r > 0\) denote by

\[
B(x_0, r) = \{x \in X : G(x_0, x, x) < r\}
\]

and by \(B(x_0, r)^G\) the G-closure of \(B(x_0, r)\).
2. Fixed Point Results for Generalized Contraction on G-Metric Space

**Theorem 2.1.**

Let \((X, G')\) be a complete G-metric space, \(G\) another G-metric on \(X\), \(x_0 \in X\), \(r > 0\) and \(F : \overline{B(x_0, r)}^G \rightarrow X\). Suppose for any \(x, y, z \in \overline{B(x_0, r)}^G\) we have

\[
G(Fx, Fy, Fz) \leq \alpha G(x, y, z),
\]

where \(0 \leq \alpha < 1\).

In addition assume the following three properties hold:

1. \(G(x_0, Fx_0, Fx_0) < (1-\alpha)r\),

2. if \(G < G'\) then \(F\) is uniformly continuous form \((B(x_0, r), G)\) into \((X, G')\),

3. if \(G \neq G'\) then \(F\) is continuous from \((B(x_0, r)^G, G')\) into \((X, G')\).

Then \(F\) has a fixed point, that is there exists \(x \in \overline{B(x_0, r)}^G\) with \(F(x) = x\).

**Proof.** Let \(x_1 = Fx_0\). From (3), since \(\alpha < 1\), we have

\[
G(x_0, x_1, x_1) < (1-\alpha)r < r
\]

so \(x_1 \in B(x_0, r)\).

Next let \(x_2 = Fx_1\) and note that

\[
G(x_1, x_2, x_2) = G(Fx_0, Fx_1, Fx_1) \leq \alpha G(x_0, x_1, x_1)
\]

Thus

\[
G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\
\leq (1-\alpha)r + \alpha(1-\alpha)r \\
\leq (1-\alpha)r[1 + \alpha] \\
< (1-\alpha)r(1-\alpha)^{-1}
\]

\[
G(x_0, x_2, x_2) < r.
\]

That is \(x_2 \in B(x_0, r)\). Proceeding inductively we obtain

\[
G(x_n, x_{n+1}, x_{n+1}) \leq \alpha G(x_{n-1}, x_n, x_n) \\
\leq \ldots \leq \alpha^n G(x_0, x_1, x_1) \\
< \alpha^n(1-\alpha)r
\]

where \(x_n = Fx_{n-1}\) for \(n = 3, 4, 5,...\). Since \(\alpha \in [0, 1)\) it follows that \(\alpha^n \in [0, 1)\) and thus

\[
G(x_n, x_{n+1}, x_{n+1}) \leq (1-\alpha)r.
\]

The last inequality implies \(x_{n+1} \in B(x_0, r)\) and the sequence \(\{x_n\}\) is a Cauchy sequence with respect to \(G\). We claim that \(\{x_n\}\) is a Cauchy sequence with respect to \(G'\).

If \(G \geq G'\) this is trivial. Next suppose \(G < G'\). Let \(\epsilon > 0\) be given. Now from assumption-2 guarantees that there exists \(\delta > 0\) such that

\[
G'(Fx, Fy, Fy) < \epsilon
\]

where \(x, y \in B(x_0, r)\) and \(G(x, y, y) < \delta\).

From above the sequence \(\{x_n\}\) is a Cauchy sequence with respect to \(G\), so we know that there exists \(N\) with
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$$G(x_n, x_m, x_m) < \delta$$

(6)

for all $m, n \geq N$.

Now from (5) and (6) imply

$$G'(x_{n+1}, x_{m+1}, x_{m+1}) = G'(F x_n, F x_n, F x_m) < \epsilon$$

(7)

whenever $m, n \geq N$ which proves that $\{x_n\}$ is a Cauchy sequence with respect to $G'$. Now since $(X, G')$ is complete there exists $x \in B(x_0, \epsilon)'G'$ with $G'(x_n, x, x) \to 0$ as $n \to \infty$. We claim that

$$x = F x.$$

First consider the case $G \neq G'$. Notice

$$G'(x, F x, F x) \leq G'(x, x_n, x_n) + G'(x_n, F x, F x)$$

(8)

Let $n \to \infty$ and using assumption-3 we have $G(x, F x, F x) = 0$ and thus $x = F x$ in this case. Next suppose that $G = G'$ that is assumption-2 and assumption-3 are not hold. Then

$$G(x, F x, F x) \leq G(x, x_n, x_n) + G(x_n, F x, F x)$$

$$\leq G(x, x_n, x_n) + G(F x_{n-1}, F x, F x)$$

$$\leq G(x, x_n, x_n) + \alpha G(x_{n-1}, x, x)$$

as $n \to \infty$ we obtain $G(x, F x, F x) = 0$ and that $x = F x$. Thus the proof of the theorem is complete.

Next we present an homotopy result for this type of generalized contractions.

**Theorem 2.2.**

Let $(X, G')$ be a complete $G$-metric space and $G$ another metric on $X$. Let $Q \subset X$ be $G'$-closed and let $U \subset X$ be $G$-open and $U \subset Q$. Suppose $H : Q \times [0, 1] \to X$ satisfies the following five properties:

(i) $x \neq H(x, \lambda)$ for $x \in Q \setminus U$ and $\lambda \in [0, 1]$,

(ii) for any $\lambda \in [0, 1]$ and $x, y \in Q$ we have

$$G(H(x, \lambda), H(y, \lambda), H(z, \lambda)) \leq \alpha G(x, y, z)$$

(9)

with $0 < \alpha < 1$,

(iii) $H(x, \lambda)$ is continuous in $\lambda$ with respect to $G$, uniformly for $x \in Q$,

(iv) if $G < G'$ assume $H$ is uniformly continuous form $U \times [0, 1]$ endowed with the $G$-metric $G$ on $U$ into $(X, G')$,

(v) if $G \neq G'$ assume $H$ is continuous from $Q \times [0, 1]$ endowed with the $G$-metric $G'$ on $Q$ into $(X, G')$.

In addition assume $H_0$ has a fixed point. Then for each $\lambda \in [0, 1]$ we have that $H_\lambda$ has a fixed point $x_\lambda \in U$ (here $H_\lambda(\cdot) = H(\cdot, \lambda)$).

**Proof.** Let

$$A := \{\lambda \in [0, 1]; \text{there exists } x \in U \text{ such that } H(x, \lambda) = x\}.$$

Since $(H_0$ has a fixed point and (i) holds we have $0 \in A$, and so the set $A$ is homotopy. We will show $A$ is open and closed in $[0, 1]$ and so by the connectedness of $[0, 1]$ we have $A = [0, 1]$ and the proof is finished.

First we show that $A$ is closed in $[0, 1]$.

Let $(\lambda_k)$ be a sequence in $A$ with $\lambda_k \to \lambda \in [0, 1]$ as $k \to \infty$. By definition of $A$ for each $k$, there exists $x_k \in U$ such that $x_k = H(x_k, \lambda_k)$. Now we have

$$G(x_k, x_j, x_j) = G(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))$$

$$\leq G(H(x_k, \lambda_k), H(x_k, \lambda_k), H(x_k, \lambda_k)) + G(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))$$

$$\leq G(H(x_k, \lambda_k), H(x_k, \lambda_k), H(x_k, \lambda_k)) + \alpha G(x_k, x_j, x_j)$$

$$\leq (1 - \alpha)^{-1} G(H(x_k, \lambda_k), H(x_k, \lambda_k), H(x_k, \lambda_k)).$$
and (iii) guarantees that \( \{x_k\} \) is a Cauchy sequence with respect to \( G \). We claim that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \).

If \( G < G' \) this is trivial. If \( G < G' \) then

\[
G'(x_k, x, x_j, x_j) = G'(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))
\]

and (iv) guarantees that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \). (note as well that \( \{x_k\} \) is a Cauchy sequence with respect to \( G \) and \( \{\lambda_k\} \) is Cauchy sequence in \([0, 1]\)). Now since \((X, G')\) is complete there exists an \( x \in Q \) such that \( G'(x_k, x, x) \rightarrow 0 \) as \( n \rightarrow \infty \). Claim now that \( x = H(x, \lambda) \).

We consider first case \( G \neq G' \). Then

\[
G'(x, x, x, x_j) \leq G'(x, x_k, x_k) + G'(x_k, H(x, \lambda), H(x, \lambda))
\]

and (v) letting \( k \rightarrow \infty \), we have

\[
G'(x, x, x, x_j) = 0
\]

and so

\[
x = H(x, \lambda).
\]

We consider now the case \( G = G' \). Then

\[
G(x, x, x, x_j) \leq G(x, x_k, x_k) + G(x_k, H(x, \lambda), H(x, \lambda))
\]

= \( G(x, x_k, x_k) + G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda)) \)

\[
\leq G(x, x_k, x_k) + G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda))
\]

\[
+ G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda))
\]

\[
\leq G(x, x_k, x_k) + aG(x_k, x, x) + G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda)).
\]

Letting \( k \rightarrow 0 \) and using (iii) we have

\[
G(x, x, x, x_j) \leq 0.
\]

So we have

\[
G(x, x, x, x_j) = 0,
\]

that is \( x = H(x, \lambda) \). Now form (i) we have \( x \in U \). Consequently \( \lambda \in A \) and so \( A \) is closed in \([0, 1]\).

We prove now \( A \) is open in \([0, 1]\).

Let \( \lambda_0 \in A \) and \( x_0 \in U \) such that \( x_0 = H(x_0, \lambda_0) \). From \( U \) \( G \)-open there exists a \( G \)-ball

\[
B(x_0, \delta) = \{ x \in X; G(x, x_0, x_0) < \delta \}, \delta > 0
\]

and \( B(x_0, \delta) \subset U \). From (iii) we have that \( H \) is uniformly continuous on \( B(x_0, \delta) \).

Let

\[
\epsilon = (1 - a)\delta > 0
\]

and using the uniform continuity of \( H \) we have there exists \( \eta = \eta(\delta) > 0 \) such that for each \( \delta \in [0, 1], |\delta - \delta_0| \leq \eta \) with

\[
G(H(x, \lambda), H(x, \lambda_0), H(x, \lambda_0)) < \epsilon
\]

for any \( x \in B(x_0, \delta) \). So this property holds for \( x = x_0 \), and then we have

\[
G(x_0, H(x_0, \lambda), H(x_0, \lambda)) = G(H(x_0, \lambda), H(x_0, \lambda), H(x_0, \lambda)) < (1 - a)\delta
\]

for \( \lambda \in [0, 1] \) and \( |\lambda - \lambda_0| \leq \eta \).

Using (ii), (iv) and (v) together with the Theorem 2.1 (in the case \( r = \delta \) and \( F = H \)) we get; there exists \( x_\lambda \in B(x_0, \delta) \subset Q \) with \( x_\lambda = H_\lambda(x_\lambda) \) for \( \lambda \in [0, 1] \) and \( |\lambda - \lambda_0| \leq \eta \) and \( x_\lambda \in U \) (i) guarantees that ) and so \( A \) contains all \( \lambda \in [0, 1] \) with \( |\lambda - \lambda_0| \leq \eta \). Consequently \( A \) is open in \([0, 1]\).

The following global result can be easily obtained from the above Theorem 2.1
Theorem 2.3.
Let \((X, G')\) be a complete G-metric space, \(G\) another G-metric on \(X\), \(x_0 \in X\), \(r > 0\) and \(F : X \to X\). Suppose for any \(x, y, z \in X\) we have
\[
G(Fx, Fy, Fz) \leq \alpha G(x, y, z),
\]  
where \(0 \leq \alpha < 1\).
In addition assume the following three properties hold:

(1) \[G(x_0, Fx_0, x_0) < (1 - \alpha)r,\]

(2) if \(G < G'\) then \(F\) is uniformly continuous form \((X, G')\) into \((X, G')\),

(3) if \(G \neq G'\) then \(F\) is continuous from \((X, G')\) into \((X, G')\).

Then \(F\) has a fixed point.

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Theorem 3.1.
Let \((X, G')\) be a complete G-metric space, \(G\) another G-metric on \(X\), \(x_0 \in X\), \(r > 0\) and \(F : \overline{B(x_0, r)^G} \to X\). Suppose for any \(x, y, z \in \overline{B(x_0, r)^G}\) we have
\[
G(Fx, Fy, Fz) \leq \alpha_1 G(x, y, z) + \alpha_2 G(x, Fx, Fx) + \alpha_3 G(y, Fy, Fy) + \alpha_4 G(z, Fz, Fz),
\]  
where \(\alpha_1, \alpha_2, \alpha_3, \alpha_4\) are non negative numbers with \(0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1\).
In addition assume the following three properties hold:

(1) \[G(x_0, Fx_0, Fx_0) < \frac{1 - \alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4} r,\]

(2) if \(G < G'\) then \(F\) is uniformly continuous form \((B(x_0, r), G)\) into \((X, G')\),

(3) if \(G \neq G'\) then \(F\) is continuous from \((B(x_0, r)^G, G')\) into \((X, G')\).

Then \(F\) has a fixed point, that is there exists \(x \in \overline{B(x_0, r)^G}\) with \(F(x) = x\).

Proof. Let \(x_1 = Fx_0\). From (13), since \(0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1\), we have
\[
G(x_0, x_1, x_1) < \left(1 - \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) r < r
\]  
so \(x_1 \in B(x_0, r)\).

Next let \(x_2 = Fx_1\) and note that
\[
G(x_1, x_2, x_2) = G(Fx_0, Fx_1, Fx_1) \leq \alpha_1 G(x_0, x_1, x_1) + \alpha_2 G(x_0, Fx_0, Fx_0) + \alpha_3 G(x_1, Fx_1, Fx_1) + \alpha_4 G(x_1, Fx_1, Fx_1) \\
\leq \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4} G(x_0, x_1, x_1) \\
G(x_1, x_2, x_2) \leq \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4} \left(1 - \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4}\right) r.
\]

Let \(0 \leq \beta = \frac{\alpha_1 + \alpha_2}{1 - \alpha_3 - \alpha_4} < 1\).
Thus
\[ G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \]
\[ \leq (1 - \beta)r + \beta(1 - \beta)r \]
\[ \leq (1 - \beta)r[1 + \beta] \]
\[ < (1 - \beta)r[1 + \beta + \beta^2 + \beta^3 + ...] \]
\[ < (1 - \beta)r(1 - \beta) \]
\[ G(x_0, x_2, x_2) < r. \]

That is \( x_2 \in B(x_0, r) \). Proceeding inductively we obtain
\[ G(x_n, x_{n+1}, x_{n+1}) = G(Fx_{n-1}, Fx_n, Fx_n) \]
\[ \leq \beta G(x_{n-1}, x_n, x_n) \]
\[ \leq ... \leq \beta^n G(x_0, x_1, x_1) \]
\[ < \beta^n(1 - \beta)r \]
where \( x_n = Fx_{n-1} \) for \( n = 3, 4, 5, ... \). Since \( \beta \in [0, 1) \) it follows that \( \beta^n \in [0, 1) \) and thus
\[ G(x_n, x_{n+1}, x_{n+1}) \leq (1 - \beta)r. \] (14)

The last inequality implies \( x_{n+1} \in B(x_0, r) \) and the sequence \( \{x_n\} \) is a Cauchy sequence with respect to \( G \). We claim that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \).

If \( G \geq G' \) this is trivial. Next suppose \( G < G' \). Let \( \epsilon > 0 \) be given. Now from assumption - 2 guarantees that there exists \( \delta > 0 \) such that
\[ G'(Fx, Fx') < \epsilon \] (15)
where \( x, y \in B(x_0, r) \) and \( G(x, y, y) < \delta \).

From above the sequence \( \{x_n\} \) is a Cauchy sequence with respect to \( G \), so we know that there exists \( N \) with
\[ G(x_n, x_m, x_m) < \delta \] (16)
for all \( m, n \geq N \).

Now from (15) and (16) imply
\[ G'(x_{n+1}, x_{m+1}, x_{m+1}) = G'(Fx_n, Fx_m, Fx_m) < \epsilon \] (17)
whenever \( m, n \geq N \) which proves that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \). Now since \( \langle X, G' \rangle \) is complete there exists \( x \in \overline{B}(x_0, r) \) with \( G'(x_n, x, x) \to 0 \) as \( n \to \infty \). We claim that
\[ x = Fx \]

First consider the case \( G \neq G' \). Notice
\[ G'(x, Fx, Fx) \leq G'(x, x, x) + G'(x, Fx, Fx) \] (18)
Let \( n \to \infty \) and using assumption-3 we have \( G(x, Fx, Fx) = 0 \), and thus \( x = Fx \) in this case. Next suppose that \( G = G' \) that is assumption-2 and assumption-3 are not hold. Then
\[ G(x, Fx, Fx) \leq G(x, x, x) + G(x, Fx, Fx) \]
\[ \leq G(x, x, x) + G(Fx_{n-1}, Fx, Fx) \]
\[ \leq G(x, x, x) + \beta G(x_{n-1}, x, x) \]
as \( n \to \infty \) we obtain \( G(x, Fx, Fx) = 0 \) and that \( x = Fx \). Thus the proof of the theorem is complete.

Now we present an homotopy result for this type of generalization contractions.

**Theorem 3.2.**

Let \( \langle X, G' \rangle \) be a complete \( G \)-metric space and \( G \) another metric on \( X \). Let \( Q \subset X \) be \( G' \)-closed and let \( U \subset X \) be \( G \)-open and \( U \subset Q \). Suppose \( H : Q \times [0, 1] \to X \) satisfies the following five properties:
(i) \( x \neq H(x, \lambda) \) for \( x \in Q \setminus U \) and \( \lambda \in [0, 1] \).

(ii) for any \( \lambda \in [0, 1] \) and \( x, y \in Q \) we have

\[
G(H(x, \lambda), H(y, \lambda), H(z, \lambda)) \leq \alpha_1 G(x, y, z)
+ \alpha_2 G(x, H(x, \lambda), H(y, \lambda))
+ \alpha_3 G(y, H(y, \lambda), H(y, \lambda))
+ \alpha_4 G(z, H(z, \lambda), H(z, \lambda))
\]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are non-negative numbers with \( 0 \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1 \).

(iii) \( H(x, \lambda) \) is continuous in \( \lambda \) with respect to \( G \), uniformly for \( x \in Q \).

(iv) if \( G < G' \) assume \( H \) is uniformly continuous form \( U \times [0, 1] \) endowed with the \( G \)-metric \( G \) on \( U \) into \( (X, G') \).

(v) if \( G \neq G' \) assume \( H \) is continuous from \( Q \times [0, 1] \) endowed with the \( G \)-metric \( G' \) on \( Q \) into \( (X, G') \).

In addition assume \( H_0 \) has a fixed point. Then for each \( \lambda \in [0, 1] \) we have that \( H_0 \) has a fixed point \( x_\lambda \in U \) (here \( H_0(x) = H(\cdot, \lambda) \)).

**Proof.** Let

\[
A := \{ \lambda \in [0, 1]; \text{there exists } x \in U \text{ such that } H(x, \lambda) = x \}.
\]

Since \( H_0 \) has a fixed point and (i) holds we have \( 0 \in A \), and so the set \( A \) is homotopy. We will show \( A \) is open and closed in \([0, 1]\) and so by the connectedness of \([0, 1] \) we have \( A = [0, 1] \) and the proof is finished.

First we show that \( A \) is closed in \([0, 1]\).

Let \( \lambda_k \) be a sequence in \( A \) with \( \lambda_k \to \lambda \in [0, 1] \) as \( k \to \infty \). By definition of \( A \) for each \( k \), there exists \( x_k \in U \) such that \( x_k = H(x_k, \lambda_k) \). Now we have

\[
G(x_k, x_j, x_j) = G(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))
\leq G(H(x_k, \lambda_k), H(x_k, \lambda), H(x_j, \lambda_j))
+ G(H(x_k, \lambda), H(x_j, \lambda_j), H(x_j, \lambda_j))
\leq G(H(x_k, \lambda_k), H(x_k, \lambda), H(x_k, \lambda))
+ \alpha G(x_k, x_j, x_j)
\leq (1 - \alpha)^{-1} G(H(x_k, \lambda_k), H(x_k, \lambda), H(x_k, \lambda)).
\]

and (iii) guarantees that \( \{x_k\} \) is a Cauchy sequence with respect to \( G \). We claim that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \).

If \( G \geq G' \) this is trivial. If \( G < G' \) then

\[
G'(x_k, x_j, x_j) = G'(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))
\]

and (iv) guarantees that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \). (note as well that \( \{x_k\} \) is a Cauchy sequence with respect to \( G \) and \( \{\lambda_k\} \) is Cauchy sequence in\([0, 1]\)).

Now since \( (X, G') \) is complete there exists an \( x \in Q \) such that \( G'(x_k, x) \to 0 \) as \( n \to \infty \). Claim now that \( x = H(x, \lambda) \).

We consider first case \( G \neq G' \).

Then

\[
G'(x, H(x, \lambda), H(x, \lambda)) \leq G'(x, x_k, x_k) + G'(x_k, H(x, \lambda), H(x, \lambda))
= G(x, x_k, x_k)
+ G(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
\leq G(x, x_k, x_k)
+ G(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
+ G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda))
+ (1 - \alpha_3 - \alpha_4) G(x, H(x, \lambda), H(x, \lambda))
\]

together with (v), letting \( k \to \infty \), we have \( G'(x, H(x, \lambda), H(x, \lambda)) = 0 \) and so \( x = H(x, \lambda) \).

We consider now the case \( G = G' \). Then

\[
G(x, H(x, \lambda), H(x, \lambda)) \leq G(x, x_k, x_k)
+ G(x_k, H(x, \lambda), H(x, \lambda))
= G(x, x_k, x_k)
+ G(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
\leq G(x, x_k, x_k)
+ G(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
+ G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda))
(1 - \alpha_3 - \alpha_4) G(x, H(x, \lambda), H(x, \lambda)) \leq G(x, x_k, x_k) + \alpha G(x_k, x_k)
+ \alpha G(x_k, H(x_k, \lambda_k), H(x_k, \lambda_k))
+ G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda)).
\]
Letting $k \to 0$ and using (iii) we have

$$(1 - a_3 - a_4)G(x, H(x, \lambda), H(x, \lambda)) \leq 0.$$ 

So we have $G(x, H(x, \lambda), H(x, \lambda)) = 0$, that is $x = H(x, \lambda)$. Now form (i) we have $x \in U$. Consequently $\lambda \in A$ and so $A$ is closed in $[0,1]$. We prove now $A$ is open in $[0,1]$. Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. From $U$-open there exists a $G$-ball $B(x_0, \delta) = \{x \in X; G(x, x_0, x_0) < \delta\}, \delta > 0$ and $B(x_0, \delta) \subset U$. From (iii) we have that $H$ is uniformly continuous on $B(x_0, \delta)$. Let $\epsilon = (1 - \alpha)\delta > 0$ and using the uniform continuity of $H$ we have there exists $\eta = \eta(\delta) > 0$ such that for each $\delta \in [0,1]$ $|\delta - \delta_0| \leq \eta$ with $G(H(x, \lambda), H(x, \lambda_0), H(x, \lambda_0) < \epsilon$ for any $x \in B(x_0, \delta)$. So this property holds for $x = x_0$, and then we have

$$G(x_0, H(x_0, \lambda), H(x_0, \lambda)) = G(H(x_0, \lambda), H(x_0, \lambda), H(x_0, \lambda))$$

$$< \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right)\delta$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| \leq \eta$.

Using (ii), (iv) and (v) together with the Theorem 3.1 (in the case $r = \delta$ and $F = H_3$) we get; there exists $x_3 \in B(x_0, \delta)^{G'} \subset Q$ with $x_3 = H_3(x, \lambda)$ for all $\lambda \in [0,1]$ and $|\lambda - \lambda_0| \leq \eta$. But $x_3 \in U$ (i) guarantees that and so $A$ contains all $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$. Consequently $A$ is open in $[0,1]$.

Following result can be easily obtain from the Theorem 3.1

**Theorem 3.3.**

Let $(X, G')$ be a complete $G$-metric space, $G$ another $G$-metric on $X$, $x_0 \in X$, $r > 0$ and $F : X \to X$. Suppose for any $x, y, z \in X$ we have

$$G(Fx, Fy, Fz) \leq a_1G(x, y, z) + a_2G(x, Fx, Fx) + a_3G(y, Fy, Fy) + a_4G(z, Fz, Fz),$$

(20)

where $a_1, a_2, a_3, a_4$ are non negative numbers with $0 \leq a_1 + a_2 + a_3 + a_4 < 1$. In addition assume the following three properties hold:

1. $$G(x_0, Fx_0, Fx_0) < \left(1 - \frac{a_1 + a_2}{1 - a_3 - a_4}\right)r,$$

(21)

2. if $G < G'$ then $F$ is uniformly continuous form $(X, G')$ into $(X, G')$,

3. if $G \neq G'$ then $F$ is continuous from $(X, G')$ into $(X, G')$.

Then $F$ has a fixed point.

**4. Fixed Point Result for Cirić Generalized Contractions**

**Theorem 4.1.**

Let $(X, G')$ be a complete $G$-metric space, $G$ another $G$-metric on $X$, $x_0 \in X$, $r > 0$ and $F : B(x_0, r)^{G'} \to X$. Suppose for any $x, y, z \in B(x_0, r)^{G'}$ we have

$$G(Fx, Fy, Fz) \leq \alpha \max\{G(x, y, z), G(x, Fx, Fx), G(y, Fy, Fy), G(z, Fz, Fz)\},$$

(22)

where $0 \leq \alpha < 1$. In addition assume the following three properties hold:

1. $$G(x_0, Fx_0, Fx_0) < (1 - \alpha)r,$$

(23)

2. if $G < G'$ then $F$ is uniformly continuous form $B(x_0, r), G)$ into $(X, G')$,.
(3) if \( G \neq G' \) then \( F \) is continuous from \((B(x_0, r)')^G', G')\) into \((X, G')\).

Then \( F \) has a fixed point, that is there exists \( x \in B(x_0, r)' \) with \( F(x) = x \).

\textbf{Proof.} Let \( x_1 = Fx_0 \). From \((23)\), since \( \alpha < 1 \), we have

\[
G(x_0, x_1, x_1) < (1 - \alpha) r < r
\]

so \( x_1 \in B(x_0, r) \).

Next let \( x_2 = Fx_1 \) and note that

\[
G(x_1, x_2, x_2) = G(Fx_0, Fx_1, Fx_1) \\
\leq \alpha \max\{G(x_0, x_1, x_1), G(x_0, Fx_0, Fx_0) \}\] \\n\leq a G(x_0, x_1, x_1) \\
G(x_1, x_2, x_2) \leq \alpha (1 - \alpha) r.
\]

Thus

\[
G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\
\leq (1 - \alpha) r + \alpha (1 - \alpha) r \\
\leq (1 - \alpha) r[1 + \alpha] \\
< (1 - \alpha) r[1 + \alpha + \alpha^2 + \alpha^3 + ...] \\
< (1 - \alpha) r\left(1 - \alpha^{-1}\right)
\]

\[
G(x_0, x_2, x_2) \leq r.
\]

That is \( x_2 \in B(x_0, r) \). Proceeding inductively we obtain

\[
G(x_n, x_{n+1}, x_{n+1}) = G(Fx_{n-1}, Fx_n, Fx_n) \\
\leq \alpha G(x_{n-1}, x_n, x_n) \\
\leq \alpha^n G(x_0, x_1, x_1) \\
< \alpha^n(1 - \alpha)r
\]

where \( x_n = Fx_{n-1} \) for \( n = 3, 4, 5,... \). Since \( \alpha \in [0, 1) \) it follows that \( \alpha^n \in [0, 1) \) and thus

\[
G(x_n, x_{n+1}, x_{n+1}) \leq (1 - \alpha)r. \quad (24)
\]

The last inequality implies \( x_{n+1} \in B(x_0, r) \) and the sequence \( \{x_n\} \) is a Cauchy sequence with respect to \( G \). We claim that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \).

If \( G \geq G' \) this is trivial. Next suppose \( G < G' \). Let \( \epsilon > 0 \) be given. Now from assumption 2 guarantees that there exists \( \delta > 0 \) such that

\[
G'(Fx, Fy, Fy) < \epsilon \quad (25)
\]

where \( x, y \in B(x_0, r) \) and \( G(x, y, y) < \delta \).

From above the sequence \( \{x_n\} \) is a Cauchy sequence with respect to \( G \), so we know that there exists \( N \) with

\[
G(x_n, x_m, x_m) < \delta \quad (26)
\]

for all \( m, n \geq N \).

Now from \((25)\) and \((26)\) imply

\[
G'(x_{n+1}, x_{m+1}, x_{m+1}) = G'(Fx_n, Fx_m, Fx_m) < \epsilon \quad (27)
\]
whenever \( m, n \geq N \) which proves that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \). Now since \((X, G')\) is complete there exists \( x \in B(x_0, r)^{G'} \) with \( G'(x_n, x, x) \to 0 \) as \( n \to \infty \). We claim that

\[
x = Fx,
\]

First consider the case \( G \neq G' \). Notice

\[
G'(x, Fx, Fx) \leq G'(x, x_n, x_n) + G'(x_n, Fx, Fx)
\]

(28)

Let \( n \to \infty \) and using assumption-3 we have \( G(x, Fx, Fx) = 0 \), and thus \( x = Fx \) in this case. Next suppose that \( G = G' \) that is assumption-2 and assumption-3 are not hold. Then

\[
G(x, Fx, Fx) \leq G(x, x_n, x_n) + G(x_n, Fx, Fx)
\]

\[
\leq G(x, x_n, x_n) + G(Fx_{n-1}, Fx, Fx)
\]

\[
\leq G(x, x_n, x_n) + \alpha G(x_{n-1}, x, x)
\]

as \( n \to \infty \) we obtain \( G(x, Fx, Fx) = 0 \) and that \( x = Fx \). Thus the proof of the theorem is complete. \( \square \)

Now we present an homotopy result for this type of generalized contractions.

**Theorem 4.2.**

Let \((X, G')\) be a complete \( G' \)-metric space and \( G \) another metric on \( X \). Let \( Q \subset X \) be \( G' \)-closed and let \( U \subset X \) be \( G \)-open and \( U \subset Q \). Suppose \( H : Q \times [0, 1] \to X \) satisfies the following five properties:

(i) \( x \neq H(x, \lambda) \) for \( x \in Q \setminus U \) and \( \lambda \in [0, 1] \).

(ii) for any \( \lambda \in [0, 1] \) and \( x, y \in Q \) we have

\[
G(H(x, \lambda), H(y, \lambda), H(z, \lambda)) \leq \alpha \max\{G(x, y, z), G(x, H(x, \lambda), H(x, \lambda)), G(y, H(y, \lambda), H(y, \lambda)), G(z, H(z, \lambda), H(z, \lambda))\}
\]

(29)

where \( 0 \leq \alpha < 1 \),

(iii) \( H(x, \lambda) \) is continuous in \( \lambda \) with respect to \( G \), uniformly for \( x \in Q \),

(iv) if \( G < G' \) assume \( H \) is uniformly continuous form \( U \times [0, 1] \) endowed with the \( G \)-metric \( G \) on \( U \) into \((X, G')\),

(v) if \( G \neq G' \) assume \( H \) is continuous from \( Q \times [0, 1] \) endowed with the \( G \)-metric \( G' \) on \( Q \) into \((X, G')\).

In addition assume \( H_0 \) has a fixed point. Then for each \( \lambda \in [0, 1] \) we have that \( H_\lambda \) has a fixed point \( x_\lambda \in U \) (here \( H_\lambda(.) = H(., \lambda) \)).

**Proof.** Let

\[
A := \{ \lambda \in [0, 1] ; \text{there exists } x \in U \text{ such that } H(x, \lambda) = x \}.
\]

Since \( H_0 \) has a fixed point and (i) holds we have \( 0 \in A \), and so the set \( A \) is homotopy. We will show \( A \) is open and closed in \([0, 1]\) and so by the connectedness of \([0, 1]\) we have \( A = [0, 1] \) and the proof is finished

First we show that \( A \) is closed in \([0, 1]\).

Let \( (\lambda_k) \) be a sequence in \( A \) with \( \lambda_k \to \lambda \in [0, 1] \) as \( k \to \infty \). By definition of \( A \) for each \( k \), there exists \( x_k \in U \) such that \( x_k = H(x_k, \lambda_k) \). Now we have

\[
G(x_k, x_j, x_j) = G(H(x_k, \lambda_k), H(x_j, \lambda_j), H(x_j, \lambda_j))
\]

\[
\leq G(H(x_k, \lambda_k), H(x_j, \lambda_k), H(x_k, \lambda_k))
\]

\[
+ G(H(x_k, \lambda_k), H(x_k, \lambda_j), H(x_j, \lambda_j))
\]

\[
\leq G(H(x_k, \lambda_k), H(x_k, \lambda_k), H(x_k, \lambda_j)) + \alpha G(x_k, x_j, x_j)
\]

\[
\leq (1 - \alpha)^{-1} G(H(x_k, \lambda_k), H(x_k, \lambda_k), H(x_k, \lambda_j)).
\]

and (iii) guarantees that \( \{x_k\} \) is a Cauchy sequence with respect to \( G \). We claim that \( \{x_n\} \) is a Cauchy sequence with respect to \( G' \).

If \( G \geq G' \) this is trivial. If \( G < G' \) then
and (iv) guarantees that \( \{x_n\} \) is a Cauchy sequence with respect to \( G^{'}. \) (note as well that \( \{x_n\} \) is a Cauchy sequence with respect to \( G \) and \( \{\lambda_k\} \) is Cauchy sequence in\([0, 1]) \). Now since \((X, G^{'})\) is complete there exists an \( x \in Q \) such that \( G^{'}(x, x, x) \to 0 \) as \( n \to \infty \). Claim now that \( x = H(x, \lambda) \).

We consider first case \( G \neq G^{'} \). Then

\[
G^{'}(x, H(x, \lambda), H(x, \lambda)) \leq G^{'}(x, x_k, x_k) + G^{'}(x_k, H(x, \lambda), H(x, \lambda))
\]

\[
= G^{'}(x, x_k, x_k) + G^{'}(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
\]

\[
\leq G(x, x_k, x_k) + G(H(x_k, \lambda_k), H(x, \lambda), H(x, \lambda))
\]

\[
\quad + G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda)) + G(H(x, \lambda_k), H(x, \lambda), H(x, \lambda)).
\]

Letting \( k \to 0 \) and using (iii) we have

\[
G(x, H(x, \lambda), H(x, \lambda)) \leq 0.
\]

So we have

\[
G(x, H(x, \lambda), H(x, \lambda)) = 0,
\]

that is \( x = H(x, \lambda) \). Now form (i) we have \( x \in U \). Consequently \( \lambda \in A \) and so \( A \) is closed in \([0, 1]) \.

We prove now \( A \) is open in \([0, 1]) \.

Let \( \lambda_0 \in A \) and \( x_0 \in U \) such that \( x_0 = H(x_0, \lambda_0) \). From \( U \) \( G \)-open there exists a \( G \)-ball

\[
B(x_0, \delta) = \{ x \in X; G(x, x_0, x_0) < \delta \}, \delta > 0
\]

and

\[
B(x_0, \delta) \subseteq U.
\]

From (iii) we have that \( H \) is uniformly continuous on \( B(x_0, \delta) \).

Let \( \epsilon = (1 - a)\delta > 0 \) and using the uniform continuity of \( H \) we have there exists \( \eta = \eta(\delta) > 0 \) such that for each \( \delta \in [0, 1], |\delta - \delta_0| \leq \eta \) with

\[
G(H(x, \lambda), H(x, \lambda_0), H(x, \lambda_0) < \epsilon
\]

for any \( x \in B(x_0, \delta) \). So this property holds for \( x = x_0, \) and then we have

\[
G(x_0, H(x_0, \lambda), H(x_0, \lambda)) = G(H(x_0, \lambda), H(x_0, \lambda), H(x_0, \lambda)) < (1 - a)\delta
\]

for \( \lambda \in [0, 1] \) and \( |\lambda - \lambda_0| \leq \eta \).

Using (ii), (iv) and (v) together with the Theorem 4.1 (in the case \( r = \delta \) and \( F = H_j ) \) we get; there exists \( x_k \in B(x_0, \delta) \cap Q \) with \( x_k = H_j(x_k) \) for \( k \in [0, 1] \) and \( |\lambda - \lambda_0| \leq \eta \). But \( x_k \in U \) (i) guarantees that \) and so \( A \) contains all \( \lambda \in [0, 1] \) with \( |\lambda - \lambda_0| \leq \eta \). Consequently \( A \) is open in \([0, 1]) \.

Following result can be easily obtained from Theorem 4.1
Theorem 4.3.
Let $(X, G')$ be a complete G-metric space, $G$ another G-metric on $X$, $x_0 \in X$, $r > 0$ and $F : X \to X$. Suppose for any $x, y, z \in X$ we have

$$G(Fx, Fy, Fz) \leq \alpha \max\{G(x, y, z), G(x, Fx, Fx),$$
$$G(y, Fy, Fy), G(z, Fz, Fz)\},$$

where $0 \leq \alpha < 1$.

In addition assume the following three properties hold:

1. $$G(x_0, Fx_0, Fx_0) < (1 - \alpha) r,$$

2. if $G < G'$ then $F$ is uniformly continuous form $(X, G')$ into $(X, G')$.

3. if $G \neq G'$ then $F$ is continuous from $(X, G')$ into $(X, G')$.

Then $F$ has a fixed point.

References