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The unsteady state finite volume numerical grid technique for multidimensional problems

Research Article

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Abstract: In this paper numerical technique has been used to solve multidimensional unsteady state heat conduction equation with Dirichlet boundary conditions. We focus on finite volume numerical technique for solution of multidimensional time dependent heat conduction problems. Finally, the efficiency of this technique is tested for some heat conduction problems with known analytical solutions and the numerical results obtained show that the technique produces accurate results.

MSC: 76M12 • 35K05

Keywords: Analytical solutions • Dirichlet boundary condition • Finite volume technique • Unsteady state heat equation

1. Introduction

Mathematical model of heat conduction problems must often be solved numerically. There are numerous discretization assumptions and techniques originating from general solution procedures for partial differential equations [5]. The numerical solution of multidimensional partial differential equation is obtained with various conditions using different methods [9]. The most common procedures, which have found problems in computational fluid dynamics (CFD), will be described here.

There are three main approximation grid techniques are available- A commonly used numerical technique is the finite difference method (FDM), described in references [1], [2] and [7]. The another numerical technique called the finite element method (FEM) developed originally for the solution of structural problem, has been applied to the solution of heat conduction problems and other details about this technique can be seen in the reference [1], [2] and [8] and the next popular numerical technique is finite volume method (FVM) was originally developed as a special finite difference formulation; for more detailed see [3] ,[4] and [6]. These techniques are based on the idea of first discretizing the heat conduction equation and then solving the resulting algebraic equations. The discretization is accomplished by regarding the medium as constituted by a collection of cells or volumes of finite size. The nodes are usually associated with each cell thus producing a grid of points. The separation between any two nodes is the grid spacing. Temperature at each cell is then represented by the temperature at the corresponding nodal location. A computer and a MS excel are then used to solve the resulting algebraic problem. Out of the available numerical gird techniques, finite volume technique is one of the most flexible and versatile technique for solving the multidimensional time dependent problems in CFD.

The remainder of the paper is organised as follows. In Section 2, a short review of finite volume techniques with the help of TDMA (Tri-Diagonal Matrix Algorithm) is given. In Section 3, formulation the two and three dimensional heat flow problems with Dirichlet boundary conditions. In section 4, Numerical examples are presented to illustrate the efficiency of the developed scheme. In Section 5, the numerical solutions obtained by this technique are compared with exact solution. Finally, Section 6 concludes the paper.

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2. Finite volume grid technique

The finite volume Discretizations technique is based on the idea of regarding the computation domain as subdivided into a collection of finite volumes. In this technique, each finite volume is represented by a line, an area and a volume in one, two, and three dimensional respectively. Nodes, located inside each finite volume, become the locus of computational values. For each node, the rectangle faces are formed by drawing perpendiculars through the midpoints between contiguous nodes. The Discretizations are obtained by integrating the original partial differential equation over the span of each finite volume. The method is easily extended to nonlinear problems. The Finite Volume analysis involves three basic steps.

1. The problem domain is defined and divided the solution domain into discrete control volume. Let us place a numbers of nodal points in the given space and domain is divided in such way that, each node is surrounded by the control volume or grid and the physical boundaries coincide with the control volume boundaries.

2. The integration of the governing equation over the control volume to yield a discretised equation at its nodal point.

3. Solve the set of discretised equations using TDMA solver.

2.1. Finite volume discretizations

We may generalise the approach by mean of a weighting parameter between 0 and 1 and write the standard form of discretised equations for multidimensional unsteady state heat conduction problems are given by Eq. (1). For $i =$ W,E,N,S,T and B boundary sides.

$$a_p T_p = \sum_i a_i \left(\theta T_i + (1-\theta) T_i^0\right) + \left(a_p^0 - (1-\theta)\sum_i a_i\right) T_p^0 + S_T \tag{1}$$

$$a_p = \theta \sum_i a_i - \theta S_p + a_p^0 \tag{2}$$

$$a_p^0 = \rho c \frac{\triangle V}{\triangle t}$$

For the fixed value T, $S_T = \dfrac{2k}{\triangle} T$ and $S_p = \dfrac{-2k}{\triangle}$, where $\triangle = \triangle x$ or $\triangle y$ or $\triangle z$

For two dimensional problems, $a_w = a_E = \dfrac{k \triangle y}{\triangle x}$ and $a_N = a_S = \dfrac{k \triangle x}{\triangle y}$

For three dimensional problems, $a_w = a_E = \dfrac{k \triangle y \triangle z}{\triangle x}$, $a_N = a_S = \dfrac{k \triangle x \triangle z}{\triangle y}$ and $a_T = a_B = \dfrac{k \triangle x \triangle y}{\triangle z}$

The exact form of the final discretised equation depends on the value of θ. When $\theta = 0$, the resulting scheme is called explicit. When $0 < \theta \leq 1$, the resulting schemes are called implicit. The extreme case of $\theta = 1$ is termed fully implicit and the case corresponding to $\theta = \dfrac{1}{2}$ is called Crank-Nicolson scheme. The standard formulation of finite volume discretizations equation along north-south (n-s) lines as shown in Table 1 and Table 2. These all type of

Table 1. Formulation of FV Discretizations equation in explicit scheme

Problems	Explicit Scheme
2D	$-a_S T_S + a_p T_p -_N T_N = a_W T_W + a_E T_E + \left[a_p^0 - (a_W + a_E + a_S + a_N)\right] T_P^0 + S_T$
3D	$-a_S T_S + a_p T_p -_N T_N = a_W T_W + a_E T_E + a_T T_T + a_B T_B \left[a_p^0 - (a_W + a_E + a_S + a_N + a_T + a_B)\right] T_P^0 + S_T$

Table 2. Formulation of FV Discretizations equation in Fully Implicit Scheme

Problems	Fully Implicit Scheme
2D	$-a_S T_S + a_p T_p -_N T_N = a_W T_W + a_E T_E + a_p^0 T_p^0 + S_T$
3D	$-a_S T_S + a_p T_p -_N T_N = a_W T_W + a_E T_E + a_T T_T + a_B T_B + a_p^0 T_p^0 + S_T$

schemes is very simple and convenient and is frequently used in CFD problems. However it suffers from a series drawback. We see the term multiplying T_p^0 can become negative when

$$a_p^0 < \sum_i a_i$$

This type of behavior is not possible with a parabolic partial differential equation. We can avoid this by requiring $a_p^0 > \sum_i a_i$. For a uniform mesh and constant properties, this restriction can be shown for two and three dimensional problems respectively.

$$\triangle t < \frac{\rho c (\triangle x)^2}{4k} \text{ and } \triangle t < \frac{\rho c (\triangle x)^2}{6k}$$

2.2. Tri diagonal matrix algorithm(TDMA)

The tri diagonal matrix algorithm (TDMA), also known also Thomas algorithm, is a simplified form of Gaussian elimination that can be used to solve tri diagonal system of equations. A tri-diagonal system for n unknowns may be written as

$$\begin{aligned}
&T_1 &= d_1\\
&-a_2 T_1 + b_2 T_2 - c_3 T_3 &= d_2\\
&\quad -a_3 T_2 + b_3 T_3 - c_3 T_4 &= d_3\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\\
&-a_{n-1} T_{n-2} + b_{n-1} T_{n-1} - c_{n-1} T_n &= d_{n-1}\\
&\qquad -a_n T_{n-1} + b_n T_n - c_n T_{n+1} &= d_n\\
&\qquad\qquad T_{n+1} &= d_{n+1}
\end{aligned}$$

In the above set of equations $T_1$and T_{n+1} are known boundary values. The general form of any single equation for $i = 1 - - - - - - n$

$$-a_i T_{i-1} + b_i T_i - c_i T_{i+1} = d_i \tag{3}$$

The equations of the above system can be rewritten as

$$T_2 = \frac{c_2}{b_2} T_3 + \frac{a_2}{b_2} T_1 + \frac{d_2}{b_2} \tag{4}$$

$$T_3 = \frac{c_3}{b_3} T_4 + \frac{a_3}{b_3} T_2 + \frac{d_3}{b_3} \tag{5}$$

$$T_n = \frac{c_n}{b_n} T_{n+1} + \frac{a_n}{b_n} T_{n-1} + \frac{d_n}{b_n} \tag{6}$$

The TDMA is based on the Gaussian elimination procedure and consist of two parts - a forward elimination phase and a backward substitution phase. The forward elimination process start by removing T_2 from Eq. (5) by substitution from Eq. (4) to get

$$T_3 = \left(\frac{c_3}{b_3 - a_3 \frac{c_2}{b_2}} \right) T_4 + \left(\frac{a_3 \left(\frac{a_2}{b_2} T_1 + \frac{d_2}{b_2} \right) + d_3}{b_3 - a_3 \frac{c_2}{b_2}} \right) \tag{7}$$

If we let $A_2 = \frac{c_2}{b_2}$ and $B_2 = \frac{a_2}{b_2} T_1 + \frac{d_2}{b_2}$ then, the Eq. (7) can be written as

$$T_3 = \left(\frac{c_3}{b_3 - a_3 A_2} \right) T_4 + \left(\frac{a_3 B_2 + d_3}{b_3 - a_3 A_2} \right) \tag{8}$$

If we let $A_s = \frac{c_3}{b_3 - a_3 A_2}$ and $B_s = \frac{a_3 B_2 + d_3}{b_3 - a_3 A_2}$, from the above system obtain

$$\begin{aligned}
A_i &= \frac{c_1}{b_1} && if\, i = 1\\
&= \frac{c_i}{b_i - a_i A_{i-1}} && if\, i = 2, 3 - - - - - - - (n-1)
\end{aligned}$$

$$B_i = \frac{d_1}{b_1} \qquad if\, i = 1$$
$$= \frac{a_i B_{i-1} + d_i}{b_i - a_i A_{i-1}} \qquad if\, i = 2, 3 ------- n$$

then, the Eq. (8) can be rewritten as

$$T_3 = A_3 T_4 + B_3 \tag{9}$$

The Eq. (9) can be used to eliminate and the procedure can be repeated up to the last equation of the system. This constitutes the forward elimination process. For the back substitution, we use the general form of the Eq. (9).

$$T_i = A_i T_{i+1} + B_i \qquad if\, i = (n-1), (n-2), ------, 1 \tag{10}$$

Note, the conditions of diagonal dominance for the above system that are sufficient for stability of the tri-diagonal elimination can actually be relaxed. In fact, one can only require that the coefficients of the above system satisfy the inequalities:

$$|b_1| \geq |c_1|$$
$$|b_i| \geq |a_i| + |c_i| \qquad if\, i = 2, 3, ----, (n-1)$$
$$|b_n| \geq |c_n|$$

3. Problems Formulation

3.1. For two dimensional Problems

Consider two dimensional unsteady state heat conduction equations with Dirichlet boundary conditions; the mathematical formulation of this problem is given by

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial t}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial t}\right) \qquad in \quad 0 \leq x, y \leq 2 \tag{11}$$

The initial and the boundary conditions are

$$T(x, y, 0) = f(x, y), \qquad 0 < x, y < 2, t = 0$$
$$T(0, y, t) = f_1(y, t) \qquad 0 < y < 2, t > 0$$
$$T(2, y, t) = f_2(y, t) \qquad 0 < y < 2, t > 0$$
$$T(x, 0, t) = g_1(x, t) \qquad 0 < x < 2, t > 0$$
$$T(x, 2, t) = g_2(x, t) \qquad 0 < x < 2, t > 0$$

Where f, f_1, f_2, g_1 and g_2 are known functions. The solution region with boundary sides is shown in Fig. 1 .

3.2. For three dimensional problems

Consider three dimensional unsteady state heat conduction equations with Dirichlet boundary conditions; the mathematical formulation of this problem is given by

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial t}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial t}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial t}\right) \qquad in \quad 0 \leq x, y, z \leq 1 \tag{12}$$

The initial and the boundary conditions are

$$T(x, y, z, 0) = f(x, y, z), \qquad 0 < x, y, z < 1, t = 0$$
$$T(0, y, z, t) = f_1(y, z, t) \qquad 0 < y, z < 1, t > 0$$
$$T(1, y, z, t) = f_2(y, z, t) \qquad 0 < y, z < 1, t > 0$$
$$T(x, 0, z, t) = g_1(x, z, t) \qquad 0 < x, z < 1, t > 0$$
$$T(x, 1, z, t) = g_2(x, z, t) \qquad 0 < x, z < 1, t > 0$$
$$T(x, y, 0, t) = h_1(x, y, t) \qquad 0 < x, y < 1, t > 0$$
$$T(x, y, 1, t) = h_2(x, y, t) \qquad 0 < x, y < 1, t > 0$$

Where $f, f_1, f_2, g_1, g_2, h_1$ and h_2 are known functions. The solution region with boundary sides is shown in Fig. 2. For three dimensional problems, the TDMA solver is applied line by line on the selected plane and then the calculation is moved to the next plane, scanning the solution region plane by plane. In this problem, there are four planes. These planes are numbered as I, II, III, and IV from bottom to top as shown in fig. 2. Using the TDMA procedure values of the along a selected north- south line are computed. The calculation is moved to the next line and subsequently swept through the whole plane until all unknown values on each line have been calculated. After completion the calculation of the plane-I, the process is moved on to the next plane-II and then continue up to plane-IV. The solution region is divided into four XY planes with girds as shown in Fig. 3.

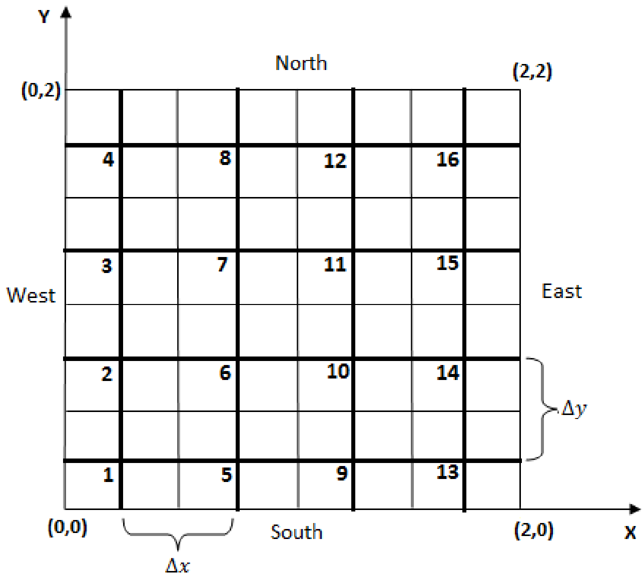

Fig. 1. Solution region with boundary sides for two dimensional problems

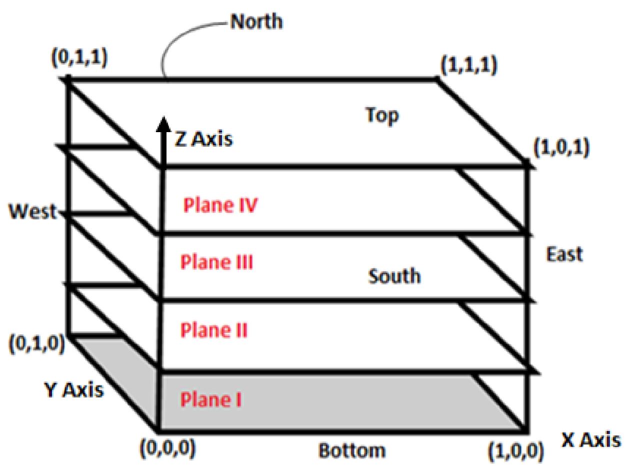

Fig. 2. Solution region with boundary sides for three dimensional problems

4. Numerical examples

The finite volume technique is apply on three selected examples in which the exact solutions of are known to us in order to test the efficiency and adaptability of the proposed technique. The computed solution is found for the entire interior grid points are given in Tables 3-5.

Example 4.1.
Let us consider the two dimensional unsteady state heat conduction equations as shown by Eq. (11) with initial and the boundary conditions are

$$T(x,y,0) = cos\left[\pi\left(\frac{(x-y)}{2}\right)\right] - cos\left[\pi\left(\frac{(x+y)}{2}\right)\right]$$

$$T(0,y,t) = T(x,0,t) = T(2,y,t) = T(x,2,t) = 0$$

The Analytical solution of this problem is given by

$$T(x,y,t) = 2sin\left(\frac{\pi x}{2}\right)sin\left(\frac{\pi y}{2}\right)e^{\frac{-(2\pi^2 kt)}{4}}$$

and its finite volume numerical solution is obtained at the time $t = 40s$ as shown in Table 3.

Example 4.2.
Let us consider the three dimensional unsteady state heat conduction equations as shown by Eq. (12) with initial

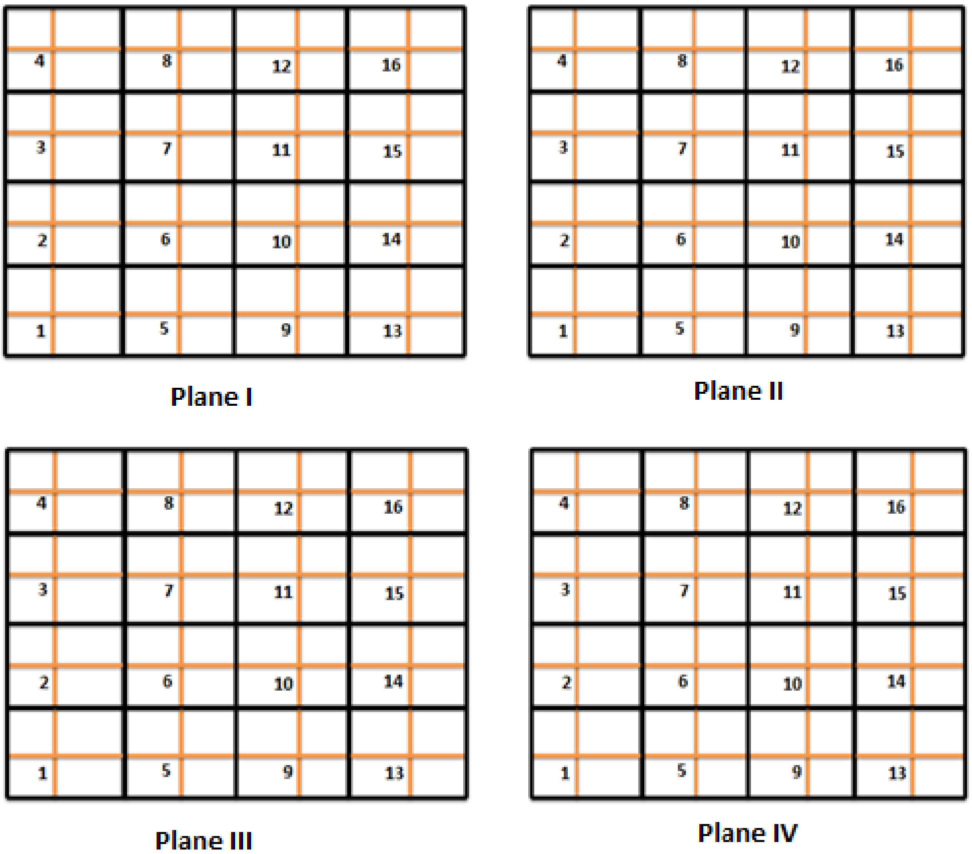

Fig. 3. The solution region is divided into four XY planes with girds and nodes

and the boundary conditions are
$T\left(x, y, z, 0\right) = e^{x+y+z}$
$T\left(0, y, z, t\right) = e^{y+z+3t}$ and $T\left(1, y, z, t\right) = e^{1+y+z+3t}$
$T\left(x, 0, z, t\right) = e^{x+z+3t}$ and $T\left(x, 1, z, t\right) = e^{1+x+z+3t}$
$T\left(x, y, 0, t\right) = e^{x+y+3t}$ and $T\left(x, y, 1, t\right) = e^{1+x+y+3t}$
The Analytical solution of this problem is given by $T\left(x, y, z, t\right) = e^{x+y+z+3t}$ and its finite volume numerical solution is obtained at the time $t = 30s$ as shown in Table 4.

Example 4.3.

Let us consider the three dimensional unsteady state heat conduction equations as shown by Eq. (12) with initial and the boundary conditions are
$T\left(x, y, z, 0\right) = \left(1 - y - z\right) e^{x}$
$T\left(0, y, z, t\right) = \left(1 - y - z\right) e^{t}$ and $T\left(1, y, z, t\right) = \left(1 - y - z\right) e^{1+t}$
$T\left(x, 0, z, t\right) = \left(1 - z\right) e^{x+t}$ and $T\left(x, 1, z, t\right) = -z e^{x+t}$
$T\left(x, y, 0, t\right) = \left(1 - y\right) e^{x+t}$ and $T\left(x, y, 1, t\right) = -y e^{x+t}$
The Analytical solution of this problem is given by $T\left(x, y, z, t\right) = \left(1 - y - z\right) e^{x+t}$ and its finite volume numerical solution is obtained at the time $t = 30s$ as shown in Table 5.

Table 3. Comparison between Finite Volume solution and Analytical solution at $t = 40s$ for example I

Nodes	Finite Volume Numerical Solutions		Analytical Solution
	Explicit Scheme	Fully Implicit Scheme	
1	0.2924	0.2926	0.2926
2	0.7062	0.7066	0.7065
3	0.7067	0.7070	0.7069
4	0.2935	0.2937	0.2937
5	0.7062	0.7066	0.7065
6	1.7059	1.7062	1.7059
7	1.7071	1.7074	1.7070
8	0.7089	0.7093	0.7092
9	0.7067	0.7070	0.7069
10	1.7071	1.7074	1.7070
11	1.7082	1.7085	1.7082
12	0.7094	0.7097	0.7096
13	0.2935	0.2937	0.2937
14	0.7089	0.7093	0.7092
15	0.7094	0.7097	0.7096
16	0.2948	0.2948	0.2948

Table 4. Comparison between Finite Volume solution and Analytical solution at $t = 30s$ for example II

Nodes	Finite Volume Numerical Solutions (FMS)				Analytical Solution			
	I	II	III	IV	I	II	III	IV
1	1.4553	1.8687	2.3995	3.0811	1.4551	1.8684	2.3991	3.0805
2	1.8687	2.3995	3.0811	3.9563	1.8684	2.3991	3.0805	3.9554
3	2.3995	3.0811	3.9562	5.0799	2.3991	3.0805	3.9554	5.0789
4	3.0811	3.9557	5.0792	6.5176	3.0805	3.9554	5.0789	6.5214
4	1.8682	2.3988	3.0802	3.9551	1.8684	2.3991	3.0805	3.9554
6	2.3988	3.0802	3.9551	5.0785	2.3991	3.0805	3.9554	5.0789
7	3.0802	3.9551	5.0784	6.5210	3.0805	3.9554	5.0789	6.5214
8	3.9551	5.0785	6.5210	8.3732	3.9554	5.0789	6.5214	8.3737
9	2.3988	3.0802	3.9550	5.0784	2.3991	3.0805	3.9554	5.0789
10	3.0802	3.9551	5.0784	6.5210	3.0805	3.9554	5.0789	6.5214
11	3.9550	5.0784	6.5208	8.3731	3.9554	5.0789	6.5214	8.3737
12	5.0784	6.5210	8.3731	10.7515	5.0789	6.5214	8.3737	10.7520
13	3.0798	3.9546	5.0778	6.5201	3.0805	3.9554	5.0789	6.5214
14	3.9546	5.0778	6.5201	8.3721	3.9554	5.0789	6.5214	8.3737
15	5.0778	6.5201	8.3719	10.7500	5.0789	6.5214	8.3737	10.7520
16	6.5201	8.3721	10.7500	13.8035	6.5214	8.3737	10.7520	13.8058

Table 5. Comparison between Finite Volume solution and Analytical solution at $t = 30s$ for example III

Nodes	Finite Volume Numerical Solutions (FMS)				Analytical Solution			
	I	II	III	IV	I	II	III	IV
1	0.850	0.566	0.283	0.000	0.849	0.566	0.283	0.000
2	0.566	0.283	0.000	-0.283	0.566	0.283	0.000	-0.283
3	0.283	0.000	-0.283	-0.566	0.283	0.000	-0.283	-0.566
4	0.000	-0.283	-0.566	-0.850	0.000	-0.283	-0.566	-0.849
5	1.091	0.727	0.363	0.000	1.091	0.727	0.363	0.0000
6	0.727	0.363	0.000	-0.363	0.727	0.363	0.000	-0.363
7	0.363	0.000	-0.363	-0.727	0.363	0.000	-0.363	-0.727
8	0.000	-0.363	-0.727	-1.091	0.000	-0.363	-0.727	-1.091
9	1.401	0.934	0.467	0.000	1.401	0.934	0.467	0.000
10	0.934	0.467	0.000	-0.467	0.934	0.467	0.000	-0.467
11	0.467	0.000	-0.467	-0.934	0.467	0.000	-0.467	-0.934
12	0.000	-0.467	-0.934	-1.401	0.000	-0.467	-0.934	-1.401
13	1.799	1.199	0.599	0.000	1.799	1.199	0.599	0.000
14	1.199	0.599	0.000	-0.599	1.199	0.599	0.000	-0.599
15	0.599	0.000	-0.599	-1.199	0.599	0.000	-0.599	-1.199
16	0.000	-0.599	-1.199	-1.799	0.000	-0.599	-1.199	-1.799

5. Results and discussion

Table 3 shows the numerical (explicit and fully implicit scheme) and analytical results at the time $t = 40s$ and the Fig. 4 shows the comparison in a graphical form for two dimensional heat conduction problems. The Table 4 and 5 we give finite volume numerical solution at $t = 30s$ obtained using fully implicit methods with the analytical solution. Whereas the explicit scheme gives unrealistic oscillation at the step size, the implicit scheme gives results that are in reasonable agreement with the analytical solution.

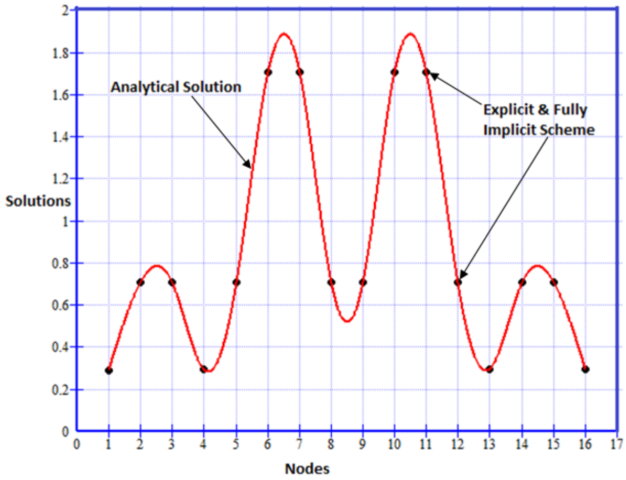

Fig. 4. Graphical Comparison between Numerical (Explicit and Fully Implicit Scheme) and analytical Solution at time $t = 40s$ for example I

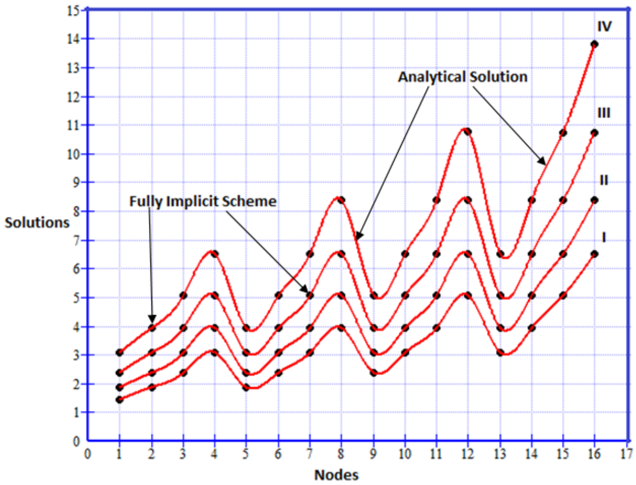

Fig. 5. Graphical Comparison between Numerical (Fully Implicit Scheme) and analytical solution of planes I, II, III, and IV at time $t = 30s$ for example II

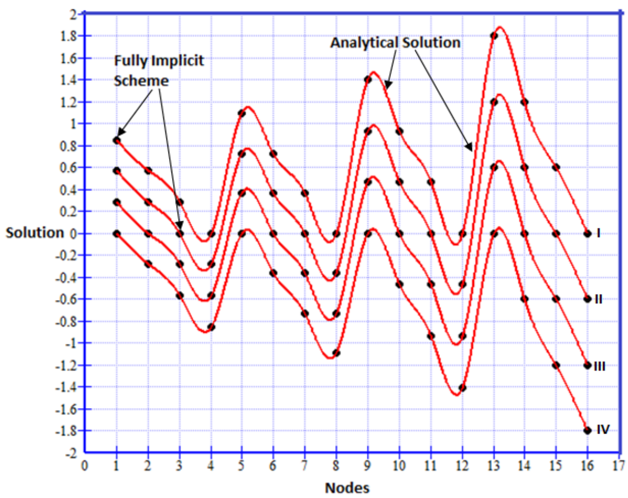

Fig. 6. Graphical Comparison between Numerical (Fully Implicit Scheme) and analytical solution of planes I, II, III, and IV at time $t = 30s$ for example III

6. Conclusions

In this paper, the finite volume numerical grid technique for unsteady state heat conduction problems have been studied and obtained the numerical solution for two and three dimensional problems with Dirichlet boundary conditions. All the numerical calculations obtained with finite control volume grids for multidimensional problems using MS excel. Also, the TDMA technique have been used for solving algebraic equations and the results obtained by this technique are all in good agreement with the analytical solutions and the total error less than 1 under study. Moreover this technique is efficient, reliable, accurate and easier to implement in MS excel as compared to the other costly techniques.

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