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Finite source Markovian inventory system with bonus service for certain customers

Research Article

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Abstract: We consider a perishable inventory system with bonus service for certain customers, wherein the demand of a customer is satisfied only after performing some service on the item which is assumed to be of random duration. The maximum inventory level is S. The ordering policy is $(0, S)$ policy and the lead time is zero. The life time of each items is assumed to be exponential.The customers arrive according a quasi-random distribution. Arriving customers form a single waiting line based on the order of their arrivals. The first $R \geq 1$ customers who arrive after the system becomes empty must wait for service to begin. We have assumed that these R customers received an additional service, called bonus service, in addition to the essential service rendered to all customers. The service times are exponentially distributed. The joint probability distribution of the number of customers in the waiting hall and the inventory level is obtained for the steady state case. Some important system performance measures and the long-run total expected cost rate are derived in the steady state.

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1. Introduction

Research on queueing systems with inventory control has captured much attention of researchers over the last decades. In this system, customers arrive at the service facility one by one and require service. In order to complete the customer service, an item from the inventory is needed. A served customer departs immediately from the system and the on - hand inventory decreases by one at the moment of service completion. This system is called a queueing - inventory system [19]. Berman and Kim [4] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [7] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [8] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [5] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [6] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. The M/M/1 queueing - inventory system with backordering was investigated by Schwarz and Daduna [18]. The authors derived the system steady state behavior under Π(1) reorder policy which is (0, Q) policy with an additional threshold 1 for the queue

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Notations

B_0	Magnetic induction
I	Identity matrix
$\mathbf{e}$	a column vector of appropriate dimension containing all ones
$\mathbf{0}$	Zero matrix
$[A]_{ij}$	entry at $(i,j)^{th}$ position of a matrix A
δ_{ij}	$\begin{cases} 1 & \text{if } j=i \\ 0 & \text{otherwise} \end{cases}$
$\bar{\delta}_{ij}$	$1-\delta_{ij}$
l	$R+2$
r	$2R+2$
x	$(N+R+1)+2\sum_{j=1}^{R}(N-j)$
y_j	$(N-(R-(j-1)))$, where $j=1,2,\ldots,R$
z_j	$N-(R-j)$, where $j=1,2,\ldots,R$

length as a decision variable. Krishnamoorthy and Anbazhagan [15] introduced a perishable (s,S) inventory system with N policy. Anbazhagan et al. [2] investigated a continuous review (s,S) inventory system at a service facility with two types of services and finite waiting hall. They have derived the joint probability distribution of both the inventory level and the number of customers in the system in the steady state case.

William Gray and Meckinley Scott [11] has introduced a queuing model in which the management finds it unprofitable to operate a service counter unless a certain number, $R+1$, of customers are available during each busy period. Thus the first R, customers who arrive after the system becomes empty must wait for service to begin. In [12], they have studied an M/G/l type vacation queuing model exceptional service for certain customers. They assumed that service times have general distributions. Also the parameter of the Poisson process for the arrival distributions is assumed to depend on the class of states the system is in at the times of arrival of customers. Recently Jeganathan [13] analyzed a perishable inventory model with bonus service for certain customers, infinite populations and $N+1$ Policy. The joint distribution of the number of demands in the waiting hall and the inventory level was obtained in steady state cases. Pandit [16] considered the supply-demand model as a system involving fuzzy parameters. Ertefaie and Meshkani [9] studied Bayesian analysis of software reliability models with reference prior and they compare the results obtained from using conjugate and reference priors.

In this paper we model this situation by assuming a finite source perishable inventory model with bonus service for certain customers and $(0,S)$ ordering policy is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry is also a problem which motivated me to create the present stochastic model. The problem we consider is more relevant to the real life situation. Queueing systems with classical waiting lines and finite population have been reviewed in detail by Takagi [23]. For the theory of queueing system with finite source, the often quoted articles Falin and Artalejo [10], Alfa and Sapna [1] and Sztrik [22] and for the recent articles see Artalejo and Lapez-Herrero [3] and Jinbiao Wu et al. [14]. In the case of inventory modeling with continuous review finite-source inventory system, the first paper was by Sivakumar [21] who assumed the exponential life time for the items, exponential lead time for the supply of the ordered items and exponential retrial rate for the customers in the orbit. In a very recent paper, Shophia Lawrence et al. [20] considered a perishable inventory system with service facility and finite source, where the service time, the lead time are assumed to have Phase type distribution.

The rest of the paper is organized as follows. In the next section, the problem formulation and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are proposed in section 3. Some key system performance measures are derived in section 4. In section 5, we derived the total expected cost rate. The last section is meant for conclusion.

2. Problem formulation

Consider a service facility in which perishable items are stocked and the items are delivered to the demanding customers. The demands are generated by a finite number of homogeneous sources N and the demand time points form a quasi-random distribution with parameter λ. That is, the probability that any particular source generates a demand in any interval $(t,t+dt)$ is $\lambda dt + o(dt)$ as $dt \to 0$ if the source is idle at time t, and zero if the source is in the service facility at time t, independently of the behavior of any other sources. Customers arrive at the server form a single waiting line and are served in the order of their arrivals; that is, the first-come, first served discipline. Suppose that the server can serve only one customer at a time, and that the service is independent of the arrival of the customers.

We assumed that after the waiting area becomes empty service does not begin until $R+1$ customers arrive (i.e., the service starts only when the customer level reaches a prefixed level $R+1<N$ starting from the epoch at which no customer is left behind in the system). In our model the first R customers who arrive after the system became empty have to wait, but each of them receives bonus (i.e., special service) service. The first R customers will be called initial customers, and the others are regular customers. The maximum capacity of the inventory is S. The demand is for single item per customer. The items are issued to the demanding customers only after some random time due to some service on it. The latter type of service referred to as essential service (main service). The first R customers (initial customers) receive two stages of service, namely essential service (i.e., stage 1) and bonus service or additional service on their item (stage 2). Regular customers receive only essential service (i.e., stage 1). Each initial customer receives bonus service immediately after receiving the essential service. The essential service of a customer (initial customers/regular) is assumed to be exponentially distributed with parameter μ_1. The bonus service times of initial customers is assumed to be exponentially distributed with parameter μ_2.

Life time of each item has negative exponential distribution with parameter $\gamma>0$. Note that in this model we have assumed that the servicing item also perish. We assume that the replenishment of inventory is instantaneous (i.e: $(0,S)$ ordering policy with zero lead time). Various stochastic processes involved in the system are independent of each other.

2.1. Classification of states:

The states of the system are divided into classes as follows:

1. $E_a=\{E_{aa}=(i_1,0,0)\mid i_1=1,2,\ldots,S,\}$ where E_{aa} is the state in which i_1 items are in the inventory, but no customers in the system and server is idle.
2. $E_b=\{E_{bb}=(i_1,0,i_3)\mid i_1=1,2,\ldots,S,\ i_3=1,2,\ldots,R,\}$ where E_{bb} is the state in which i_1 items are in the inventory and i_3 customers are in the system, but service has not begin (i.e., server is idle).
3. $E_c=\{E_{cc}=(i_1,i_2,i_3)\mid i_1=1,2,\ldots,S,\ i_2=1,2,\ldots,R,\ l-i_2\le i_3\le N,\}$ where E_{cc} is the state in which i_1 items are in the inventory and i_3 customers are in the system while i_2th initial customer receiving essential service.
4. $E_d=\{E_{dd}=(i_1,i_2,i_3)\mid i_1=1,2,\ldots,S,\ i_2=R+j,1\le j\le R,\ r-i_2\le i_3\le N,\}$ where E_{dd} is the state in which i_1 items are in the inventory and i_3 customers are in the system while jth initial customer receiving bonus service.
5. $E_e=\{E_{ee}=(i_1,2R+1,i_3)\mid i_1=1,2,\ldots,S,\ i_3=1,2,\ldots,N,\}$ where E_{ee} is the state in which i_1 items are in the inventory and i_3 customers are in the system after all initial customers have been served (i.e., the server is providing essential service to a regular customer).

3. Analysis

Let $L(t)$, $Y(t)$ and $X(t)$ respectively, denote the inventory level, the server status and the number of customers in the waiting hall at time t. Further, server status $Y(t)$ be defined as follows:

$$Y(t):\quad \begin{cases} 0, & \text{if the server is idle at time t,}\\ 1, & \text{if the server is providing essential service to a first initial customer at time t,}\\ 2, & \text{if the server is providing essential service to a second initial customer at time t,}\\ 3, & \text{if the server is providing essential service to a third initial customer at time t,}\\ \vdots & \vdots \\ R-1, & \text{if the server is providing essential service to a (R-1)th initial customer at time t,}\\ R, & \text{if the server is providing essential service to a Rth initial customer at time t,}\\ R+1, & \text{if the server is providing bonus service to a first initial customer at time t,}\\ R+2, & \text{if the server is providing bonus service to a second initial customer at time t,}\\ \vdots & \vdots \\ 2R-1, & \text{if the server is providing bonus service to a (R-1)th initial customer at time t,}\\ 2R, & \text{if the server is providing bonus service to a Rth initial customer at time t,}\\ 2R+1, & \text{if the server is providing essential service to a regular customer at time t,} \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $Z(t) = \{(L(t), Y(t), X(t)), t \geq 0\}$ is a continuous time Markov chain with discrete state space given by $E = E_a \cup E_b \cup E_c \cup E_d \cup E_e$.

To determine the infinitesimal generator

$$\Theta = ((\ \nu((i_1, i_2, i_3), (j_1, j_2, j_3))\ )), \quad (i_1, i_2, i_3), (j_1, j_2, j_3) \in E$$

of this process we use the following arguments :

- We first note that in a Markov process there can be at most one change in the levels of the state of the process through any one of the activities arrival, completion of service, failure of an item, instantaneous replenishment of stock.
- If the server is idle,
  - any arriving customer takes the state of the process from $(i_1, 0, i_3)$ to $(i_1, 0, i_3 + 1)$ and the intensity of this transition $\nu((i_1, 0, i_3), (i_1, 0, i_3 + 1))$ is given by $(N - i_3)\lambda$, $i_1 = 1, 2, \ldots, S$; $i_3 = 0, 1, \ldots, R - 1$.
  - the arrival of a customer makes a transition from $(i_1, 0, R)$ to $(i_1, 1, R+1)$ and the intensity of this transition $\nu((i_1, 0, R), (i_1, 1, R + 1))$ is given by $(N - i_3)\lambda$, $i_1 = 1, 2, \ldots, S$.
  - an item perishes, the inventory level is reduces by one and a transition takes place from $(i_1, 0, i_3)$ to $(0, 0, i_3)$ and due to instantaneous supply of the item, this process moves from $(i_1, 0, i_3)$ to $(S, 0, i_3)$, $i_1 = 1$, $i_3 = 0, 1, \ldots, R$, or from $(i_1, 0, i_3)$ to $(i_1 - 1, 0, i_3)$, $i_1 = 2, \ldots, S$; $i_3 = 0, 1, \ldots, R$, with intensity of transition $i_1\gamma$,
- If the server is providing essential service for the initial customer,
  - any arriving customer takes the state of the process from (i_1, i_2, i_3) to $(i_1, i_2, i_3 + 1)$, and the intensity of transition $\nu((i_1, i_2, i_3), (i_1, i_2, i_3 + 1))$ is given by $(N - i_3)\lambda$, $i_1 = 1, 2, \ldots, S$; $i_2 = 1, 2, \ldots, R$; $i_3 = l - i_2, \ldots, N - 1$.
  - at the time of essential service completion for the initial customer, the inventory level reduces by one and server immediately starts the bonus service for that initial customer and the intensity of transition $\nu((1, i_2, i_3), (S, R + i_2, i_3))$ or $\nu((i_1, i_2, i_3), (i_1 - 1, R + i_2, i_3))$, is given by μ_1, $i_1 = 2, \ldots, S$; $i_2 = 1, 2, \ldots, R$; $i_3 = l - i_2, \ldots, N$.
  - a transition from (i_1, i_2, i_3) to (S, i_2, i_3), $i_1 = 1$, $i_2 = 1, 2, \ldots, R$; $i_3 = l - i_2, \ldots, N$ or from (i_1, i_2, i_3) to $(i_1 - 1, i_2, i_3)$ will take place with intensity of transition $i_1\gamma$, $i_1 = 2, \ldots, S$; $i_2 = 1, 2, \ldots, R$; $i_3 = l - i_2, \ldots, N$.
- If the server is providing bonus service for the initial customer
  - any arrival of customer makes a transition from (i_1, i_2, i_3) to $(i_1, i_2, i_3 + 1)$ with intensity of transition $(N - i_3)\lambda$, $i_2 = R + 1, R + 2, \ldots, 2R$; $i_1 = 1, 2, \ldots, S$; $i_3 = r - i_2, \ldots, N - 1$.
  - the completion of bonus service of the initial customer makes one customer leave the system. The state of the process from (i_1, i_2, i_3) to $(i_1, i_2 - (R - 1), i_3 - 1)$, $i_2 = R + 1, R + 2, \ldots, 2R - 1$; $i_1 = 1, 2, \ldots, S$; $i_3 = r - i_2, \ldots, N$, and the bonus service completion of the last initial customer takes the state of the process from $(i_1, 2R, i_3)$ to $(i_1, 2R + 1, i_3 - 1)$, $i_1 = 1, 2, \ldots, S$; $i_3 = 2, 3, \ldots, N$, with intensity of transition μ_2.
  - a transition from (i_1, i_2, i_3) to (S, i_2, i_3), $i_1 = 1$, $i_2 = R + 1, R + 2, \ldots, 2R$, $i_3 = r - i_2, \ldots, N$; or from (i_1, i_2, i_3) to $(i_1 - 1, i_2, i_3)$, $i_1 = 2, \ldots, S$; $i_2 = R + 1, \ldots, 2R$; $i_3 = r - i_2, \ldots, R$, will takes the place when any one of i_1 items fails at a rate γ; thus intensity for this transition is $i_1\gamma$.
- If the server is providing essential service for the regular customers (after all initial customers have been served, i.e., $i_2 = 2R + 1$),
  - any arriving customer joins the queue with intensity of transition $\nu((i_1, 2R+1, i_3), (i_1, 2R+1, i_3+1))$ is given by $(N - i_3)\lambda$, $i_1 = 1, 2, \ldots, S$; $i_3 = 1, 2, \ldots, N - 1$.
  - a transition from $(i_1, 2R + 1, i_3)$ to $(S, 2R + 1, i_3)$, $i_1 = 1$, or from $(i_1, 2R + 1, i_3)$ to $(i_1 - 1, 2R + 1, i_3)$, $i_1 = 2, \ldots, S$; $i_3 = 1, 2, \ldots, N$, with intensity $i_1\gamma$ when an item perishes.
  - the completion of essential service makes one customer leave the system and decrease the inventory level by one. Thus a transition takes place from $(1, 2R + 1, i_3)$ to $(S, 2R + 1, i_3 - 1)$ or from $(i_1, 2R + 1, i_3)$ to $(i_1 - 1, 2R + 1, i_3 - 1)$ or from $(1, 2R + 1, 1)$ to $(S, 0, 0)$ or from $(i_1, 2R + 1, 1)$ to $(i_1 - 1, 0, 0)$, with intensity of transition μ_1, $i_1 = 2, \ldots, S$; $i_3 = 2, \ldots, N$.
- For other transition from (i_1, i_2, i_3) to (j_1, j_2, j_3), except $(i_1, i_2, i_3) \neq (j_1, j_2, j_3)$, the rate is zero.
- Finally, note that

$$\nu((i_1, i_2, i_3), (j_1, j_2, j_3)) = - \sum_{\substack{j_1 \\ (j_1, j_2, j_3) \neq (i_1, i_2, i_3)}} \sum_{j_2} \sum_{j_3} \nu((i_1, i_2, i_3), (j_1, j_2, j_3)).$$

Then the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$[\Theta]_{i_1 j_1} = \begin{cases} B_{i_1} & j_1 = i_1,\ i_1 = 1,2,\ldots,S \\ A_{i_1} & j_1 = i_1 - 1,\ i_1 = 2,\ldots,S-1,S \\ A_1 & j_1 = S,\ i_1 = 1, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

where the block matrices appearing in Θ are as follows: For $i_1 = 1,2,\cdots,S$,

$$[A_{i_1}]_{i_2 j_2} = \begin{cases} H_{i_1}, & j_2 = i_2, & i_2 = 0, \\ W_{i_1 i_2}, & j_2 = i_2, & i_2 = 1,2,\ldots,2R, \\ U_{i_2}, & j_2 = R + i_2, & i_2 = 1,\ldots,R, \\ L_{i_1 i_2}, & j_2 = i_2, & i_2 = 2R+1, \\ F_0, & j_2 = 0, & i_2 = 2R+1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_{i_1}]_{i_2 j_2} = \begin{cases} F_{i_1}, & j_2 = 0, & i_2 = 0, \\ C_0, & j_2 = 1, & i_2 = 0, \\ D_{i_1 i_2}, & j_2 = i_2, & i_2 = 1,\ldots,R, \\ E_{i_1 i_2}, & j_2 = i_2, & i_2 = R+1,\ldots,2R, \\ G_{i_1}, & j_2 = i_2, & i_2 = 2R+1, \\ K_{i_2}, & j_2 = i_2 - (R-1), & i_2 = R+1,\ldots,2R-1, \\ J_0, & j_2 = 2R+1, & i_2 = 2R, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_0]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 = R, \\ 0, & \text{otherwise,} \end{cases}$$

$$[J_0]_{i_3 j_3} = \begin{cases} \mu_2, & j_3 = i_3 - 1, \quad i_3 = 2,\ldots,N, \\ 0, & \text{otherwise,} \end{cases}$$

$$[F_0]_{i_3 j_3} = \begin{cases} \mu_1, & j_3 = i_3 - 1, \quad i_3 = 1, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_1 = 1,2,3,\ldots,S$

$$[H_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, & i_3 \in V_0^N, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_2 = 1,2,3,\ldots,R$

$$[W_{i_1 i_2}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, & i_3 \in V_{l-i_2}^N, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_2 = R+1, R+2,\ldots,2R$

$$[W_{i_1 i_2}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, & i_3 \in V_{r-i_2}^N, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_2 = 1,2,3,\ldots,R$

$$[U_{i_2}]_{i_3 j_3} = \begin{cases} \mu_1, & j_3 = i_3, & i_3 \in V_{l-i_2}^N, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_2 = 2R+1$

$$[L_{i_1 i_2}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, & i_3 \in V_1^N, \\ \mu_1, & j_3 = i_3 - 1, & i_3 \in V_2^N, \\ 0, & \text{otherwise,} \end{cases}$$

For $i_1 = 1,2,3,\dots,S$,

$$[F\,i_1]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{R-1}, \\ -(\lambda + i_1\gamma), & j_3 = i_3, \quad i_3 \in V_0^{R}, \\ 0, & \text{otherwise}, \end{cases}$$

For $i_2 = 1,2,3,\dots,R$

$$[D_{i_1 i_2}]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_{l-i_2}^{N-1}, \\ -(\lambda\bar{\delta}_{i_3 N} + i_1\gamma + \mu_1), & j_3 = i_3, \quad i_3 \in V_{l-i_2}^{N}, \\ 0, & \text{otherwise}, \end{cases}$$

For $i_2 = R+1, R+2,\dots,2R$

$$[E_{i_1 i_2}]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_{r-i_2}^{N-1}, \\ -(\lambda\bar{\delta}_{i_3 N} + i_1\gamma + \mu_2), & j_3 = i_3, \quad i_3 \in V_{r-i_2}^{N}, \\ 0, & \text{otherwise}, \end{cases}$$

For $i_1 = 1,2,3,\dots,S$,

$$[G_{i_1}]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ -(\lambda\bar{\delta}_{i_3 N} + i_1\gamma + \mu_1), & j_3 = i_3, \quad i_3 \in V_1^{N}, \\ 0, & \text{otherwise}, \end{cases}$$

For $i_2 = R+1, R=2,\dots,2R$,

$$[K_{i_2}]_{i_3 j_3} = \begin{cases} \mu_2, & j_3 = i_3 - 1, \quad i_3 \in V_{r-i_2}^{N}, \\ 0, & \text{otherwise}, \end{cases}$$

It can be noted that the matrices A_{i_1} and $B_{i_1}, i_1 = 1,\dots,S$ are square matrices of order x. The sub matrices $W_{i_1 i_2}, i_1 = 1,\dots,S,\ i_2 = 1,\dots,2R,\ U_{i_2}, i_2 = 1,\dots,R,\ D_{i_1 i_2}, i_1 = 1,\dots,S,\ i_2 = 1,\dots,R,\ E_{i_1 i_2}, i_1 = 1,\dots,S,\ i_2 = 1,\dots,R$, are square matrices of order y_{i_2}. $L_{i_1 i_2}, i_1 = 1,\dots,S,\ i_2 = 2R+1$ and G_{i_1} are square matrices of order N. H_{i_1}, and $F_{i_1}\, i_1 = 1,\dots,S$ are square matrices of order $R+1$. F_0, C_0, $K_{i_2}, i_2 = R+1,\dots,2R-1$, and J_0 are matrices of size $N \times (R+1)$, $(R+1)\times(N-R)$, $y_{i_2} \times z_{i_2}$ and $(N-1)\times N$.

3.1. Steady state analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X(t)), t \geq 0\}$ on the finite space E is irreducible. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[L(t) = i_1, Y(t) = i_2, X(t) = i_3 | L(0), Y(0), X(0)],$$

exists. Let $\mathbf{\Phi} = (\mathbf{\Phi^{(1)}},\dots,\mathbf{\Phi^{(S)}})$, each vector $\mathbf{\Phi^{(i_1)}}$ being partitioned as follows

$$\mathbf{\Phi^{(i_1)}} = \mathbf{\Phi^{(i_1,i_2)}},\ i_1 = 1,2,\dots,S;\ i_2 = 0,1,2,\dots,2R+1;$$

where

$$\mathbf{\Phi^{(i_1,0)}} = (\phi^{(i_1,0,0)}, \phi^{(i_1,0,1)},\dots,\phi^{(i_1,0,R)}), \qquad i_1 = 1,2,\dots,S;$$
$$\mathbf{\Phi^{(i_1,i_2)}} = (\phi^{(i_1,i_2,l-i_2)}, \phi^{(i_1,i_2,l-i_2+1)},\dots,\phi^{(i_1,i_2,N)}), \qquad i_1 = 1,2,\dots,S; i_2 = 1,2,\dots,R;$$

$$\mathbf{\Phi^{(i_1,i_2)}} = (\phi^{(i_1,i_2,r-i_2)}, \phi^{(i_1,i_2,r-i_2+1)},\dots,\phi^{(i_1,i_2,N)}), \qquad i_1 = 1,2,\dots,S; i_2 = R+1, R+2,\dots,2R;$$
$$\mathbf{\Phi^{(i_1,2R+1)}} = (\phi^{(i_1,2R+1,1)}, \phi^{(i_1,2R+1,2)},\dots,\phi^{(i_1,2R+1,N)}), \qquad i_1 = 1,2,\dots,S;$$

where $\phi^{(i_1,i_2,i_3)}$ denotes the steady state probability for the state (i_1, i_2, i_3) of the process, exists and is given by

$$\mathbf{\Phi}\Theta = \mathbf{0} \quad \text{and} \quad \sum\sum_{(i_1,i_2,i_3)}\sum \phi^{(i_1,i_2,i_3)} = 1. \tag{*}$$

Theorem 3.1.
The limiting distribution $\mathbf{\Phi}$ *is given by,*

$$\mathbf{\Phi}^{(i_1)} = \mathbf{\Phi}^{(1)}\Gamma_{i_1}, \quad i_1 = 1,\ldots,S, \quad \textit{where} \quad \Gamma_{i_1} = (-1)^{i_1-1} \mathop{\Omega}_{k=1}^{n=i_1-1} A_k B_{k+1}^{-1}, \quad i_1 = 1,2,\ldots,S$$

The value of $\mathbf{\Phi}^{(1)}$ *can be obtained from the relation*

$$\mathbf{\Phi}^{(1)}\left[B_1 + \left\{(-1)^{S-1} \mathop{\Omega}_{n=1}^{S-1} A_n B_{n+1}^{-1}\right\}\right] = 0, \; \textit{and} \quad \mathbf{\Phi}^{(1)}\left[\sum_{i_1=2}^{S}\left((-1)^{i_1-1} \mathop{\Omega}_{n=1}^{i_1-1} A_n B_{n+1}^{-1}\right) + I\right] \boldsymbol{e} = 1.$$

Proof. The first equation of $(*)$ yields the following set of equations :

$$\mathbf{\Phi}^{(i_1)} A_{i_1} + \mathbf{\Phi}^{(i_1+1)} B_{i_1+1} = \mathbf{0}, \qquad i_1 = 1,2,\ldots,S-1,$$
$$\mathbf{\Phi}^{(S)} A_S + \mathbf{\Phi}^{(1)} B_1 = \mathbf{0}.$$

Solving the above set of equations we get the required solution. □

4. System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

4.1. Expected Inventory Level

Let η_I denote the excepted inventory level in the steady state. Since $\Phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of customers in the waiting hall and the status of the server, $\Phi^{(i_1)}\mathbf{e}$ gives the probability of i_1 item in the inventory in the steady state. Hence η_I is given by

$$\eta_{II} = \sum_{i_1=1}^{S} i_1 \Phi^{(i_1)}\mathbf{e}$$

4.2. Expected reorder rate

Let η_{RR} denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from 1 to 0. This will occur when

1. a failure of one item when the server is on idle
2. a failure of one item when the server is busy (with initial customer/regular customer)
3. a essential service completion of the initial customer
4. a essential service completion of the regular customer

Hence we get

$$\eta_{RR} = \mu_1 \sum_{i_2=1}^{R} \sum_{i_3=l-i_2}^{N} \phi^{(1,i_2,i_3)} + \mu_1 \sum_{i_3=1}^{N} \phi^{(1,2R+1,i_3)} + \gamma \sum_{i_3=0}^{R} \phi^{(1,0,i_3)} +$$
$$\gamma \sum_{i_2=1}^{R} \sum_{i_3=l-i_2}^{N} \phi^{(1,i_2,i_3)} + \gamma \sum_{i_2=R+1}^{2R} \sum_{i_3=r-i_2}^{N} \phi^{(1,i_2,i_3)} + \gamma \sum_{i_3=1}^{N} \phi^{(1,2R+1,i_3)}$$

4.3. Expected perishable rate

Since $\Phi^{(i_1)}$ is the steady state probability vector for inventory level, the expected perishable rate η_{PP} is given by

$$\eta_P = i_1\gamma \sum_{i_1=1}^{S}\sum_{i_3=0}^{R} \phi^{(i_1,0,i_3)} + i_1\gamma \sum_{i_1=1}^{S}\sum_{i_2=1}^{R}\sum_{i_3=l-i_2}^{N} \phi^{(i_1,i_2,i_3)} + i_1\gamma \sum_{i_1=1}^{S}\sum_{i_2=R+1}^{2R}\sum_{i_3=r-j}^{N} \phi^{(i_1,i_2,i_3)} + i_1\gamma \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} \phi^{(i_1,2R+1,i_3)}$$

4.4. Expected number of initial customers in the waiting hall

Let η_{w_1} denote the expected number of initial customers in the steady state. Since $\phi^{(i_1,i_2,i_3)}$ is a vector of probabilities with the inventory level is i_1, the server status is i_2 and the number of customer in the waiting hall is i_3, the expected number of initial customers η_{w_1} in the steady state is given by

$$\eta_{w_1} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{R} i_3\phi^{(i_1,0,i_3)} + \sum_{i_1=1}^{S}\left(\sum_{k=0}^{R-1}\left[\sum_{i_3=1}^{R-k} i_3\phi^{(i_1,\{\sum_{i=0}^{R-1}\delta_{ki}\}k+1,i_3)}\right]\right) + \sum_{i_1=1}^{S}\left(\sum_{k=0}^{R-1}\left[\sum_{i_3=1}^{R-k} i_3\phi^{(i_1,\{\sum_{i=0}^{R-1}\delta_{ki}\}k+(R+1),i_3)}\right]\right)$$

4.5. Expected number of regular customers in the waiting hall

Let η_{w_2} denote the expected number of regular customers in the steady state. Since $\phi^{(i_1,i_2,i_3)}$ is a vector of probabilities with the inventory level is i_1, the server status is i_2 and the number of customer in the waiting hall is i_3, the expected number of regular customers η_{w_2} in the steady state is given by

$$\eta_{w_2} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N} i_3\phi^{(i_1,2R+1,i_3)} + \sum_{i_1=1}^{S}\left(\sum_{k=0}^{R-1}\left[\sum_{i_3=1}^{N-(R-k)} i_3\phi^{(i_1,\{\sum_{i=0}^{R-1}\delta_{ki}\}k+1,i_3)}\right]\right) + \sum_{i_1=1}^{S}\left(\sum_{k=0}^{R-1}\left[\sum_{i_3=1}^{N-(R-k)} i_3\phi^{(i_1,\{\sum_{i=0}^{R-1}\delta_{ki}\}k+(R+1),i_3)}\right]\right)$$

4.6. Expected waiting time of the initial customers

Let Γ_1 denote the expected waiting time of the initial customers in the waiting hall. Then by Little's formula

$$\Gamma_1 = \frac{\eta_{w_1}}{\eta_{AI}},$$

where η_{w_1} is the expected number of initial customers in the waiting hall and the effective arrival rate of the initial customers (Ross [17]), η_{AI} is given by

$$\eta_{AI} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{R-1}(N-i_3)\lambda\phi^{(i_1,0,i_3)}$$

4.7. Expected waiting time of the regular customers

Let Γ_1 denote the expected waiting time of the regular customers in the waiting hall. Then by Little's formula

$$\Gamma_2 = \frac{\eta_{w_2}}{\eta_{AR}},$$

where η_{w_2} is the expected number of regular customers in the waiting area and the effective arrival rate of the regular customers (Ross [17]), η_{AR} is given by

$$\eta_{AI} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}(N-i_3)\lambda\phi^{(i_1,2R,i_3)} + \sum_{i_1=1}^{S}\left(\sum_{k=1}^{R}\left[\sum_{i_3=k+1}^{(N-1)}(N-i_3)\lambda\phi^{(i_1,\{\sum_{i=1}^{R}\delta_{ki}\}(R-(k-1)),i_3)}\right]\right)$$
$$+\sum_{i_1=1}^{S}\left(\sum_{k=1}^{R}\left[\sum_{i_3=k+1}^{(N-1)}(N-i_3)\lambda\phi^{(i_1,\{\sum_{i=1}^{R}\delta_{ki}\}(2R-(k-1)),i_3)}\right]\right)$$

4.8. Expected waiting time of the initial and regular customers

Let Γ_b denote the expected waiting time of the initial customers and regular customers in the waiting hall. Then

$$\Gamma_b = \Gamma_1 + \Gamma_2,$$

4.9. Probability that server is idle

Let η_{SI} denote the probability that server is idle is given by

$$\eta_{PSI} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{R}\phi^{(i_1,0,i_3)}$$

4.10. Probability that server is providing essential service to the initial customers

Let η_{PSI} denote the probability that server is providing essential service to the initial customer is given by

$$\eta_{PSI} = \sum_{i_1=1}^{S} \left(\sum_{k=1}^{R} \left[\sum_{i_3=k+1}^{N} \phi^{(i_1, \{\sum_{i=1}^{R} \delta_{ki}\}(R-(k-1)), i_3)} \right] \right)$$

4.11. Probability that server is providing bonus service to the initial customers

Let η_{PSI} denote the probability that server is providing essential bonus service to the initial customer is given by

$$\eta_{PSI} = \sum_{i_1=1}^{S} \left(\sum_{k=1}^{R} \left[\sum_{i_3=k+1}^{N)} \phi^{(i_1, \{\sum_{i=1}^{R} \delta_{ki}\}(2R-(k-1)), i_3)} \right] \right)$$

4.12. Probability that server is providing essential service to the regular customers

Let η_{PSR} denote the probability that server is providing essential service to the regular customer is given by

$$\eta_{PSR} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \phi^{(i_1, 2R+1, i_3)}$$

5. Cost analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S, N, M) = c_h \eta_{II} + c_s \eta_{RR} + c_p \eta_{PP} + c_w \Gamma_b,$$

c_h : The inventory carrying cost per unit item per unit time,
c_s : Setup cost per order,
c_p : Perishable cost per unit item per unit time,
c_w : Waiting time cost of a customer per unit time,

Substituting the values of η's, we get TC(S, N, M)=

$$c_s \mu_1 \sum_{i_2=1}^{R} \sum_{i_3=l-i_2}^{M} \phi^{(1,i_2,i_3)} + c_s \mu_1 \sum_{i_3=1}^{N} \phi^{(1,2R+1,i_3)} + c_s \gamma \sum_{i_3=0}^{R} \phi^{(1,0,i_3)} + c_s \gamma \sum_{i_2=1}^{R} \sum_{i_3=l-i_2}^{N} \phi^{(1,i_2,i_3)} +$$
$$c_s \gamma \sum_{i_2=R+1}^{2R} \sum_{i_3=r-i_2}^{N} \phi^{(1,i_2,i_3)} + c_s \gamma \sum_{i_3=1}^{N} \phi^{(1,2R+1,i_3)} + c_h \sum_{i_1=1}^{S} i_1 \Phi^{(i_1)} \mathbf{e} + c_p i_1 \gamma \sum_{i_1=1}^{S} \sum_{i_3=0}^{R} \phi^{(i_1,0,i_3)} +$$
$$c_p i_1 \gamma \sum_{i_1=1}^{S} \sum_{i_2=1}^{R} \sum_{i_3=l-i_2}^{N} \phi^{(i_1,i_2,i_3)} + c_p i_1 \gamma \sum_{i_1=1}^{S} \sum_{i_2=R+1}^{2R} \sum_{i_3=r-i_2}^{N} \phi^{(i_1,i_2,i_3)} + c_p i_1 \gamma \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \phi^{(i_1,2R+1,i_3)} + c_w(\Gamma_1 + \Gamma_2)$$

6. Conclusions

The stochastic model discussed here is useful in studying a finite source stochastic inventory system with bonus service for certain customers. The joint probability distribution of the number of customers in the waiting hall and the inventory level is derived in the steady state. Various system performance measures and the long-run total expected cost rate are derived.

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