Existence and uniqueness results for impulsive neutral stochastic differential systems with nonlocal conditions

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Abstract: This paper is mainly concerned with the existence and uniqueness of mild solutions for impulsive neutral stochastic differential equations with nonlocal conditions in Hilbert spaces. The results are obtained by using fractional powers of operator in the semigroup theory and Sadovskii’s fixed point theorem. In the end as an application, an example has been presented to illustrate the results obtained.

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1. Introduction

Stochastic differential equations have been considered extensively through discussion in the finite and infinite dimensional spaces. As a matter of fact, there exist broad literature on the related to the topic and it has played an important role in many ways such as option pricing, forecast of the growth of population, etc., and as an applications which cover the generalizations of stochastic differential equations arising in the fields such as electromagnetic theory, population dynamics, and heat conduction in material with memory. Random differential and integral equations play an important role in characterizing numerous social, physical, biological and engineering problems. For more details reader may refer [1–4] and reference therein.

Byszewski and Lakshmikantham [5] and Byszewski [6] introduced nonlocal initial conditions into initial value problems and argued that the corresponding models are accurately described the phenomena. The nonlocal conditions in the different fields has been discussed in Byszewski and Byszewski and Akca [7, 8], Deng [9] and [10–12] reference therein. The deterministic equations coupled with classical initial conditions has been studied extensively both when A is linear and when A is nonlinear for more details reader may refer [13–16, 25] and references therein.

Impulsive effects exists widely in the various evolutionary process which are characterized by the fact that undergo abrupt change at certain moments of time, involved in such fields as medicine, biology, economics, bioengineering, chemical technology etc. The duration of these changes is often negligible when compared to the total duration of the process. Such changes can be reasonably approximated as being instantaneous changes of the state or as impulses. Thus the theory of impulsive differential equations has seen considerable development. For more details reader may refer [17–20, 23].

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In [21], Balsubramaniam et al. studied the existence of solutions for semilinear neutral stochastic functional differential systems with nonlocal conditions. In [22] Chang et al. examined the non-densely defined neutral impulsive differential inclusions with nonlocal conditions.

The main aim of this paper is to focus in studying the existence and uniqueness of the impulsive neutral stochastic differential systems with nonlocal conditions,

\[ d[x(t) + f_j(t, x(t), x(a_i(t)), \ldots, x(a_m(t)))dt + G(t, x(t), x(c_1(t)), \ldots, x(c_n(t)))dw(t), \quad t \in J := [0, b], t \neq t_k, k = 1, 2, \ldots, m, \]

\[ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \]

\[ x(0) = x_0 + g(x), \]

where \( A \) is the infinitesimal generator of an analytic semigroup of bounded linear operators \( \{T(t), t \geq 0\} \) on a separable Hilbert space on \( H \) with inner product \( (\cdot, \cdot) \) and norm \( \| \cdot \| \). Let \( K \) be the other separable Hilbert space with inner product \( (\cdot, \cdot_K) \) and norm \( \| \cdot \|_K \). Suppose \( \{w(t)\}_{t \geq 0} \) is a given \( K \)-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator \( Q \geq 0 \) defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \).

The fixed time \( t_k, k = 1, \ldots, n \), satisfy \( 0 \leq t_1 \leq \ldots \leq t_n \leq b \), \( x(t_k^+) \) and \( x(t_k^-) \) denote the right and left limits of \( x(t) \) at \( t = t_k \), \( \Delta x(t_k) = x(t_k^+) - x(t_k^-) \) represents the jump in the state \( x \) at time \( t_k \), where \( I_k, k = 1, 2, \ldots, m \), are bounded which determine the size of the jump. The function \( f_j, f_k, G, \) and \( g \) are the given functions to be defined later.

The structure of this paper is as follows. In Section 2, we recall some preliminaries. In Section 3, we devoted to the study of existence and uniqueness of mild solution for system (1)-(3). Finally in Section 4, an example is presented, which illustrates the main results.

2. Preliminaries

In this section, we recall a few stochastic results, lemmas and notations which are need to establish our main results.

Throughout this paper \((H, \| \cdot \|)\) and \((K, \| \cdot \|)\) denotes the two real separable Hilbert space. Let \( \mathcal{L}(K, H) \) be the set of all linear bounded operator from \( K \) into \( H \) equipped with the usual norm operator \( \| \cdot \|_\mathcal{L} \). Let \( (\Omega, \mathcal{F}, P, H) \) be the complete probability space furnished with a complete family of right continuous increasing \( \mathcal{F} \)-measurable function \( x(t) : \Omega \to H \) and a collection of random variables \( S = \{x(t, \omega) : \Omega \to H \}, t \in J \) is called stochastic process. Usually we write \( x(t) \) instead of \( x(t, \omega) \) and \( x(t) : J \to H \) in the space of \( S \). Let \( \{e_i\}_{i=1}^\infty \) be a complete orthonormal basis of \( K \). Suppose that \( \{w(t) : t \geq 0\} \) is a cylindrical \( K \)-valued wiener process with a finite trace nuclear covariance operator \( Q \geq 0 \), denote \( \text{Tr}(Q) = \sum_{i=1}^\infty \lambda_i = \lambda < \infty \), which satisfies that \( Qe_i = \lambda_i e_i \). So, actually, \( \omega(t) = \sum_{i=1}^\infty \sqrt{\lambda_i} \omega_i(t) e_i \), where \( \{\omega_i(t)\}_{i=1}^\infty \) are mutually independent one-dimensional standard Wiener processes. We assume that \( \mathcal{F}_t = \sigma(\omega(s) : 0 \leq s \leq t) \) is the \( \sigma \)-algebra generated by \( \omega \) and \( \mathcal{F} \). Let \( \Psi \in \mathcal{L}(K, H) \) and define

\[
\|\Psi\|^2_Q = \text{Tr}(\Psi^* Q \Psi) = \sum_{n=1}^\infty \| \sqrt{\lambda_n} \Psi e_n \|^2.
\]

If \( \|\Psi\|_Q < \infty \), then \( \Psi \) is called a \( Q \)-Hilbert-Schmidt operator. Let \( \mathcal{L}^2_Q(K, H) \) denote the space of all \( Q \)-Hilbert-Schmidt operators \( \Psi : K \to H \). The completion \( \mathcal{L}^2(Q, K, H) \) of \( \mathcal{L}^2(K, H) \) with respect to the topology induced by the norm \( \| \cdot \|_Q \) where \( \|\Psi\|_Q = (\Psi, \Psi) \) is a Hilbert space with the above norm topology.

Let \( A \) be the infinitesimal generator of an analytic semigroup \( T(t) \) in \( H \). Suppose that \( 0 \in \rho(A) \) where \( \rho(A) \) denote the resolvent set of \( A \) and that semigroup \( T(t) \) is uniformly bounded that is to say, \( \|T(t)\| \leq M_f \) for some constant \( M_f \geq 1 \) and for every \( t \geq 0 \). Then for \( \alpha \in (0, 1] \), it is possible to define the fractional power operator \( (-A)_\alpha \) as a closed linear invertible operator on its domain \( D((-A)_\alpha) \). Furthermore, the subspace \( D((-A)_\alpha) \) is dense in \( H \) and the expression

\[
\|x\|_\alpha = \|(-A)_\alpha x\|, \quad x \in D((-A)_\alpha),
\]

defines the norm on \( H_\alpha = D((-A)_\alpha) \). Furthermore of fractional power of operator and semigroup refer [15]. Then the following property are well known [15].

**Lemma 2.1.**

Suppose the following properties are satisfied.

(i) Let \( 0 \leq \alpha \leq 1 \). Then \( H_\alpha \) is a Banach space.

(ii) If \( 0 < \beta < \alpha \leq 1 \), then \( H_\alpha \subset H_\beta \) and the embedding is compact whenever the resolvent operator of \( A \) is compact.

(iii) For every \( 0 < \alpha \leq 1 \), there exists a positive constant \( M_\alpha > 0 \) such that
The collection of all strongly measurable, square-integrable $H$-valued random variables, denoted by $\mathcal{P} \mathcal{C}(J, L_2(\Omega, \mathcal{F}, P; H)) = \{ x : J \to L_2 : x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^-) \text{ and } x(t_k^+) \text{ exist and } x(t_k^+) = x(t_k) \}$ for each $t_k$, $k = 1, 2, 3, \ldots$, is the Banach space of piecewise continuous maps from $J$ into $L_2(\Omega, \mathcal{F}, P; H)$ satisfying the condition $\sup_{t \in J} E[|x(t)|^2] < \infty$. Let $\mathcal{P} \mathcal{C}(J, L_2(\Omega, \mathcal{F}, P; H)) \subseteq \mathcal{P} \mathcal{C}(J, L_2(\Omega, H))$ be the closed subspace of $\mathcal{P} \mathcal{C}(J, L_2(\Omega, \mathcal{F}, P; H))$ consisting of measurable, $\mathcal{F}_t$-adapted and $H$-valued processes $x(t)$. Then $\mathcal{P} \mathcal{C}(J, L_2(\Omega, H))$ is a Banach space endowed with the norm $\|x\|_{\mathcal{P} \mathcal{C}} = \sup_{t \in J} \{ E[|x(t)|^2] : x \in \mathcal{P} \mathcal{C}(J, L_2(\Omega, H)) \}$.

Here we discuss existence of mild solutions of the system (1)-(3) is studied with the following assumption:

(H1) There exists a constant $\beta \in (0, 1)$ such that $f_j : [0, b] \times H^{n+1} \to H$ is a continuous function, and $M_{f_j}, \overline{M}_{f_j} > 0$ such that $(-A)^{\beta} f_j$ satisfies the Lipschitz condition:

$$
\|( -A)^{\beta} f_j(s_1, x_0, x_1, \ldots, x_m) - ( -A)^{\beta} f_j(s_2, y_0, y_1, \ldots, y_m) \| \\
\leq M_{f_j} \|s_1 - s_2\| + \max_{i=0,1,\ldots,m} \|x_i - y_i\|,
$$

for any $0 \leq s_1, s_2 \leq b$, $x_i, y_i \in H$, $i = 0, 1, \ldots, m$. Moreover, the inequality

$$
\|( -A)^{\beta} f_j(t, x_0, x_1, \ldots, x_m) \| \leq \overline{M}_{f_j} (\max \|x_i\| : i = 0, 1, \ldots, m) + 1,
$$

for any $(t, x_0, x_1, \ldots, x_m) \in J \times H^{n+1}$, $j = 1, 2, \ldots$.

(H2) The function $G : [0, b] \times H^{n+1} \to L_Q(K, H)$ satisfies the following conditions:

(i) For each $t \in [0, b]$, the function $G(t, \cdot) : H^{n+1} \to L_Q(K, H)$ is continuous and for each $(x_0, x_1, \ldots, x_n) \times H^{n+1}$ the function $G(\cdot, x_0, x_1, \ldots, x_n) : J \to L_Q(K, H)$ is $\mathcal{F}_t$- measurable;

(ii) For each positive number $l \in \mathbb{N}$, there is a positive function $h_l \in L^2(J)$ such that

$$
\sup_{\|x_0\|^2, \ldots, \|x_n\|^2 \leq l} E[\|G(t, x_0, x_1, \ldots, x_n)\|^2] \leq h_l(t)
$$

and

$$
\lim_{t \to \infty} \int_0^b \frac{h_l(s)}{l} ds = \mu < +\infty.
$$

(H3) $a_k, b_k, c_k \in C(J, J)$, $k = 1, 2, \ldots, m$. $g : \mathcal{P} \mathcal{C} \to L_2(\Omega, \mathcal{P} \mathcal{C})$ satisfies that

(i) There exist positive constants $M_g$ and $\overline{M}_g$ such that,

$$
\|g(x)\| \leq M_g \|x\|_{\mathcal{P} \mathcal{C}} + \overline{M}_g \text{ for all } x \in \mathcal{P} \mathcal{C};
$$

(ii) $g$ is completely continuous.

(H4) $I_k : \mathcal{P} \mathcal{C} \to \mathcal{P} \mathcal{C}$ is completely continuous and there exist continuous nondecreasing functions $L_k : R_+ \to R_+$ such that for each $x \in \mathcal{P} \mathcal{C}$

$$
\|I_k(x)\| \leq L_k(\|x\|_{\mathcal{P} \mathcal{C}}), \lim_{l \to \infty} \frac{L_k(l)}{l} = \lambda_k < +\infty.
$$

Our main results are based upon the following fixed point theorem [24].

**Theorem 2.1 (Sadovskii’s Fixed Point Theorem).**

Let $\Phi$ be a condensing operator on a Banach space, that is, $\Phi$ is continuous and takes bounded sets into bounded sets, and let $\alpha(\Phi, B) \leq \alpha(B)$ for every bounded set $B$ of $H$ with $\alpha(B) > 0$ of $\Phi(\Omega) \subset \Omega$ for a convex, closed, and bounded set $\Omega$ of $H$, then $\Phi$ has fixed point in $H$. (where $\alpha(\cdot)$ denotes Kuratowski’s measure of non-compactness).
3. Main results

In this section we state and prove our existence results, now we define the mild solutions for the system (1)-(3).

**Definition 3.1.**
An $\mathcal{F}$-adapted stochastic process $x(t): I \rightarrow H$ is said to be a mild solution of nonlocal Cauchy problem (1)-(3) if

1. $x_0 \in L^0_2(\Omega, H), g(x) \in L^0_2(\Omega, \mathcal{P}\mathcal{C})$;
2. $\Delta x|_{t=t_k} = I_k(x(t_k^+)), k = 1, 2, \ldots, m$;
3. for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = T(t)[x_0 + g(x) + f_1(0, u(0)), \ldots, x(a_m(0))]-f_1(t, x(t), x(a_0(t)), \ldots, x(a_m(t))]$$

$$+ \int_0^t T(t-s)f_2(s,x(s),x(b_1(s)),\ldots,x(b_m(s)))ds$$

$$- \int_0^t ATr(t-s)f_3(s,x(s),x(a_1(s)),\ldots,x(a_m(s)))ds$$

$$+ \int_0^t T(t-s)G(s,x(s),x(c_1(s)),\ldots,x(c_m(s)))dw(s) + \sum_{0 < t_k < t} T(t-t_k)I_k(x(t_k^+)), \quad t \in J.$$

**Theorem 3.1.**
Assume the conditions (H1)-(H5) are satisfied and $x(0) \in L^0_2(\Omega, H)$, then nonlocal Cauchy problem (1)-(3) has a mild solution provided that

$$L_0 = M^2_{\beta} \left( M^2_{\beta} + 1 + \frac{(M_{\beta}b^\beta)^2}{2\beta - 1} \right) < 1$$

and

$$36 \left( M^2_{\beta} [b^\beta ((M^2_{\beta} + (M_0M f_1)^2) + (2bM_0M f_2)^2 + Tr(Q)\mu + \sum_{k=1}^{m} \lambda_k] + (2M_0M f_1)^2 \right) + \frac{1}{2\beta - 1} (2M_{1-\beta}M f_1 b^\beta)^2 \right) < 1,$$

where $M_0 = \|[-A]^{-\beta}\|$ and $M_{1-\beta}$ is defined in Lemma 2.1.

**Proof.** For the sake of brevity, we rewrite that

$$(t, x(t), x(a_1(t)), \ldots, x(a_m(t))) = (t, u(t)),$$

$$(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) = (t, v(t)),$$

$$(t, x(t), x(c_1(t)), \ldots, x(c_m(t))) = (t, p(t)).$$

Consider the operator $\Phi$ on $\mathcal{P}\mathcal{C}$ defined by

$$(\Phi x)(t) = T(t)[x_0 + g(x) + f_1(0, u(0))] - f_1(t, x(t), u(t)) + \int_0^t T(t-s)f_2(s,v(s))ds$$

$$- \int_0^t ATr(t-s)f_3(s,u(s))ds + \int_0^t T(t-s)G(s,p(s))dw(s)$$

$$+ \sum_{0 < t_k < t} T(t-t_k)I_k(x(t_k^+)), \quad t \in J.$$

$$= T(t)[x_0 + f_1(0, u(0)) + g(x)] - f_1(t, x(t)) - I^u_{f_1}(t) + I^v_{f_2}(t) + I^p_{G}(t)$$

$$+ \sum_{0 < t_k < t} T(t-t_k)I_k(x(t_k^+)), \quad t \in J.$$
We shall show that the operator $\Phi$ has a fixed point, which is a solution of system (1)-(3). For each positive integer $l$, let

$$B_l = \{ x \in \mathcal{P} \mathcal{C} : E\|x(t)\|^2_{\mathcal{P} \mathcal{C}} \leq l, 0 \leq t \leq b \}.$$ 

It is clear that for each $l$, $B_l \subseteq \mathcal{P} \mathcal{C}$ is clearly a bounded closed convex set in $\mathcal{P} \mathcal{C}$. In addition to the familiar Young, Holder and Minkowski the inequalities of the form $(\sum_{i=1}^n a_i)^m \leq \sum_{i=1}^n a_i^m$ where $a_i$ are nonnegative constants $(i = 1, 2, \ldots, n)$ and $m, n \in N$ is helpful in establishing various estimates, from Lemma 2.1 and (4) together with Holder inequality, yields the following relation:

$$E\|I^u_{B_l}(t)\|^2 = \| \int_0^t (-A)T(t-s) f_1(s, u(s)) ds \|^2$$

$$\leq \| \int_0^t (-\alpha)^{-\beta} T(t-s) (-A)^{\beta} f_1(s, u(s)) ds \|^2$$

$$= 4 \int_0^t \frac{M^2_{\beta/\alpha} (\max\{E\|x_l\|^2 : i = 0, 1, \ldots, m\} + 1)}{(t-s)^{2(1-\beta)}} ds$$

and

$$E\|I^v_{B_l}(t)\|^2 = \| \int_0^t T(t-s) f_2(s, v(s)) ds \|^2$$

$$\leq \| \int_0^t (-\alpha)^{-\beta} T(t-s) (-A)^{\beta} f_2(s, v(s)) ds \|^2$$

$$= 4 b^2 M^2_{\beta/2} \int_0^t \| G(s, p(s)) \|^2 ds$$

$$\leq 4 b^2 \int_0^t h_i(s) ds.$$ 

(7)

(8)

It follows that $(\alpha) T(t-s) f_1(s, u(s)) ds$ and $T(t-s) f_2(s, v(s)) ds$ is integrable on $J$, so $\Phi$ is well defined on $B_l$. Similarly, from (H2)(ii) and together with the Itô’s formula, a computation can be performed to obtain the following:

$$E\|I^u_{B_l}(t)\|^2 = E\| T(t-s) G(s, p(s)) dw(s) \|^2$$

$$\leq Tr(Q)M^2_{\beta/2} \int_0^t \| G(s, p(s)) \|^2 ds$$

$$\leq Tr(Q)M^2_{\beta/2} \int_0^t h_i(s) ds.$$ 

(9)

Step 1. We claim that there exists a positive number $l$ such that $\Phi B_l \subseteq B_{l+1}$. If it is not true, then for each positive number $l$, there is a function $x_i(\cdot) \in B_l$, but $\| \Phi x(\cdot) \| > l$ for some $t(\cdot) \in J$, where $t(\cdot)$ denotes that $t$ is independent of $l$. However, on the other hand, we have

$$l \leq \| \Phi x(\cdot) \|$$

$$= \| T(t)x_0 + g(x) + f_1(t, u(0)) - f_1(t, u(t)) + \int_0^t T(t-s) f_2(s, v(s)) ds$$

$$\leq \int_0^t (3M^2_{\beta/2} \| x_0 \|^2_g + 4(M^2_{\beta/2} l + M_{\beta/2}^2 (l + 1)) + (2M_{\alpha} M_{\beta/2})^2 (l + 1)$$

$$+ (2bM_{\alpha} M_{\beta/2})^2 (l + 1) + \frac{1}{2\beta - 1}(2M_{\alpha} M_{\beta} b^{\beta/2})^2 (l + 1)$$

$$+ Tr(Q)M^2_{\beta/2} \int_0^t h_i(s) ds + M^2_{\beta/2} \sum_{k=1}^m L_k (l))$$

$$\leq M^* + 36 \{ 6M^2_{\beta/2} \| x_0 \|^2_g + (M_{\alpha} M_{\beta/2})^2 (l + 1) + (2M_{\alpha} M_{\beta/2})^2 (l + 1)$$

$$+ (2bM_{\alpha} M_{\beta/2})^2 (l + 1) + \frac{1}{2\beta - 1}(2M_{\alpha} M_{\beta} b^{\beta/2})^2 (l + 1)$$

$$+ Tr(Q)M^2_{\beta/2} l \int_0^t h_i(s) ds + M^2_{\beta/2} \sum_{k=1}^m L_k (l) \}.$$
where
\[
M^* = 36\left\{3M_T^2[\|x_0\|^2 + 4M_g^2 + 2(M_0\lambda_{\beta}^2) + (2bM_0\lambda_{\beta}^2)]^2 + \frac{1}{2\beta - 1}(2M_{1-\beta}\lambda_{\beta}^2)\right\}.
\]

Dividing on both sides by \( l \) and taking the lower limit as \( t \to +\infty \), we get
\[
36\left\{3M_T^2[\|x_0\|^2 + (2M_0\lambda_{\beta}^2)]^2 + (2bM_0\lambda_{\beta}^2) + Tr(Q)\left(\sum_{k=1}^m \lambda_k\right) + (2M_{1-\beta}\lambda_{\beta}^2)\right\} \geq 1.
\]

This is a contradictory to (6). Hence for positive integer \( l \), \( \Phi B_l \subseteq B_l \).

**Step 2.** Next we will show that the operator \( \Phi \) has a fixed point on \( B_l \). Now we decompose \( \Phi = \Phi_1 + \Phi_2 \) (is condensing), \( \Phi_1 \) is contraction and \( \Phi_2 \) is compact.

The operators \( \Phi_1, \Phi_2 \) are defined on \( B_l \) respectively by
\[
(\Phi_1 x)(t) = T(t)f_1(0, u(0)) - f_1(t, u(t)) - \int_0^t AT(t-s)f_1(s, u(s))ds,
\]
\[
(\Phi_2 x)(t) = T(t)x_0 + g(x) + \int_0^t T(t-s)f_2(s, v(s))ds + \int_0^t T(t-s)g(s, p(s))dw(s) + \sum_{0 < t \leq t_k \leq t_0} T(t-t_k)I_k(x(t_k^-)), \quad t \in J.
\]

We would verify that \( \Phi_1 \) is a contraction while \( \Phi_2 \) is a completely continuous operator.

To prove that \( \Phi_1 \) is a contraction, we take \( x_1, x_2 \in B_l \) arbitrarily. Then for each \( t \in J \) and by condition (H1) and (5), we have
\[
E[\|\Phi_1 x_1(t) - (\Phi_1 x_2(t))\|^2]
\leq E[\|f_1(t, u_1(t)) - f_1(t, u_2(t)) + T(t)[f_1(0, u_1(0)) - f_1(0, u_2(0))]\]
\[+ \int_0^t (-A)T(t-s)[f_1(s, u_1(s)) - f_1(s, u_2(s))]ds\|^2
\leq 9\left\{E[\|(-A)^\beta T(t)[(-A)^\beta f_1(t, u_1(t)) - (-A)^\beta f_1(t, u_2(t))]\|^2
\[+ E[\|(-A)^\beta f_1(0, u_1(0)) - (-A)^\beta f_1(0, u_2(0))]\|^2
\[+ E[\int_0^t (-A)^\beta T(t-s)[(-A)^\beta f_1(s, u_1(s)) - (-A)^\beta f_1(s, u_2(s))]ds\|^2\right\}
\leq 9\left\{M_0^2M_1^2M_0^2\sup_{0 \leq s \leq b} E[\|x_1(s) - x_2(s)\|^2 + M_0^2M_1^2\sup_{0 \leq s \leq b} E[\|x_1(s) - x_2(s)\|^2]
\[+ b\int_0^t \frac{1}{(t-s)^{2(1-\beta)}}M_0^2M_1^2E[\|x_1(s) - x_2(s)\|^2\right\}.
\]

Hence
\[
\|\Phi_1 x_1(t) - (\Phi_1 x_2(t))\|^2 \leq M_0^2M_1^2\left[M_0^2(1 + \frac{(M_{1-\beta}b^\beta)^2}{2\beta - 1})\right]\sup_{0 \leq s \leq b} \|x_1(s) - x_2(s)\|^2
\leq L_0\sup_{0 \leq s \leq b} \|x_1(s) - x_2(s)\|^2.
\]

Thus
\[
\|\Phi_1 x_1 - (\Phi_1 x_2)\|^2 \leq L_0\|x_1 - x_2\|^2,
\]

and so by assumption \( 0 \leq L_0 \leq 1 \), we see that \( \Phi_1 \) is a contraction.

To prove that \( \Phi_2 \) is compact, first we prove that \( \Phi_2 \) is continuous on \( B_l \). Let \( \{x_n\}_{n=0}^\infty \subseteq B_l \) with \( x_n \to x \) in \( B_l \), then by (H2)(i) and (H4)

(i) \( I_k, k=1,2,\ldots, m \) is continuous.

(ii) \( G(s, p_0(s)) \to G(s, p(s)), \quad n \to \infty. \)
Since
\[ E\|G(s, p_n(s)) - G(s, p(s))\| \leq 2g(s), \]
by the dominated convergence theorem, we have
\[ \Phi_2x_n \to \Phi_2x \quad \text{as} \quad n \to \infty. \]

Thus, \( \Phi_2 \) is continuous. Next, we prove that \( \{ \Phi_2 x : x \in B_1 \} \) is a family of equicontinuous functions. Let \( x \in B_1 \) and \( r_1, r_2 \in J \). Then if \( 0 < r_1 \leq r_2 \leq b \) and \( \varphi \in N_2(x) \), then for each \( t \in J \), then
\[
\| \Phi_2 x(t) - \Phi_2 x(t_1) \| \leq \| T(t_2) - T(t_1) \| \| x_0 + g(x) \| + \int_{t_1}^{t_2} \| T(t) - T(s) \| \| f_2(s, v(s)) \| ds
\]
\[
+ \int_{t_1}^{t_2} \| G(s, p_n(s)) - G(s, p(s)) \| d\| w(s) \]
\[
+ \sum_{0 \leq t_k < t} \| I_k(x(t_k^-)) - I_k(x(t_k^+)) \| d\| w(s) \]
\[
+ \sum_{t_k \leq t_1 < t_2} \| I_k(x(t_k^-)) - I_k(x(t_k^+)) \| d\| w(s) \]
\[
\to 0, \quad \text{as} \quad n \to \infty.
\]

The right-hand side is independent of \( x \in B_1 \) and tends to zero as \( t_2 - t_1 \to 0 \), since the compactness of \( \{ T(t) \}_{t \geq 0} \) implies the continuity in the uniform operator topology. Similarly, using the compactness of the set \( g(B_1) \) we can prove that the functions \( \Phi_2 x, x \in B_1 \) are equicontinuous at \( t = 0 \). Hence \( \Phi_2 \) maps \( B_1 \) into a family of equicontinuous functions.

It remains to prove that \( \Phi_2 B_1(t) \) is relatively compact for each \( t \in J \), where \( V(t) = \{ \Phi_2 x(t) : x \in B_1 \}, \ t \in J \). Obviously, by condition (H3), \( V(0) \) is relatively compact in \( B_1 \). Let \( 0 < t \leq b \) be fixed and \( 0 < \epsilon < t \). For \( x \in B_1 \), we
Assume (H1), (H5)-(H7) hold, then there exists a unique solution to

\[ \Phi_2(t) = T(t)[x_0 + g(x)] + \int_0^{t-r} T(t-s)f_2(s, v(s))\, ds + \int_0^{t-r} T(t-s)G(s, p(s))d w(s) + \sum_{0<\iota_k<s} T(t-t_k)I_k(x(t_k^-)) \]

In this part, in order to attain the uniqueness result of (1)-(3), we propose the following assumptions:

(H5) The function \( G : J \times H^{n+1} \rightarrow L_0(K, H) \) is continuous and there exists constants \( M_G > 0 \), for \( t \in J \) and \( x_i, y_i \in H \) such that

\[ E\|G(t, x_i) - G(t, y_i)\|_0 \leq M_G\|x_i - y_i\|, \quad i = 1, 2, \ldots, n. \]

(H6) The nonlocal function \( g : \mathcal{P} \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) is continuous and there exist constants \( M_g > 0 \), for \( x, y \in \mathcal{P} \mathcal{C} \) such that

\[ E\|g(x) - g(y)\| \leq M_g\|x - y\|. \]

(H7) \( I_k : \mathcal{P} \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) is continuous and there exist constants \( M_{I_k} > 0 \), for \( x, y \in \mathcal{P} \mathcal{C} \) such that

\[ E\|I_k(x) - I_k(y)\| \leq M_{I_k}\|x - y\|, \quad k = 1, 2, \ldots, m. \]

Theorem 3.2.
Assume (H1), (H5)-(H7) hold, then there exists a unique solution to (1)-(3) if \( \sum_{k=1}^{m} M_{I_k} < 1 \).
Let $x, y \in B_1$ then we have that, for each $t \in J$

$E\|\Phi(x)(t) - \Phi(y)(t)\|^2$

$\leq 36\left\{E\|T(t)[x_0 + g(x_1) + f_1(0, u_1(0))] - T(t)[x_0 + g(x_2) + f_1(0, u_2(0))]\|^2$

$+ E\|f_1(t, u_1(t)) - f_1(t, u_2(t))\|^2 + E\|\int_0^t T(t - s)[f_2(s, v_1(s)) - f_2(s, v_2(s))]ds\|^2$

$+ E\|\sum_{0 \leq l \leq t} T(t - l_k)(I_k(x(t_k^-)) - I_k(x(t_k^-)))\|^2\}\}

$\leq 36\left\{3M^2_0M^2_M \sum_{k=1}^m E\|x_1(t) - x_2(t)\|^2 + M^2_0M^2_M \sum_{k=1}^m E\|x_1(t) - x_2(t)\|^2$

$+ b\int_0^t M^2_\beta \left(\frac{M^2_0M^2_M}{2\beta - 1} + \left(M^2_\beta + 1\right)(M^2_0 M^2_M)\right)\sup_{0 \leq s \leq b} E\|x_1(t) - x_2(t)\|^2$

$\leq 36\left\{M^2_0M^2_\beta + M^2_\beta \sum_{k=1}^m M^2_{k}\right\} \sum_{k=1}^m M^2_{k}\sup_{0 \leq s \leq b} E\|x_1(t) - x_2(t)\|^2$

$\leq 36\left\{M^2_0M^2_\beta + M^2_\beta \sum_{k=1}^m M^2_{k}\right\} \sum_{k=1}^m M^2_{k}\sup_{0 \leq s \leq b} E\|x_1(t) - x_2(t)\|^2$

$\leq q\|x_1 - x_2\|^2$.

If we define $q = 36\left\{M^2_0M^2_\beta + M^2_\beta \sum_{k=1}^m M^2_{k}\right\} \sum_{k=1}^m M^2_{k}\sup_{0 \leq s \leq b} E\|x_1(t) - x_2(t)\|^2$

This shows that operator $\Phi$ is a contraction. Consequently, the operator $\Phi$ satisfies all the assumption and it has a unique fixed point which is a solution to (1)-(3). The proof is completed.

**4. Examples**

In this section we provide the example to illustrate our previous abstract result.

$d\left[x(t, z) + \int_0^\pi \mu_1(\xi, z) x(t \sin t, \xi) d\xi\right] = \left[\frac{\partial^2}{\partial x^2} x(t, z) - \int_0^\pi \mu_2(\delta, z) x(t \sin t, \delta) d\delta\right] dt$

$h(t, x(t \sin t, z))d\beta(t)$, $0 \leq t \leq b$, $0 \leq z \leq \pi$, $t \neq t_k$, $k = 1, 2, \ldots, m$,

$x(t, 0) = x(t, \pi) = 0$,

$x(t_k^+) - x(t_k^-) = I_k(z(t_k^-))$, $k = 1, 2, \ldots, m$,

$x(t, z) + \sum_{i=1}^p \int_0^\pi k(x, \xi) x(t_i, \xi) d\xi = x_0(z)$, $0 \leq z \leq \pi$. 

(10)

(11)

(12)

(13)
where \( p \) is a positive integer, \( 0 \leq b \leq \pi, 0 < t_0 < \cdots < t_p < 1, \text{ and } 0 < t_1 < t_2 < \cdots < t_m < b . \) The function \( x_0(x) \in H = L^2([0, \pi]) \), \( k(z, x) \in L^2([0, \pi] \times [0, \pi]) \), \( \beta(t) \) denotes a standard one dimensional Wiener process in \( H \) defined on a Probability space \( (\Omega, \mathcal{F}, P) \) and to rewrite the system (10)-(13) in abstract form of (1)-(3), let \( H = L^2([0, \pi]) \) and \( A \) is defined by \( Ay = y'' \).

We consider the operator \( A : D(A) \subseteq H \) defined by, \( D(A) = \{ y \in H, y, y' \) are absolutely continuous, \( y'' \in H, y(0) = y(\pi) = 0 \}. \)

Then \( A \) infinestsimal generates a strongly continuous semigroup \( T(t) \) which is compact, analytic and self-adjoint. Furthermore, \( A \) has a discrete spectrum, the eigenvalues are \( n^2, \; n \in N \), with the corresponding normalized eigenvectors \( x_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz) \).

Then the following properties hold:

(a) If \( y \in D(A) \), then
\[
Ay = \sum_{n=1}^{\infty} n^2 \left< y, x_n \right> x_n.
\]

(b) For each \( y \in H \),
\[
A^{-\frac{1}{2}} y = \sum_{n=1}^{\infty} \frac{1}{n} \left< y, x_n \right> x_n, \quad \text{In Particular,} \quad \| A^{-\frac{1}{2}} \|^2 = 1.
\]

(c) The operator \( A^{-\frac{1}{2}} \) is given by
\[
A^{-\frac{1}{2}} y = \sum_{n=1}^{\infty} n \left< y, x_n \right> x_n,
\]

on the space \( D(A^{-\frac{1}{2}}) = \{ y(\cdot) \in H, \sum_{n=1}^{\infty} \left< y, x_n \right> x_n \in H \} \).

We assume that the following conditions hold:

(i) The function \( \mu_1, \mu_2 \) is measurable and
\[
\int_0^{\pi} \int_0^{\pi} \mu_1^2(\xi, z) d\xi dz < \infty \quad \text{and} \quad \int_0^{\pi} \int_0^{\pi} \mu_2^2(\delta, z) d\delta dz < \infty.
\]

(ii) The function \( \frac{\partial}{\partial \xi} \mu_1(\xi, z) \) is measurable, \( \mu_1(\xi, 0) = \mu_1(\xi, \pi) = 0 \), and let
\[
N_1 = \left[ \int_0^{\pi} \int_0^{\pi} \left( \frac{\partial}{\partial \xi} \mu_1(\xi, z) \right)^2 d\xi dz \right]^\frac{1}{2} < \infty.
\]

(iii) The function \( \frac{\partial}{\partial \xi} \mu_2(\delta, z) \) is measurable, \( \mu_2(\delta, 0) = \mu_2(\delta, \pi) = 0 \), and let
\[
N_2 = \left[ \int_0^{\pi} \int_0^{\pi} \left( \frac{\partial}{\partial \xi} \mu_2(\delta, z) \right)^2 d\delta dz \right]^\frac{1}{2} < \infty.
\]

(iv) \( \beta(t) \) denotes a one dimensional standard Brownian motion.

(v) For the function \( h : J \times R \to R \) the following three conditions are satisfied:

(1) For each \( t \in [0, b] \), \( h(t, \cdot) \) is continuous.

(2) For each \( x \in \mathcal{P} \mathcal{C} \), \( h(\cdot, x) \) is measurable.

(3) There are positive functions \( h_1, h_2 \in L^1(J) \) such that
\[
|h(t, x)| \leq h_1(t) \| x \| + h_2(t), \quad \forall (t, x) \in [0, b] \times H.
\]

(vi) The function \( I_k(\mathcal{P} \mathcal{C}, \mathcal{P} \mathcal{C}) \), \( k = 1, 2, \ldots, m \) and there exist nondecreasing functions \( L_k \in (J, R_+), k = 1, 2, \ldots, m \) such that for each \( x \in \mathcal{P} \mathcal{C} \)
\[
\|I_k(x)\|^2 \leq L_k(\| x \|^2).
\]
We define $f_1 : J \times H \to L(K, H)$ and $f_2 : J \times H \to L(K, H)$ by
\[
f_1(t, x) = Z_1(x),
f_2(t, x) = Z_2(x),
\]
and
\[
G(t, x) = h(t, x(z)),
\]
and
\[
g(w(t)) = \sum_{i=0}^{\infty} Kw(t_i), \quad w \in \mathcal{P}'C,
\]
respectively, where
\[
Z_1(x)(z) = \int_{0}^{\pi} \mu_1(\xi, z)x(\xi)d\xi,
\]
\[
Z_2(x)(z) = \int_{0}^{\pi} \mu_2(\delta, x(\delta)d\delta
\]
and
\[
K(x)(z) = \int_{0}^{\pi} k(z, \xi)x(\xi)d\xi.
\]
Then $G$ satisfies condition (H2) while $g$ verifies (H3) (noting that $K : \mathcal{P}'C \to \mathcal{P}'C$ is completely continuous). From (i) it is clear that $Z_1$ and $Z_2$ is a bounded linear operators on $H$. Furthermore, $Z_1(x) \in D[A^{1/2}]$, and $Z_2(x) \in D[A^{1/2}]$, then $\|A^{1/2}Z_1\|^2 \leq N_1, \|A^{1/2}Z_2\|^2 \leq N_2$. In fact, from the definition of $Z_1$, $Z_2$ and (ii), (iii) it follows that
\[
\langle Z_1(x), n \rangle = \int_{0}^{\pi} x_n(z)\left[ \int_{0}^{\pi} \mu_1(\xi, z)x(\xi)d\xi \right]d\xi
\]
\[
= \frac{1}{n} \sqrt{\frac{2}{\pi}} \langle Z(x), \cos(nz) \rangle,
\]
where $Z$ is defined by
\[
Z(x)(z) = \int_{0}^{\pi} \frac{\partial}{\partial \xi} \mu_1(\xi, z)x(\xi)d\xi
\]
and
\[
\langle Z_2(x), n \rangle = \int_{0}^{\pi} x_n(z)\left[ \int_{0}^{\pi} \mu_2(\delta, z)x(\delta)d\delta \right]d\xi
\]
\[
= \frac{1}{n} \sqrt{\frac{2}{\pi}} \langle Z(x), \cos(nz) \rangle,
\]
where $Z$ is defined by
\[
Z(x)(z) = \int_{0}^{\pi} \frac{\partial}{\partial \delta} \mu_2(\delta, z)x(\delta)d\delta.
\]
From (ii) and (iii) we know that $Z : H \to H$ is a bounded linear operator with $\|Z\|^2 \leq N_1$ and $\|Z\|^2 \leq N_2$. Hence $\|A^{1/2}Z_1(z)\|^2 = \|Z(z)\|^2$ and $\|A^{1/2}Z_2(z)\|^2 = \|Z(z)\|^2$, which implies the assertion. Therefore, the conditions (H1)-(H4) are all satisfied. Hence from Theorem 3.1, system (10)-(13) admits a mild solution on $f$ under the above assumptions additionally provided in (5) and (3) hold.

References