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Result on fixed point theorem in Hilbert space

Research Article

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Abstract: This paper presents a result which is the generalization of the Banach contraction principle in the Hilbert space, involving four rational square terms in the inequality. Furthermore, we obtained the corollary of Koparde and Waghmode by taking vanishing values to some constants in the end of this result.

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Keywords: Hilbert space • Closed subset • Cauchy sequence • Completeness

1. Introduction

Ever since research in the discipline of Fixed point Theory and Approximation Theory were initiated by Banach in 1922 and then several mathematicians contributed to the growth of this area of knowledge has been extensively reported in the treatises of Nadler [8], Sehgal [10],Kannan [5], Zamfirescu [15], Wong [14], etc. Also some generalizations of Banach fixed point theorem were given by D.S.Jaggi [4], Fisher [2], Khare[6]. Ganguly and Bandyopadhyay [3], Koparde and Waghmode [7], Pandhare [9],Veerapandi and Anil Kumar [13] investigated the properties of fixed points of family of mappings on complete metric spaces and in Hilbert spaces.

Motivated by the above results, the result which is found here is the refinement and sharpens some of the generalizations of Singh. Th. Manihar [11], Smart [12] and Dass and Gupta [1] results. The theorem follows with the statement:

2. Theorem

Theorem 2.1.

Let X be a closed subset of a Hilbert space and $T : X \to X$ be a self mapping satisfying the following condition

$$||Tx - Ty||^2 \le a_1 \frac{||y - Ty||^2[1 + ||x - Tx||^2]}{1 + ||x - y||^2} + a_2 \frac{||x - Tx||^2[1 + ||y - Ty||^2]}{1 + ||x - y||^2}$$

$$+ a_3 \frac{||x - Ty||^2[1 + ||y - Tx||^2]}{1 + ||x - y||^2} + a_4 \frac{||y - Tx||^2[1 + ||x - Ty||^2]}{1 + ||x - y||^2} + a_5 ||x - y||^2$$

for each $x, y \in X$ and $x \neq y$, where a_1, a_2, a_3, a_4, a_5 are non-negative reals with $0 \le a_1 + a_2 + a_3 + 4a_4 + a_5 < 1$. Then T has a unique fixed point in X.

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Proof. For some $x_0 \in X$, we define a sequence $\{x_n\}$ of iterates of T as follows

$$x_1 = T\,x_0, x_2 = T\,x_1, x_3 = T\,x_2, ..., \text{ i.e } x_{n+1} = T\,x_n, \text{ for } n = 0,1,2,3, ...$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. For this consider

$$||x_{n+1} - x_n||^2 = ||T\,x_n - T\,x_{n-1}||^2$$

Then by using the hypothesis, we have

$$||x_{n+1} - x_n||^2 \le a_1 \frac{||x_{n-1} - T\,x_{n-1}||^2[1 + ||x_n - T\,x_n||^2]}{1 + ||x_n - x_{n-1}||^2} + a_2 \frac{||x_n - T\,x_n||^2[1 + ||x_{n-1} - T\,x_{n-1}||^2]}{1 + ||x_n - x_{n-1}||^2}$$

$$+ a_3 \frac{||x_n - T\,x_{n-1}||^2[1 + ||x_{n-1} - T\,x_n||^2]}{1 + ||x_n - x_{n-1}||^2} + a_4 \frac{||x_{n-1} - T\,x_n||^2[1 + ||x_n - T\,x_{n-1}||^2]}{1 + ||x_n - x_{n-1}||^2} + a_5 ||x_n - x_{n-1}||^2$$

which implies that

$$(1 - a_2 - 2\,a_4)||x_{n+1} - x_n||^2 + (1 - a_1 - a_2)||x_{n+1} - x_n||^2\,||x_n - x_{n-1}||^2 \le \left\{(a_1 + 2\,a_4 + a_5) + a_5\,||x_n - x_{n-1}||^2\right\}||x_n - x_{n-1}||^2$$

resulting in

$$||x_{n+1} - x_n||^2 \le p(n)||x_n - x_{n-1}||^2$$

where

$$p(n) = \frac{(a_1 + 2\,a_4 + a_5) + a_5\,||x_n - x_{n-1}||^2}{(1 - a_2 - 2\,a_4) + (1 - a_1 - a_2)||x_n - x_{n-1}||^2},\ for\ n = 1,2,3,...$$

Clearly $p(n) < 1$, for all n as $0 \le a_1 + a_2 + a_3 + 4\,a_4 + a_5 < 1$. Repeating the same argument, we find some $S < 1$, such that

$$||x_{n+1} - x_n||^2 \le \lambda^n\,||x_1 - x_0||,\ where\ \lambda = S^2$$

Letting $n \to \infty$, we obtain $||x_{n+1} - x_n|| \to 0$. It follows that $\{x_n\}$ is a Cauchy sequence in X. So by completeness of X there exists a point $\mu \in X$ such that $x_n \to \mu\ as\ n \to \infty$.
Also $\{x_{n+1}\} = \{T\,x_n\}$ is subsequence of $\{x_n\}$ converges to the same limit μ. Since T is continuous, we obtain

$$T(\mu) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T\,x_n = \lim_{n\to\infty} x_{n+1} = \mu$$

Hence μ is a fixed point of T in X. Next, we show the uniqueness of μ; if T has another fixed point v, $\mu \ne v$, then

$$||\mu - v||^2 = ||T\,\mu - T\,v||^2$$

$$\le a_1 \frac{||v - T\,v||^2\left[1 + ||\mu - T\,\mu||^2\right]}{1 + ||\mu - v||^2} + a_2 \frac{||\mu - T\,\mu||^2\left[1 + ||v - T\,v||^2\right]}{1 + ||\mu - v||^2} + a_3 \frac{||\mu - T\,v||^2\left[1 + ||v - T\,\mu||^2\right]}{1 + ||\mu - v||^2}$$

$$+ a_4 \frac{||v - T\,\mu||^2\left[1 + ||\mu - T\,v||^2\right]}{1 + ||\mu - v||^2} + a_5\,||\mu - v||^2$$

which inturn, implies that

$$||\mu - v||^2 \le (a_3 + a_4 + a_5)||\mu - v||^2$$

This gives a contradiction; for $a_3 + a_4 + a_5 < 1$. Thus μ is a unique fixed point of T in X. □

Remark 2.1.
Here is our interest, we can get corollary [1] of Koparde and Waghmode [7] by setting $a_3 = a_4 = 0$ in the above result.

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