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Rate of convergence for some linear positive operators for bounded variation functions

Research Article

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Abstract: Gupta and Ahmad [1] introduced the modified Beta operators $B_n(f,x)$ and estimated some direct results in simultaneous approximation. In the present paper, we study certain integral modification of the well known modified Beta-Stancu operators with the weight function of Beta basis function. We establish rate of convergence for these operators for functions having derivatives of bounded variation.

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1. Introduction

Gupta and Ahmad [1] defined modified Beta operators so as to approximation Lebesque integrable function on $[0,\infty)$ as

$$B_n(f,x)=\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)dt,\qquad x\in[0,\infty) \tag{1}$$

where

$$b_{n,k}(x)=\frac{(n+k)!}{k!(n-1)!}\frac{x^k}{(1+x)^{n+k+1}}$$

and

$$p_{n,k}(t)=\frac{(n+k-1)!}{k!(n-1)!}\frac{t^k}{(1+t)^{n+k}} \tag{2}$$

In 1983, the Stancu type generalization of Bernstein operators was given in [2]. In 2010 in [3], the authors have studied the Stancu type generalization of the q-analogue of classical Baskakov operators. On similar type of operators some approximation properties have been discussed by Maheshwari [4], [5] and Mohammad-Abdul Rahman [6] etc. Motivated by the recent work on Stancu type operators, here we propose the Stancu type generalization of modified-Beta Stancu operators, for $0\le\alpha\le\beta$ as

$$B_{n,\alpha,\beta}(f,x)=\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt,\qquad x\in[0,\infty) \tag{3}$$

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Although these operators are similar to the generalized summation integral type operators introduced by Srivastava and Gupta [7], but the approximation properties of the operators (3) are different from those introduced in [7]. Here in summation and integration we are taking different basis function.
We define

$$\beta_n(x,t)=\frac{n-1}{n}\int_0^t\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(s)ds, \tag{4}$$

then we have,

$$\beta_n(x,\infty)=1. \tag{5}$$

In [1], the author studied some direct results in simultaneous approximation for the operators (3). Gupta et al. [8] studied the approximation properties of these operators in $Lp-$ norm and they obtained some direct results for the linear combinations of the operators. The rate of convergence for function having derivatives of bounded variation on Bernstein polynomials were studied by Bojanic and Cheng [9], [10]. Here we continue our studies on these operators and consider the following class of functions.
Let $DB_r(o,\infty)$, where $r\geq 0$ denotes the class of absolutely continuous function f, which is defined on the interval (o,∞), satisfies the condition $f(t)=O(t^r), t\to\infty$ and having a derivative f' on the interval $(0,\infty)$, coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all function $f\in BD_r(0,\infty)$ possess for each real value $c>0$ the equation:

$$f(x)=f(c)+\int_c^x\psi(t)dt,\qquad x\geq c.$$

Gupta-Agrawal [11] studied a certain integral modification of the well-know Baskakov operators with the weight function of Beta basis function and establish the rate of convergence for these operators for functions having derivaties of bounded variation. Gupta et al. [12] obtained an interesting result on the rate of convergence of certain Durrmeyer type operators in ordinary and simultaneous approximation. In [13] some direct local and global approximation theorems were given for the q Bernsein-Durrmeyer operators. Ispir et al. [14] estimated similar results for Kantorovich operators for functions with derivatives of bounded variation. This motivate us to study the modified Beta Stancu operators and estimate the rate of convergence of the operators (3) for functions having derivatives of bounded variation.

2. Moment estimation and auxiliary results

In this section we estimate moments and mention certain basic results.

Lemma 2.1 ([1]).
Let the function $T_{n,m}(x), m\in N\bigcup 0$, we define

$$T_{n,\alpha,\beta,}=\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt, \tag{6}$$

then

$$T_{n,0}(x)=1, T_{n,1}(x)=\frac{x[3n+\beta(2-n)]+n+\alpha(n-2)}{(n-2)(n+\beta)}$$

and

$$\begin{aligned}T_{n,2}(x)=&\frac{x^2(2n^2(n+7)-n^2\beta(7-n)+n\beta(n\beta-5\beta+18)+6\beta^2)}{(n+\beta)^2(n-2)(n-3)}\\&+\frac{x(2n^2(n+5)+2\alpha n^2(5-2n)-2n^2\beta(1+\alpha)+2n(5\alpha\beta+3\beta-9\alpha)-12\alpha\beta]}{(n+\beta)^2(n-2)(n-3)}\\&+\frac{2n^2+n^2\alpha(\alpha+2)-n\alpha(5\alpha+6)+6\alpha^2}{(n+\beta)^2(n-2)(n-3)}\end{aligned} \tag{7}$$

and we have the recurrence relation

$$\begin{aligned}(n-m-2)\left(\frac{n+\beta}{n}\right)T_{n,m+1}(x)=&x(1+x)[T'_{n,m}(x)+mT_{n,m-1}(x)]\\&+\left[(m+nx+1+x)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2m-2)\right]T_{n,m}(x)\\&-\left(\frac{\alpha}{n+\beta}-x\right)\left[1-\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)\right]mT_{n,m-1}(x).\end{aligned} \tag{8}$$

Consequently for each $x\in[0,\infty)$,

$$T_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right). \tag{9}$$

Corollary 2.1.
From Lemma 2.1 and using Cauchy -Schwarz inequality, it follows that

$$B_{n,\alpha,\beta}\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|,x\right)\leq[T_{n,2}(x)]^{1/2}\leq\sqrt{\frac{Ax(x+1)}{n+\beta}} \tag{10}$$

where $A>2$.

Lemma 2.2.
Let $x\in[0,\infty)$, *then for* $A>2$ *and* n *sufficiently large, we have*

$$\begin{aligned}&(i)\quad \beta_{n,\alpha,\beta}(x,y)\leq\frac{Ax(x+1)}{(n+\beta)(x-y)^2},\qquad y\in[0,x),\\ &(ii)\quad 1-\beta_{n,\alpha,\beta}(x,z)\leq\frac{Ax(x+1)}{(n+\beta)(z-x)^2},\qquad z\in[x,\infty).\end{aligned} \tag{11}$$

Proof. First, we prove first inequality by using Lemma 2.1, we get

$$\begin{aligned}\beta_{n,\alpha,\beta}(x,y)=&\frac{n-1}{n}\int_0^y\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\\ \leq&\frac{n-1}{n}\int_0^y\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^2}{(x-y)^2}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\\ \leq&(x-y)^{-2}T_{n,2}(x)\leq\frac{Ax(x+1)}{(n+\beta)(x-y)^2}.\end{aligned} \tag{12}$$

The proof of the second inequality is similar, we omit the details. □

3. Direct estimates

In this section, we prove the following main theorem.

Theorem 3.1.
Let $f\in DB_r(0,\infty)$, $r\in N$, *and* $x\in(0,\infty)$. *Then for* $A>2$ *and for* n *sufficiently large, we have*

$$\begin{aligned}|B_{n,\alpha,\beta}(f,x)-f(x)|\leq&\frac{A(x+1)}{n+\beta}\left(\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}\left(\frac{\partial f}{\partial x}\right)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}\left(\frac{\partial f}{\partial x}\right)\right)\\ &+\frac{A(x+1)}{x(n+\beta)}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\ &+\sqrt{\frac{Ax(x+1)}{n+\beta}}\left(M2^rO(n^{-r/2}+|f'(x^+)|\right)\\ &+\frac{1}{2}\sqrt{\frac{Ax(x+1)}{n+\beta}}|f'(x^+)-f'(x^-)|\\ &+\frac{1}{2}|f'(x^+)-f'(x^-)|\frac{x[3n+\beta(2-n)]+n+\alpha(n-2)}{(n-2)(n+\beta)}\end{aligned} \tag{13}$$

where the auxiliary function $\dfrac{\partial f}{\partial x}$ *is given by*

$$\frac{\partial f}{\partial x}=\begin{cases}f(t)-f(x^-), & t\in[0,x)\\ 0, & t=x\\ f(t)-f(x^+), & 0\in[x,\infty)\end{cases} \tag{14}$$

where $\bigvee_a^b f(x)$ *denotes the total variation of* $\dfrac{\partial f}{\partial x}$ *on* $[a,b]$.

Proof. By the application of mean value theorem, we have

$$B_{n,\alpha,\beta}(f,x)-f(x)=\frac{n-1}{n}\int_0^{\infty}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)\left(f\left(\frac{nt+\alpha}{n+\beta}\right)-f(x)\right)dt$$
$$=\int_0^{\infty}\int_x^t\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)(f'(u)du)dt. \tag{15}$$

Using the identity

$$f'(u)=\frac{1}{2}[f'(x^+)+f'(x^-)]+(f')_x(u)+\frac{1}{2}[f'(x^+)-f'(x^-)]sgn(u-x)$$
$$+\left[f'(x)-\frac{1}{2}[f'(x^+)+f'(x^-)]\right]\chi_x(u), \tag{16}$$

it is easily verified that if we substitute the above $f'(u)$ in (15), the last term of the identity vanishes. Also

$$\frac{n-1}{n}\int_0^{\infty}\left(\int_x^t\frac{1}{2}[f'(x^+)-f'(x^-)]sgn(u-x)du\right)\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt$$
$$=\frac{1}{2}[f'(x^+)-f'(x^-)]B_{n,\alpha,\beta}\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|,x\right),$$
$$\frac{n-1}{n}\int_0^{\infty}\left(\int_x^t\frac{1}{2}[f'(x^+)+f'(x^-)]du\right)\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt$$
$$=\frac{1}{2}[f'(x^+)+f'(x^-)]B_{n,\alpha,\beta}\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|,x\right). \tag{17}$$

Therefore

$$|B_{n,\alpha,\beta}(f,x)-f(x)|\le\left|\int_x^{\infty}\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\right.$$
$$\left.-\int_0^{x}\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\right|$$
$$+\frac{1}{2}|f'(x^+)-f'(x^-)|B_{n,\alpha,\beta}\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|,x\right)$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|B_{n,\alpha,\beta}\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|,x\right). \tag{18}$$

Thus with the application of Lemma 2.1, and corollary 2.1, (18), we have

$$|B_{n,\alpha,\beta}(f,x)-f(x)|\le|C_n(f,x)+D_n(f,x)|+\frac{1}{2}|f'(x^+)-f'(x^-)|+\sqrt{\frac{Ax(x+1)}{n+\beta}}$$
$$+\frac{1}{2}|f'(x^+)+f'(x^-)|\frac{x[3n+\beta(2-n)]+n+\alpha(n-2)}{(n-2)(n+\beta)}. \tag{19}$$

To complete the proof of the theorem, it will be enough to estimate the terms $C_n(f,x)$ and $D_n(f,x)$. Applying integration by parts, using Lemma 2.2 and taking $y=x-\frac{x}{\sqrt{n}}$, we have

$$|D_n(f,x)|=\left|\int_0^x\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)dt\beta_n(x,t)dt\right|$$

$$\int_0^x\beta_n(x,t)\left(\frac{\partial f}{\partial x}\right)(t)dt\le\left(\int_0^y+\int_y^x\right)\left|\left(\frac{\partial f}{\partial x}\right)(t)\right||\beta_n(x,t)|dt$$
$$\le\frac{Ax(x+1)}{n+\beta}\int_0^y\bigvee_t^x\left(\frac{\partial f}{\partial x}\right)\frac{1}{\left(x-\frac{nt+\alpha}{n+\beta}\right)^2}dt+\int_y^x\bigvee_t^x\left(\frac{\partial f}{\partial x}\right)dt$$
$$\le\frac{Ax(x+1)}{n+\beta}\int_0^y\bigvee_t^x\left(\frac{\partial f}{\partial x}\right)\frac{1}{\left(x-\frac{nt+\alpha}{n+\beta}\right)^2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\left(\frac{\partial f}{\partial x}\right). \tag{20}$$

Let $u=\frac{x}{x-t}$. then we have

$$\frac{Ax(x+1)}{n+\beta}\int_0^y \bigvee_t^x\left(\frac{\partial f}{\partial x}\right)\frac{1}{\left(x-\frac{nt+\alpha}{n+\beta}\right)^2}dt=\frac{Ax(x+1)}{n+\beta}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\left(\frac{\partial f}{\partial x}\right)du$$
$$\leq \frac{Ax(x+1)}{n+\beta}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{\sqrt{k}}}^{x}\left(\frac{\partial f}{\partial x}\right). \tag{21}$$

Thus

$$|D_n(f,x)|\leq \frac{Ax(x+1)}{n+\beta}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{\sqrt{u}}}^{x}\left(\frac{\partial f}{\partial x}\right)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\left(\frac{\partial f}{\partial x}\right). \tag{22}$$

On the other hand, we have

$$|C_n(f,x)|=\left|\frac{n-1}{n}\int_x^\infty\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)dt\right|$$
$$=\left|\frac{n-1}{n}\int_{2x}^\infty\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)dt+\int_x^{2x}\left(\int_x^t\left(\frac{\partial f}{\partial x}\right)(u)du\right)dt(1-\beta_n(x,t))\right|$$
$$\leq\left|\frac{n-1}{n}\int_{2x}^\infty\left(f\left(\frac{nt+\alpha}{n+\beta}\right)-f(x)\right)\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)dt\right|$$
$$+|f'(x^+)|\left|\frac{n-1}{n}\int_{2x}^\infty(\frac{nt+\alpha}{n+\beta}-x)\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)dt\right|+\left|\int_x^{2x}\left(\frac{\partial f}{\partial x}\right)(u)du\right||1-\beta_n(x,2x)|$$
$$+\int_x^{2x}\left|\left(\frac{\partial f}{\partial x}\right)\left(\frac{nt+\alpha}{n+\beta}\right)\right||1-\beta_n(x,t)|dt$$

$$\leq\frac{M}{x}\left(\frac{n-1}{n}\right)\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r\left|\frac{nt+\alpha}{n+\beta}-x\right|dt$$
$$+\frac{|f(x)|}{x^2}\left(\frac{n-1}{n}\right)\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt$$
$$+|f'(x^+)|\left(\frac{n-1}{n}\right)\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|dt$$
$$+\frac{A(x+1)}{(n+\beta)x}|f(2x)-f(x)-xf'(x^+)|$$
$$+\frac{A(x+1)}{(n+\beta)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}\left(\frac{\partial f}{\partial x}\right)+\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}\left(\frac{\partial f}{\partial x}\right). \tag{23}$$

Next applying Holder inequality, Lemma 2.1, and corollary 2.1 we proceed as follows for the estimation of the first two terms in the right hand side of (23),

$$\frac{M}{x}\left(\frac{n-1}{n}\right)\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r\left|\frac{nt+\alpha}{n+\beta}-x\right|dt$$
$$+\frac{|f(x)|}{x^2}.\left(\frac{n-1}{n}\right)\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt$$
$$\leq\frac{M}{x}\left(\frac{n-1}{n}\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^{2r}dt\right)^{1/2}\left(\int_0^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt\right)^{1/2}$$
$$+\frac{|f(x)|}{x^2}\left(\frac{n-1}{n}\int_{2x}^\infty\sum_{k=0}^\infty b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt\right)$$
$$\leq M2^rO(n^{-r/2})\frac{\sqrt{Ax(x+1)}}{\sqrt{(n+\beta)}}+|f(x)|\frac{A(x+1)}{(n+\beta)x}. \tag{24}$$

Also the third term of the right side of (23) is given by

$$
\begin{aligned}
&|f'(x^+)|\left(\frac{n-1}{n}\right)\int_{2x}^{\infty}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r\left|\frac{nt+\alpha}{n+\beta}-x\right|dt\\
&\le |f'(x^+)|\left(\frac{n-1}{n}\right)\int_{0}^{\infty}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^r\left|\frac{nt+\alpha}{n+\beta}-x\right|dt\\
&\le |f'(x^+)|\frac{\sqrt{Ax(x+1)}}{\sqrt{(n+\beta)}} \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
C_n(f,x) \le\ & M2^r O\left(n^{-r/2}\right)\frac{\sqrt{Ax(x+1)}}{\sqrt{(n+\beta)}}+|f(x)|\frac{A(x+1)}{(n+\beta)x}\\
&+|f'(x^+)|\frac{\sqrt{Ax(x+1)}}{\sqrt{(n+\beta)}}\\
&+\frac{A(x+1)}{(n+\beta)x}|f(2x)-f(x)-xf'(x^+)|\\
&+\frac{A(x+1)}{(n+\beta)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}\left(\frac{\partial f}{\partial x}\right)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\left(\frac{\partial f}{\partial x}\right). \qquad (26)
\end{aligned}
$$

Collecting the estimates (13), (19), (20), (22), (24) and (26), we get the desired result.
This completes the proof of the theorem. □

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