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Solving problems with the help of computers: A fuzzy logic approach

Research Article

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Abstract: In the present paper we develop a fuzzy model for the Problem Solving (PS) process by representing its main stages as fuzzy subsets of a set of linguistic labels characterizing the solvers' performance in each of these stages and we use the total possibilistic uncertainty of the corresponding fuzzy system as a measure of the individuals' PS skills. Classroom experiments are also provided illustrating our results in practice. The outcomes of these experiments suggest that the use of computers as a tool for PS enhances the students' abilities in solving real world problems. This has been crossed by us and by other researchers in earlier papers.

MSC: 03E72 • 68T20

Keywords: Problem solving • Computational thinking • Fuzzy sets • Measures of uncertainty

1. Introduction

The importance of Problem Solving (PS) has been realised for such a long time that in a direct or indirect way affects nowadays our daily lives. Volumes of research have been written about PS and attempts have been made by many educationists and psychologists to make it accessible to all in various degrees (e.g. see [1]). Several definitions have been given during the years about PS. Among all these definitions perhaps Martinez's [2] definition carries the modern message about PS: "PS can be defined as the pursuit of a goal when the path to that goal is uncertain. In other words, it's what you do when you don't know what you're doing".

The PS process is a complicated situation. In fact, it is the human mind at the end that has to be applied in a problematic situation and solve the problem. But human thinking can vary from a very simple and mundane thought to a very sophisticated and complex one. The nature of the problem dictates the level of thinking. The higher-order thinking, termed as critical thinking, involves abstraction, uncertainty, application of multiple criteria, reflection, and self-regulation. The complexity of critical thinking is evident from the fact that there is no definition that is universally accepted. At any case, a great number of critical thinking skills as identified by are agreed upon by many authors. Some of these skills are: analysis and synthesis, making judgements, decision making, reaching to warranted conclusions and generalisations, etc (e.g. see [3]).

With the explosion of information technology and moving away from an industrial society to a knowledge society, the attitude to think critically became a prerequisite (necessary condition) for solving non-routine problems. However, it is not always a sufficient condition too, especially when tackling complicated technological problems of our everyday life, where computers are frequently used as a supporting tool. In this case the need for computational thinking (CT) is another prerequisite for PS. Computation is an increasingly essential tool for doing scientific

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research. It is expected that future generations of scientists and engineers will need to engage and understand computing in order to work effectively with computational systems, technologies and methodologies. CT, named for its extensive use of computer science techniques [4], is a type of analytical thinking that employs mathematical and engineering thinking to understand and solve complex problems within the constraints of the real world. The main characteristics of CT include:

- Analyzing and logically organizing data
- Data modelling, data abstractions, and simulations
- Formulating problems such that computers may assist
- Identifying, testing, and implementing possible solutions
- Automating solutions via algorithmic thinking
- Generalizing and applying this process to other problems

The relationship between CT and critical thinking, the two modes of thinking in solving problems, has not been clearly established yet. In a recent paper we attempted to shed some light into this relationship [5]. The conclusions of the above study can be summarized with the help of Fig. 1, where a 3 - dimensional model for the PS process is presented. According to this model the existing knowledge serves as the connecting tool between critical and computational thinking, while the problem's solution appears to be the "product" of a simultaneous application of the above three components (knowledge, critical and computational thinking) to the PS process. This approach is based on the hypothesis that, when the already existing knowledge is adequate, the necessary for the problem's solution new knowledge is obtained through critical thinking, while CT is applied to design and to obtain the solution.

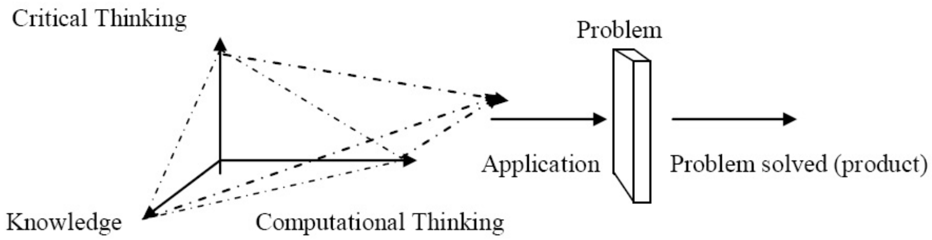

Fig. 1. The 3-dimensional model for the PS process

The type of each problem dictates the order of the application of the above three components, which (order) can have in certain, relatively simple, cases the linear form of Fig. 2. The above model can be used in formulating the

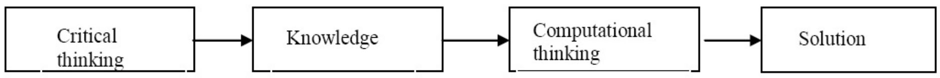

Fig. 2. The linear PS model

PS process of the complex problems of our everyday life and especially of the complicated technological problems.

The present paper aims at using principles of fuzzy logic to develop a mathematical model representing the PS process and at obtaining a fuzzy measure for assessing the students' CT skills. The text is organized as follows: In the next section we develop our fuzzy model for PS, while in the third section we establish a fuzzy measure of uncertainty that will be used in assessing students' PS skills. In the fourth section two classroom experiments, performed recently at the Graduate Technological Educational Institute of Western Greece, are presented illustrating the use of our model in practice. Finally our last section (fifth) is devoted to our final conclusions and to a short discussion about our future plans for further research on the subject.

2. The fuzzy model for the PS process

Carlson and Bloom [6] drawing from the large amount of literature related to PS developed a broad taxonomy to characterize major PS attributes that have been identifying as relevant to PS success. This taxonomy gave genesis to their "Multidimensional Problem-Solving Framework" (MPSF). The MPSF includes four phases: Orientation, Planning, Executing and Checking. Carlson and Bloom [6] observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases the solvers obtained linearly the solution of a problem (i.e. they ran through this cycle only once). Thus embedded in MPSF are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases (planning, executing and checking). It has been also observed that, when contemplating various solution approaches during the planning phase of the PS process, the solvers were at times engaged in a conjecture-imagine-evaluate (accept/reject) sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning. For more details about MPSF and a scheme giving a graphical representation of the PS cycle (Figure 1, p. 54) see [6].

Notice that the MPSF, as well as similar models for the PS process developed earlier by other researchers (e.g. Polya [7], Schoenfeld [8, 9], Voskoglou, & Perdikaris [10], etc) are helpful in understanding the ideal behaviour of a problem-solver. However the reality is not like that. Recent research on problems of mathematical modelling reports that students in school take individual routes when tackling these problems, associated with their individual learning styles [11–13], etc). On the other hand, from the teacher's point of view there usually exists an uncertainty about the degree of success of students in each of the stages of the PS process. All these gave us the impulsion to introduce principles of *Fuzzy Logic* (FL) and of Uncertainty theory for representing in a more realistic way the PS process. For general facts on fuzzy sets and on uncertainty theory we refer to the book of Klir and Folger [14]. The definition of fuzzy sets and some basic but useful comments related to it can be also found in the Introduction of [15], where we have used a different FL approach (the center of gravity defuzzification technique) for assessing the bridge players' performance.

Our starting point for the development of our FL model for the PS process will be the Carlson's and Bloom's MPSF, but in order to make the presentation of our model technically simpler, we shall consider the first two stages of MPSF, i.e. orientation and planning as a joint stage. In fact, orientation, although it needs some attention, is actually the preliminary stage of the PS process, which in forming the *assumed real system*[1] could be considered without loss of generality as a sub-stage of planning.

Let us now consider a group of n students, $n \geqslant 2$, during the PS process in classroom. Denote by $S_i, i = 1,2,3$ the stages of orientation/planning, executing and checking of the MPSF and by a, b, c, d, and e the linguistic labels of negligible, low, intermediate, high and very high success respectively of a student in each of the S_i's. Set $U = \{a, b, c, d, e\}$. We are going to correspond to each stage S_i a fuzzy subset A_i of U. We are going to correspond to each stage S_ia fuzzy subset A_i of U. For this, if $n_{ia}, n_{ib}, n_{ic}, n_{id}$ and n_{ie} denote the number of students that faced negligible, low, intermediate, high and complete success at stage S_i respectively, $i = 1,2,3$, we define the *membership function* m_{A_i} for each x in U, as follows:

$$m_{A_i}(x) = \begin{cases} 1 & if \quad \frac{4n}{5} < n_{ix} \leqslant n \\ 0.75 & if \quad \frac{3n}{5} < n_{ix} \leqslant \frac{4n}{5} \\ 0.5, & if \quad \frac{2n}{5} < n_{ix} \leqslant \frac{3n}{5} \\ 0.25 & if \quad \frac{n}{5} < n_{ix} \leqslant \frac{2n}{5} \\ 0 & if \quad 0 \leqslant n_{ix} \leqslant \frac{n}{5} \end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form : $A_i = \{(x, m_{A_i}(x)) : x \in U\}, \quad i = 1,2,3.$

In order to represent all possible *student profiles (overall states)* during the PS process we consider a fuzzy relation, say R, in U^3 (i.e. a fuzzy subset of U^3) of the form: $R = \{(s, m_R(s)) : s = (x, y, z) \in U^3\}$.
To determine properly the membership function m_R in the above fuzzy relation we give the following definition:

Definition 2.1.

A profile $s = (x, y, z)$, with x, y, z in U, is said to be well ordered if x corresponds to a degree of success equal or greater than y, and y corresponds to a degree of success equal or greater than z.

[1] *With the term assumed real system we mean a simplified form of the real system, which is ready for mathematical amendment ([16], section 3)*

For example, (c, c, a) is a well ordered profile, while (b, a, c) is not.

We define now the *membership degree* of a profile s to be $m_g(s) = m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$, if s is well ordered, and zero otherwise. We emphasize that according to the above definition a well ordered profile could have membership degree zero if at least one of the three components $m_{A_1}(x), m_{A_2}(y)$ and $m_{A_3}(z)$is zero, but a profile which is not well ordered has always membership degree zero, even if all the above components are non zero. It is logical to accept this, because, if for example profile (b, a, c) possessed a nonzero membership degree, then it could be possible for a student, who has failed during the executing stage (i.e. who has not obtained a correct solution), to check satisfactorily the solution obtained.

Next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the probability p_s of the profile s can be defined in a way analogous to crisp data, i.e. by

$$p_s = \frac{m_s}{\sum\limits_{s \in U^3} m_s}$$

However, according to Shackle [17] and many other researchers after him, human reasoning can be formalized more adequately by possibility rather, than by probability theory. The possibility r_s of s is defined by

$$r_s = \frac{m_s}{\max\{m_s\}}$$

where $\max\{m_s\}$ denotes the maximal value of m_s , for all s in U^3. In other words, the possibility of the profile s expresses the relative membership degree of s with respect to $\max\{m_s\}$. Following Shackle's view we shall use here the possibilities instead of probabilities of the students' profiles.

In several cases it is useful to study the combined results of behaviour of k different student groups, $k \geqslant 2$, during the solution of the same problems, e.g. when we want to study the performance of the students of a school in terms of its classes, etc. For this we introduce the *fuzzy variables* $A_1(t), A_2(t)$ and $A_3(t)$, with $i = 1, 2, 3 ... k$. The values of these variables represent fuzzy subsets of U corresponding to the stages of the PS for each of the k student groups; e.g. $A_1(2)$ represents the fuzzy subset of U corresponding to the stage of planning for the second group ($t = 2$). More explicitly, a fuzzy variable is characterized in general by a triple of the form $(t, U, A(t))$, where t is its name, U is the set of the discourse, A is a fuzzy subset of U and $A(t)$ is also a fuzzy subset of U that represents the restriction imposed by t on A [18].

It becomes evident that, in order to measure the degree of evidence of the combined results of the k groups, it is necessary to define the possibility $r(s)$ of each student profile s with respect to the membership degrees of s for all student groups. For this, we introduce the *pseudo-frequencies* $f(s) = \sum\limits_{t=1}^{k} m_s(t)$ and we define $r(s) = \frac{f(s)}{\max\{f(s)\}}$ where $\max\{f(s)\}$ denotes the maximal pseudo-frequency. Obviously the same method could be applied when one wants to study the behaviour of a student group during the solution of k different problems.

3. Measuring the uncertainty in PS

In classical probability and information theory a system's uncertainty is measured by the Shannon's entropy [19]. For use in a fuzzy environment this measure is expressed in the form ([20], pp. 20):

$$H = -\frac{1}{\ln n} \sum_{s=1}^{n} m_s \ln m_s$$

where n denotes the total number of the system's entities involved in the corresponding process. The sum in the above formula is divided by $\ln n$ (the natural logarithm of n) in order to be normalized. Thus H takes values in the real interval [0, 1]. The value of H measures the *system's total probabilistic uncertainty* and the associated to it information. As it becomes evident from the above formula, H is calculated directly from the membership degrees of all profiles s without being necessary to calculate their probabilities p_s first.

However, as said in the previous section, here we shall make use of the possibility and not of the probability theory. Within the domain of possibility theory uncertainty is distinguished to *strife* (or *discord*), which expresses conflicts among the various sets of alternatives, and to *non-specificity* (or imprecision), which expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([20], p.28). For a better intuitive understanding of the above two types of uncertainty we give the following simple example:

Example 3.1.
Let U be the set of the humans' ages, whose elements are the integers from 0 to 150 and let Y = young, A = adult and O = old (these characterizations are given according to each person's outer appearance) be fuzzy subsets of U defined by the membership functions m_Y, m_A and m_O respectively. Then, given x in U, there usually exists a degree of uncertainty about the reasonable values that the membership degrees $m_Y(x), m_A(x)$ and $m_O(x)$ could take, resulting to a conflict among the fuzzy subsets Y, A and O of U.

For example, if $x = 1$, then there is no uncertainty at all, since in this caser we must take $m_Y(x) = 1$ and $m_A(x) = m_O(x) = 0$. On the contrary, if for example $x = 18$, then $m_O(x) = 0$, but there is an uncertainty about the values of $m_Y(x)$ and $m_A(x)$. In fact, values like $m_Y(x) = 0.8$ and $m_A(x) = 0.3$ seem to be close to the reality, but they are not the only ones, e.g. we could also accept the values like $m_Y(x) = 0.7$ and like $m_A(x) = 0.5$, etc. [2] The existing conflict is higher, if, for example, $x = 50$. In fact, in this case we could have $m_Y(x) \neq 0$, since sometimes people being 50 years old look much younger than others aged 40 or even 30 years. However, there exist also people aged 50 who look older from others aged 70, or even 80 years, which means that we could also have $m_Y(x) = 0$. All the above are examples of the type of uncertainty that we have called strife or discord.

On the other hand the non-specificity (or imprecision) is connected to the question: How many x in U should have non zero membership degrees in Y, A and O respectively? In other words, the existing in this case uncertainty creates a conflict among the 'sizes' of the fuzzy subsets of U. Notice that the cardinality of a fuzzy subset, say B, of U is defined $\sum_{x\in U} m_B(x)$ of all membership degrees of the elements of U in B. Therefore, non-specificity is actually connected with a conflict among the cardinalities of the fuzzy subsets of U.

Strife is measured by the function $ST(r)$ on the ordered possibility distribution $r : r_1 = 1 \geqslant r_2 \geqslant \ldots. r_n \geqslant r_{n+1}$ of the student group defined by ([20], p.28)

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{m}(r_i - r_{i+1})\log\left(\frac{i}{\sum_{j=1}^{i} r_j}\right)\right]$$

In the same way non-specificity is measured by the function ([20], p.28)

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{m}(r_i - r_{i+1})\log i\right]$$

Thus, the sum $T(r) = ST(r) + N(r)$ measures the *total possibilistic uncertainty* for ordered possibility distributions. But, as it is well known from the classical information theory, the amount of information obtained by an action can be measured by the reduction of uncertainty resulting from this action (e.g. see [19]). In other words the amount of uncertainty regarding some situation represents the total amount of potential information in this situation. Accordingly students' uncertainty during the PS process is connected to students' capacity in obtaining relevant information. Therefore the existing total possibilistic uncertainty $T(r)$ in a student group after the end of the PS process can be adopted as a measure of students' PS skills. The lower is the value of $T(r)$, which means greater reduction of the initially existing uncertainty, the better the student group's performance.

4. Applications of the fuzzy model in classroom

Exploratory investigations have demonstrated how exposure to computers enhances the students' abilities for approaching real world problems ([5], [21-23], etc). In order to investigate further the above fact, but also to illustrate the use of our fuzzy model in practice, we performed recently the following two experiments in which the subjects were students of the Graduate Technological Educational Institute ($T.E.I.$) of Western Greece.

Notice that the $T.E.I.$'s are relatively new institutions of Higher Education in Greece, founded in 1983. However, a number of their Departments (like the Departments of Physiotherapy, Nursing, Graphic Arts and Design, Accounting, etc) offering more practical specializations and hence better opportunities to find a job, have become nowadays more popular than most of the Departments of the Schools of Technological Applications (prospective engineers),

[2] *We recall that the sum of the membership degrees $m_Y(x) + m_A(x) + m_O(x)$ need not be equal to 1, as it happens with the probabilities*

which are similar to the corresponding Departments of the traditional Polytechnic Schools of the Universities. As a result, the minimum grades of the Panhellenic Exams required for entrance in the Engineering Departments of the $T.E,I.$'s are at least 10-20 % lower than the corresponding bases of the above popular Departments (for more details about the structure of the Higher Education in Greece see for example [25]).
In the first of the above mentioned experiments our subjects were 35 students of the Department of Accounting of the School of Management and Economics being at their first term of studies. The lectures in the mathematical courses for the students of this School are performed in the classical way on the board including a number of exercises and of problems connecting mathematics with real world applications. The students themselves participate to the solution procedure of these exercises and problems. The basic tool for performing our experiment was a list of 10 problems involving mathematical modelling (see Appendix) that were given to the students for solution (time allowed 3 hours). The mathematical topics related to these problems (elementary and linear algebra, differential and integral calculus, elementary differential equations and probability theory) had been taught at the mathematics course of the first term.
Before starting the experiment we gave the proper instructions to students emphasizing among the others that we are interested for all their efforts (successful or not) during the PS process, and therefore they must keep records on their papers for all of them. Further, and in order to prevent the students' deliberate and /or unconscious bias towards recording the PS process in the required way, we clarified to them that in assessing their papers we shall take under consideration not only their final results (correct or not), but also their ability to describe accurately their previous (if any) unsuccessful attempts. This manipulation enabled us in obtaining more realistic data from our experiment for each stage of the PS process than those based on students' final results only. The following sample from the students' papers concerning the solution of Problem 10 of our list (see Appendix) is an example illustrating the positive way that students responded in most cases to our instructions and the way used for evaluating the data obtained from their responses:

"The area of the surface of a cylindrical tower is given by $E = 2\pi R\upsilon$, where R denotes the radius of its base and ÏĚ denotes its height. Also the volume of the tower is given by $V = \pi R^2\upsilon = 180\pi$. Therefore $\upsilon = \frac{90}{R}$ and $V(R) = 90\pi R$. Taking the derivative $V'(R) = 90\pi \neq 0$...? WRONG! The areas of the roof and of the ground floor of the tower must be also added in calculating the area of its total surface. We have $E = 2\pi R\upsilon + 2\pi R^2 = 180\pi$, therefore $\upsilon = \frac{90-R^2}{R}$ and $V = V(R) = \pi R^2 \frac{90-R^2}{R} = 90\pi R - \pi R^3$. Taking the derivative $V'(R) = 90\pi - 3\pi R^2 = 0$ we $R = \sqrt{30}m$ (the negative root is rejected) and $\upsilon = \frac{60}{\sqrt{30}} = 2\sqrt{30}m$"

Using a scale from 0 to 10 the mark awarded to this student for the solution of Problem 10 was 8, since he did not check that $V''(\sqrt{30}) < 0$ before giving his final answer. The overall mark for each student was the mean of the marks awarded for each of the 10 problems. Apart from the mark and for the needs of our experiment we kept the following record for the above student's PS procedure in Problem 10: S_1 = correct after one wrong attempt, S_2 = correct and S_3 = wrong.

Remark 4.1.

The student's first attempt, although mathematically was wrong, it had an intuitive basis. In fact, one can hardly include to the surface of a building its roof and its ground floor. Some other students included only the roof of the tower and they found another solution, which was considered to be correct (the mark awarded in those cases was 7).

For describing the criteria used for the characterization of the students' performance at **each stage** of the PS process let us denote by C the total number of a student's correct attempts and by W the total number of his/her wrong attempts at a certain stage of the PS process when solving the 10 problems of the given list. Then a student was considered to have at this particular stage:

- Negligible success, if $C \leqslant 1$.
- Low success, if $1 < C \leqslant 4$.
- Intermediate success, if $4 < C \leqslant 8$ and $C < W$.
- High success, if $4 < C \leqslant 8$ and $C \geqslant W$.
- Very high success, if $8 < C \leqslant 10$.

It is reasonable to accept that the above criteria give a realistic representation of the students' performance, which is based on their total attempts and not only on their final results during the PS process. We emphasize once more that using these criteria we didn't consider each of the 10 problems as a whole; instead, we examined separately the students' progress at each stage of the PS process for each problem.

Inspecting students' papers we found with respect to the above criteria that 15, 12 and 8 students had intermediate, high and very high success respectively at the stage of orientation/planning. Therefore we obtained that $n_{1a} = n_{1b} = 0$, $n_{1c} = 15$, $n_{1d} = 12$ and $n_{1e} = 8$. Thus, by the definition of $m_{A_i}(x)$, orientation/planning corresponds to a fuzzy subset of U of the form:$A_1 = \{(a,0),(b,0),(c,\ 0.5),(d,\ 0.25),(e,\ 0.25)\}$. In the same way we represented the stages of executing and checking as fuzzy sets in U by $A_2 = \{(a,0),(b,0),(c,\ 0.5),(d,\ 0.25),(e,0)\}$ and $A_3 = \{(a,\ 0.25),(b,\ 0.25),(c,\ 0.25),(d,0),(e,0)\}$ respectively.

Next, using the definition given in the previous section we calculated the membership degrees of the 5^3(number of the ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles (see column of $m_s(1)$ in table 1). For example, for $s = (c,c,a)$ one finds that $m_s = m(c).\ m(c).\ m(a) = (0.5).(0.5).(0.25) = 0.06225$. It turns out that (c,c,a) was one of the profiles of maximal membership degree and therefore the possibility of each s in U^3 is given by $r_s = \frac{m_s}{0{,}06225}$. Calculating the possibilities of all profiles (see column of rs(1) in Table 1) one finds that the ordered possibility distribution for the student group is:
$r_1 = r_2 = 1, r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0{,}5, r_9 = r_{10} = r_{11} = r_{12} = r_{13} = r_{14} = 0{,}258,\ r_{15} = r_{16} = \ldots\ldots.. = r_{125} = 0.$
Thus with the help of a calculator we found that

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{14}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right] \approx \frac{1}{0.301}\left[0.5\log\frac{2}{2} + 0.242\log\frac{8}{5} + 0.258\log\frac{14}{6.548}\right]$$
$$\approx 3.32\,[(0{,}242).(0.204) + (0.258).(0.33)] \approx 0.445$$

and

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{14}(r_i - r_{i+1})\log i\right] = \frac{1}{\log 2}\left[0.5\log 2 + 0.242\log 8 + 0.258\log 14\right]$$
$$\approx .5 + 3.(0.242) + (0.857).1.146 \approx 2.208$$

Therefore we finally obtained that $T(r) \approx 2.653$.

A few days later we performed the same experiment with a group of 30 students from the Department of Civil Engineering of the School of Technological Applications being also at their first term of studies. The topics taught at the mathematics course of the first term for this Department were the same with those taught (by the same instructor, four hours/week for both Departments) to the students of the Department of Accounting. The only difference was that about the 1/3 of the lectures and the exercises of mathematical courses for the students of the School of Technological Applications are performed in a computer laboratory, where the instructor presents the corresponding mathematical topics with the help of computers, while the students themselves, divided in small groups, use known mathematical software to solve the problems.

In a way analogous to the first experiment we found that
$A_1 = \{(a,0),(b,\ 0.25),(c,\ 0.5),(d,\ 0.25),(e,0)\}$, $A_2 = \{(a,0.25),(b,0.25),(c,0.5),(d,\ 0),(e,0)\}$ and
$A_3 = \{(a,0.25),(b,0.25),(c,0.25),(d,0),(e,0)\}$.
Then we calculated the membership degrees of all possible profiles of the student group (see column of $m_s(2)$ in table 1). As it turned out the maximal membership degree was again 0.06225, therefore the possibility of each profile s is given by the same formula as for the first group. Calculating the possibilities of all profiles (see column of $r_s(2)$ in Table 1) we found that the ordered possibility distribution of the second group is:

$$r : r_1 = r_2 = 1, r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0{,}5, r_9 = r_{10} = r_{11} = r_{12} = r_{13} = 0{,}258, r_{14} = r_{15} = \ldots\ldots = r_{125} = 0$$

Further, working as in the previous experiment we found that $T(r) = 0.432 + 2.179 = 2.611$. Thus, since $2.611 < 2.653$, it turns out that the second group had in general a slightly better performance than the first one. This, combined to the fact that: i) The students of the first group (Accounting) obtained better grades in general at the Panhellenic exams for entrance in higher education, and ii) The students of both groups had the same exposure to computational tools prior to the experiment (in high school), gives an indication (not very strong because of the small difference between the two groups performance) that the use of computers as a tool for PS enhances the students' abilities in solving real world problems.

Finally, for studying the combined results of behaviours of the two groups, we introduced the fuzzy variables $A_i(t), i = 1,2,3$ and $t = 1,2$, as we have described in the previous section. Then the pseudo-frequency of each student profile s is given by $f(s) = m_s(1) + m_s(2)$ (see corresponding column in Table 1). It turns out that the highest pseudo-frequency is 0.124 and therefore the possibility of each student's profile is given by

$$r(s) = \frac{f(s)}{0.124}$$

The possibilities of all profiles having non-zero pseudo-frequencies are presented in the last column of Table 1. The outcomes in Table 1 are with accuracy up to the third decimal point.

Table 1. Profiles with non zero membership degrees

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	$F(s)$	$r(s)$
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0.031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	b	a	0	0	0.016	0.258	0.016	0.129
d	b	b	0	0	0.016	0.258	0.016	0.129
d	c	a	0.031	0.5	0.031	0.5	0.062	0.5
d	c	b	0.031	0.5	0.031	0.5	0.062	0.5
d	c	c	0.031	0.5	0.031	0.5	0.062	0.5
e	c	a	0.031	0.5	0	0	0.031	0.25
e	c	b	0.031	0.5	0	0	0.031	0.25
e	c	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.258	0	0	0.016	0.129
e	d	b	0.016	0.258	0	0	0.016	0.129
e	d	c	0.016	0.258	0	0	0.016	0.129

5. Conclusions and discussion

The study performed in this paper can be summarized with the following conclusions:

- We developed a fuzzy model for PS representing the main stages of the PS process as fuzzy subsets of a set of linguistic labels characterizing the solvers' performance at each of these stages We also used the total possibilistic uncertainty as a tool for measuring the students' PS abilities.
- Our model, apart from the quantitative information (possibilities, value of the total possibilistic uncertainty, etc), gives also through the study of all students' profiles a qualitative view of their performance during the PS process. Another advantage of it is that it gives the opportunity to the user for studying the combined results of performance of two or more groups of students in solving the same problems, or alternatively the performance of the same group in solving different problems.
- On the other hand, the characterization of the problem solvers' performance in terms of a set of linguistic labels which have no clear boundaries is a disadvantage of our fuzzy model, since it depends on the user's personal criteria, which must be carefully checked for their credibility before applying them in practice.
- Two classroom experiments, performed recently at the Graduate Technological Educational Institute of Western Greece, were also presented illustrating the use of our model in practice. The outcomes of the above experiments, combined with the results of analogous experiments performed by the author and by other researchers, and reported in earlier papers, suggest that the use of computers as a tool for PS enhances the students' abilities in solving real world problems.

One could claim that our fuzzy model is more useful than the traditional ones [6-9] for a deeper study and understanding of the PS process. However, this claim, based on qualitative criteria only, is subject to quantitative (statistical) verification. This is one of our future priorities through further empirical research and the comparison of the outcomes of our fuzzy method with the corresponding outcomes of traditional measuring methods that have or can be applied for PS.

Also further research is needed for the PS process itself. In fact, although many studies have investigated and compared the characteristics of novice and expert problem solvers (e.g. [6], [9], [25-26], etc), some of the qualitative differences appearing among them still do not appear to be completely understood. It is hoped that the use of our fuzzy model as a tool in future experimental research on PS could lead to practical ways of restoring the weaknesses appearing to novices with respect to the expert problem solvers.

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Appendix

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20 cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water ?

Problem 2: A car dealer has a mean annual demand of 250 cars, while he receives 30 new cars per month. The annual cost of storing a car is 100 euros and each time he makes a new order he pays an extra amount of 2200 euros for general expenses (transportation, insurance etc). The first cars of a new order arrive at the time when the last car of the previous order has been sold. How many cars must he order in order to achieve the minimum total cost ?

Problem 3: An importation company codes the messages for the arrivals of its orders in terms of characters consisting of a combination of the binary elements 0 and 1. If it is known that the arrival of a certain order will take place from 1st until the 16^{th} of March, find the minimal number of the binary elements of each character required for coding this message.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet ($A = 1, B = 2, C = 3$ etc). Let us correspond also to each word consisting of 4 letters a 2×2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix $E = \begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message ?

Problem 5: The demand function $P(Q_d) = 25 - {Q_d}^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s) = 2Q_s + 1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0, P_0), the producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour ?

Problem 8: The rate of increase of the population of a country is analogous to the number of its inhabitants. If the population is doubled in 50 years, in how many years it will be tripled ?
(**Answer:** In $50\frac{\ln 3}{\ln 2} \approx 79$ years)

Problem 9: A company circulates for first time in market a new product, say K. Market's research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week. i) Find the market's share for K two weeks after its first circulation, provided that the market's conditions remain unchanged. ii) Find the market's share for K in the long run, i.e. when the consumers' preferences will be stabilized.

Problem 10: Among all the cylindrical towers having a total surface of $180\pi m^2$, which one has the maximal volume?