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Analytical approximations of the porous medium equations by reduced differential transform method

Research Article

A. Taghavi[1], A. Babaei[1,*], A. Mohammadpour[1,2]

[1] *Department of Mathematics, University of Mazandaran, P.O. Box: 47416-95447, Babolsar, Mazandaran, Iran*

[2] *Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran*

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Abstract: In this paper, the reduced differential transform method is used for solving the initial value problem of the porous medium equation that usually occurs in nonlinear problems of heat and mass transfer and also in biological systems. A complete description of the method is derived. Finally, to show the efficiency of the presented method, four examples are given.

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Keywords: Reduced differential transform method • Porous medium equation • Unsteady heat transfer

1. Introduction

Nonlinear phenomena occurs in a wide range of apparently different contexts in nature, for instance biological, economical, chemical and physical systems [1–6]. There are well-known methods which successfully applied to construct exact solutions for a wide range of nonlinear equations [7–11]. In this paper, we consider the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \tag{1}$$

usually called the porous medium equation [3]. The default settings are: $u = u(x,t)$ is a non-negative scalar function of space $x \in R$ and time $0 < t < \infty$ and m is a constant rational number. This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory, and flows in porous media. For example, it describes unsteady heat transfer in a quiescent medium with the heat diffusivity being a power-law function of temperature [3]. Eq. (1) has also applications to many physical systems including the fluid dynamics of thin films [4]. Murray [5] describes how this model has been used to represent "population pressure" in biological systems.
In the range of exponents $m < 0$, Since the diffusion coefficient u^m goes to infinity as $u \to 0$, the Eq. (1) is called the fast diffusion equation. In the other hand, Eq. (1) with $n > 0$ is known as the slow diffusion equation.
We solve the initial value problem given by (1) and the initial condition

$$u(x,0) = f(x), \tag{2}$$

by the differential transformation method [12–18]
The given problem can be transformed into a recurrence relation, using differential transformation operations, which leads to a series solution.

* Corresponding author.
E-mail address: Babaei@umz.ac.ir

2. Reduced differential transform method

In this section we review the basic definitions and operations of RDTM which was introduced in [14].
Consider a function of two variables $u(x,t)$ and suppose that it can be represented as product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of differential transform [16], function $u(x,t)$ can be represented as

$$u(x,t)=\sum_{k=0}^{\infty} F_i x^i \sum_{j=0}^{\infty} G_j t^j = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{3}$$

where $U_k(x)$ is called t-dimensional spectrom function of $u(x,t)$.

Definition 2.1.
If function $u(x,t)$ is analytic and differentiated continuosly with respect to time t and space x in the domain of interest, then let

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{4}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function which is called T-function in brief.
The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty} U_k(x)t^k. \tag{5}$$

Combining (4) and (5) gives the solution as

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{6}$$

For more illustration, consider the general nonlinear partial differential equation:

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),$$

with initial condition

$$u(x,0)=f(x), \tag{7}$$

where $L=\frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term. According to the RDTM and Table 1 , we can construct

$$(k+1)U_{k+1}(x)=G_k(x)-RU_k(x)-NU_k(x),$$

where $U_k(x)$, $RU_k(x)$, $NU_k(x)$ and $G_k(x)$ are the transformation of the functions $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$ respectively.
In real application, by consideration of $U_0(x)=f(x)$ as transformaiton of initial condition (7), the function $u(x,t)$ can be written by a finite series of Eq. (6) as

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n} U_k(x)t^k$$

where n is order of approximation solution. The exact solutoin is given by

$$u(x,t)=\lim_{n\longrightarrow\infty}\tilde{u}_n(x,t).$$

The fundamental operations of reduced differential transform that can be deduced from Eqs. (4) and (5) are listed in below [12–17].

Table 1. Some basic reduced differential transformations

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}$
$u(x,t) = v(x,t) + w(x,t)$	$U_k(x) = V_k(x) + W_k(x)$
$u(x,t) = c\,v(x,t)$	$U_k(x) = c\,V_k(x)$ (c is a constant)
$u(x,t) = v(x,t)w(x,t)$	$U_k(x) = \sum_{k_1=0}^{k} V_{k_1}(x) W_{k-k_1}(x)$
$u(x,t) = \frac{\partial}{\partial t} v(x,t)$	$U_k(x) = (k+1)V_{k+1}(x)$
$u(x,t) = \frac{\partial^m}{\partial x^m} v(x,t)$	$U_k(x) = \frac{\partial^m}{\partial x^m} V_k(x)$
$u(x,t) = v^m(x,t)\frac{\partial}{\partial x} w(x,t)$	$U_k(x) = \sum_{k_m=0}^{k}\sum_{k_{m-1}=0}^{k_m}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\partial}{\partial x} W_{k_1} V_{k_2-k_1} V_{k_3-k_2} \cdots V_{k_{m-1}-k_{m-2}} V_{k-k_{m-1}}$

3. Application of reduced differential transform method

According to the RDTM and Table1, we can construct the following iteration for the Eq. (1) as:

$$(k+1)U_{k+1}(x) = \frac{\partial}{\partial x}\left(\sum_{k_m=0}^{k}\sum_{k_{m-1}=0}^{k_m}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\partial}{\partial x} U_{k_1} U_{k_2-k_1} U_{k_3-k_2}\cdots U_{k_{m-1}-k_{m-2}} U_{k-k_{m-1}}\right). \tag{8}$$

From the initial condition (2), we can get the $U_0(x)$ and afterwards the $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximation solution as:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k,$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\longrightarrow\infty} \tilde{u}_n(x,t).$$

Now, we apply the presented method for solving some initial value problems of porous medium equations. [19–22]

Example 3.1.
Consider the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\frac{1}{3}}\frac{\partial u}{\partial x}\right), \\ u(x,0) = (\frac{1}{3}x - 3)^{\frac{1}{3}}, \ -\infty < x < +\infty, \end{cases} \tag{9}$$

with the exact solution $u(x,t) = (\frac{1}{3}x + \frac{1}{3}t - 3)^3$. By using the transformation $v = u^{\frac{1}{3}}$, the problem (9) becomes

$$\begin{cases} \frac{\partial v}{\partial t} = v.\frac{\partial^2 v}{\partial x^2} + 3\left(\frac{\partial v}{\partial x}\right)^2, \\ v(x,0) = \frac{1}{3}x - 3, \ -\infty < x < +\infty. \end{cases} \tag{10}$$

By using the basic properties of the reduced differential transform and Table1, we can find transformed form of Problem (10) as:

$$(k+1)V_{k+1}(x) = \sum_{k_1=0}^{k}\frac{\partial^2}{\partial x^2} V_{k_1}(x) V_{k-k_1}(x) + 3\sum_{k_1=0}^{k}\frac{\partial}{\partial x} V_{k_1}(x)\frac{\partial}{\partial x} V_{k-k_1}(x) \tag{11}$$

and

$$V_0(x) = \frac{1}{3}x - 3. \tag{12}$$

By substituding of Eq. (12) into (11) we have:

$$V_1(x)=\frac{1}{3},$$
$$V_k(x)=0, \qquad k=2,3,\cdots.$$

The differential inverse transform of $V_k(x)$ gives the exact solution as:

$$v(x,t)=\sum_{k=0}^{\infty}V_k(x,t)t^k=(\frac{1}{3}x-3)+(\frac{1}{3})t.$$

Hence, we take the exact solution of the problem (9) as

$$u(x,t)=(\frac{1}{3}x+\frac{1}{3}t-3)^3.$$

Example 3.2.

In this example, we consider the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-\frac{4}{3}}\frac{\partial u}{\partial x}\right), \\ u(x,0)=(2x)^{-\frac{3}{4}}, \; -\infty<x<+\infty, \end{cases} \tag{13}$$

with the exact solution $u(x,t)=(2x-3t)^{-\frac{3}{4}}$. By using the transformation $v=u^{-\frac{4}{3}}$, the problem (13) becomes

$$\begin{cases} \frac{\partial v}{\partial t}=v.\frac{\partial^2 v}{\partial x^2}-\frac{3}{4}\left(\frac{\partial v}{\partial x}\right)^2, \\ v(x,0)=2x, \; -\infty<x<+\infty. \end{cases} \tag{14}$$

Being in a similar way with the first example, we apply the reduced differential transform, and achieve the transformed form of (14) as

$$(k+1)V_{k+1}(x)=\sum_{k_1=0}^{k}\frac{\partial^2}{\partial x^2}V_{k_1}(x)V_{k-k_1}(x)-\frac{3}{4}\sum_{k_1=0}^{k}\frac{\partial}{\partial x}V_{k_1}(x)\frac{\partial}{\partial x}V_{k-k_1}(x). \tag{15}$$

Substitution of $V_0(x)=2x$ into (15) gives:

$$V_1(x)=-3,$$
$$V_k(x)=0, \qquad k=2,3,\cdots.$$

The differential inverse transform of $V_k(x)$ gives the exact solution as:

$$v(x,t)=\sum_{k=0}^{\infty}V_k(x,t)t^k=2x-3t.$$

Hence, we take the exact solution of the problem (13) as:

$$u(x,t)=(2x-3t)^{\frac{-3}{4}}$$

Example 3.3.

In this example, we will consider the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right), \\ u(x,0)=\frac{1}{x}, \quad -\infty<x<+\infty, \end{cases} \tag{16}$$

with the exact solution $u(x,t)=\dfrac{1}{x-t}$. By using the transformation $v=u^{-\frac{4}{3}}$, the problem (16) becomes

$$\begin{cases} \frac{\partial v}{\partial t}=v.\frac{\partial^2 v}{\partial x^2}-\left(\frac{\partial v}{\partial x}\right)^2, \\ v(x,0)=x, \; -\infty<x<+\infty. \end{cases} \tag{17}$$

Now we apply the reduced differential transform for (17) and find $it's$ transformed form as:

$$(k+1)V_{k+1}(x)=\sum_{k_1=0}^{k}\frac{\partial^2}{\partial x^2}V_{k_1}(x)V_{k-k_1}(x)-\sum_{k_1=0}^{k}\frac{\partial}{\partial x}V_{k_1}(x)\frac{\partial}{\partial x}V_{k-k_1}(x). \tag{18}$$

After substituting $V_0(x) = x$ into (18), we obtain the next terms of $V_k(x)$ as:

$$\begin{aligned} V_1(x) &= -1, \\ V_k(x) &= 0, \qquad k = 2, 3, \cdots. \end{aligned}$$

The differential inverse transform of $V_k(x)$ gives the exact solution as:

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x,t) t^k = x - t.$$

At last, we get the following exact solution :

$$u(x,t) = (2x - 3t)^{-1}.$$

Example 3.4.
Let us consider the following nonlinear initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) + 2u, \\ u(x,0) = x^{-1}, \; -\infty < x < +\infty. \end{cases} \tag{19}$$

An exact solution of (19) in [22] is

$$u(x,t) = x^{-1} e^{2t}.$$

By using the transformation $v = u^{-2}$ the problem (3.4) becomes

$$\begin{cases} \frac{\partial v}{\partial t} = v.\frac{\partial^2 v}{\partial x^2} - \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 - 4v, \\ v(x,0) = x^2, \; -\infty < x < +\infty. \end{cases} \tag{20}$$

Taking reduced differential transform of Problem (20), the following is obtained:

$$(k+1)V_{k+1}(x) = \sum_{k_1=0}^{k} \frac{\partial^2}{\partial x^2} V_{k_1}(x) V_{k-k_1}(x) - \frac{1}{2}\sum_{k_1=0}^{k} \frac{\partial}{\partial x} V_{k_1}(x) \frac{\partial}{\partial x} V_{k-k_1}(x) - 4V_k(x). \tag{21}$$

By using the recurrence relation (21) and the transformed initial condition $V_0(x) = x^2$, we obtain the following terms of $V_k(x)$:

$$V_1(x) = -4x^2, \; V_2(x) = \frac{(-4)^2}{2!}x^2, \; V_3(x) = \frac{(-4)^3}{3!}x^2, \; \cdots, \; V_k(x) = \frac{(-4)^k}{k!}x^2.$$

Finally, the differential inverse transform of $V_k(x)$ gives:

$$\widetilde{v}_n(x,t) = \sum_{k=0}^{n} \frac{(-4)^k}{k!} x^2 t^k.$$

Therefore, the exact solution is given as:

$$v(x,t) = \lim_{n \longrightarrow \infty} \widetilde{v}_n(x,t) = x^2 e^{-4t}.$$

Hence we take the exact solution of problem (19) as:

$$u(x,t) = v(x,t)^{-\frac{1}{2}} = x^{-1} e^{2t}.$$

4. Conclusion

In this paper, the reduced differential transform method (RDTM) has been successfully applied for some problems of porous medium equations. It can be concluded that, RDTM is a very powerful and efficient technique for finding exact solutions for wide classes of problems and can be applied to many complicated linear and nonlinear problems, such as porous medium equation, and does not require linearization, discretization or perturbation.

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