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On the properties of generalized Fibonacci like polynomials

Research Article

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Abstract: The Fibonacci polynomial has been generalized in many ways,some by preserving the initial conditions,and others by preserving the recurrence relation.In this article,we study new generalization $\{M_n\}(x)$, with initial conditions $M_0(x) = 2$ and $M_1(x) = m(x) + k(x)$, which is generated by the recurrence relation $M_{n+1}(x) = k(x)M_n(x) + M_{n-1}(x)$ for $n \geq 2$, where $k(x), m(x)$ are polynomials with real coefficients.We produce an extended Binet's formula for $\{M_n\}(x)$ and,thereby identities such as Simpson's,Catalan's,d'Ocagene's,etc.using matrix algebra.Moreover, we present sum formulas concerning this new generalization.

MSC: 11B39 • 11B83

Keywords: Fibonacci polynomials • Lucas polynomials • Recurrence relation • Matrix algebra

1. Introduction

Definition 1.1.

The Fibonacci polynomial $F_n(x), n = 1,2,3,...$ is defined as, $F_{n+1}(x) = k(x)F_n(x) + F_{n-1}(x)$ with $F_0(x) = 0, F_1(x) = 1$ for $n \geq 1$.

Definition 1.2.

The Lucas polynomial $L_n(x), n = 1,2,3,...$ is defined as, $L_{n+1}(x) = k(x)L_n(x) + L_{n-1}(x)$ with $L_0(x) = 2, L_1(x) = k(x)$ for $n \geq 1$.

In this paper, we study the new generalization $\{M_n(x)\}$, with initial conditions $M_0(x) = 2$ and $M_1(x) = m(x) + k(x)$, which is generated by the recurrence relation, $M_{n+1}(x) = k(x)M_n(x) + M_{n-1}(x) for, n \geq 2$, where $k(x), m(x)$ are polynomials with real coefficients.
Many authors have studied relationship between the Fibonacci sequence,its certain generalizations and matrix properties (for more details see [1]-[11] and Fibonacci polynomials.In this paper we define a new generalization of the Fibonacci polynomial and give identities and sum formulas concerning this new generalization.

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2. The generalized Fibonacci like polynomial

Now, a new generalization of the Fibonacci polynomial is introduced and few terms of this sequence are given.

Definition 2.1.

For any polynomial with real coefficients $k(x) \geq 1$ and $m(x) \geq 0$ the Generalized Fibonacci like polynomial $M_n(x), n \geq 1$ is defined by: $M_{n+1}(x) = k(x)M_n(x) + M_{n-1}(x)$, for $n \geq 1$,with $M_0(x) = 2, M_1(x) = m(x) + k(x)$.

Characteristic equation of the initial recurrence relation is- $r^2 - k(x)r - 1 = 0$, and characteristic roots are-

$$r_1 = \frac{k(x) + \sqrt{k^2(x)+4}}{2} \tag{1}$$

and

$$r_2 = \frac{k(x) - \sqrt{k^2(x)+4}}{2} \tag{2}$$

Characteristic roots verify the properties,

$$r_1 - r_2 = \sqrt{k^2(x)+4}, r_1 + r_2 = k(x), r_1.r_2 = -1. \tag{3}$$

It is clear from the definition of the Generalized Fibonacci like polynomial it satisfy,

$$M_n(x) = m(x)F_n(x) + L_n(x), for\, n \geq 0, \tag{4}$$

where $F_n(x)$ and $L_n(x)$ are Fibonacci polynomials and Lucas polynomials respectively.
First few terms of the generalized Fibonacci like polynomial:

$$M_0(x) = 2,$$
$$M_1(x) = m(x) + k(x),$$
$$M_2(x) = k^2(x) + m(x)k(x) + 2,$$
$$M_3(x) = k^3(x) + m(x)k^2(x) + 3k(x) + m(x),$$
$$M_4(x) = k^4(x) + m(x)k^3(x) + 4k^2(x) + 2m(x)k(x) + 2,$$
$$M_5(x) = k^5(x) + m(x)k^4(x) + 5k^3(x) + 3m(x)k^2(x) + 5k(x) + m(x),$$
$$M_6(x) = k^6(x) + m(x)k^5(x) + 6k^4(x) + 4m(x)k^3(x) + 9k^2(x) + 3m(x)k(x) + 2,$$
$$M_7(x) = k^7(x) + m(x)k^6(x) + 7k^5(x) + 5m(x)k^4(x) + 14k^3(x) + 6m(x)k^2(x) + 7k(x) + m(x),$$
$$M_8(x) = k^8(x) + m(x)k^7(x) + 8k^6(x) + 6m(x)k^5(x) + 20k^4(x) + 10m(x)k^3(x) + 16k^2(x) + 4m(x)k(x) + 2.$$

3. Fundamental properties of the generalized Fibonacci Like polynomials

Theorem 3.1.

The $M_n(x), n \in N$ is given by,

$$M_n(x) = \frac{[m(x) + r_1 - r_2]r_1^n - [m(x) + r_2 - r_1]r_2^n}{r_1 - r_2}, \tag{5}$$

Proof. (1) Using Binet's formula for Fibonacci polynomials and Lucas polynomials, we have

$$F_n(x) = \frac{r_1^n - r_2^n}{r_1 - r_2},$$
$$L_n(x) = r_1^n + r_2^n,$$
$$M_n(x) = m(x)(x)F_n(x) + L_n(x),$$
$$M_n(x) = m(x)(\frac{r_1^n - r_2^n}{r_1 - r_2}) + (r_1^n + r_2^n),$$
$$M_n(x) = \frac{[m(x) + r_1 - r_2]r_1^n - [m(x) + r_2 - r_1]r_2^n}{r_1 - r_2}$$

hence proof. □

Proof. (2) The general form of the generalized Fibonacci like polynomials may be expressed in the form,

$$M_n(x) = A(x)r_1^n + B(x)r2^n$$

for some polynomials A(x) and B(x). The polynomials A(x) and B(x) can be determined by the initial conditions,

$$M_0 = 2 = A(x) + B(x), M_1 = m(x) + k(x) = A(x)r_1 + B(x)r_2,$$

Solving above equation system for A(x) and B(x),we get,

$$A(x) = \frac{m(x) + k(x) - 2r_2}{r_1 - r_2}$$
$$B(x) = \frac{2r_1 - [m(x) + k(x)]}{r_1 - r_2},$$

Using,

$$2r_1 = k(x) + \sqrt{k^2(x) + 4}, 2r_2 = k(x) - \sqrt{k^2(x) + 4}$$

Therefore we write,

$$M_n(x) = \frac{[m(x) + \sqrt{k^2(x) + 4}]r_1^n}{r_1 - r_2} - \frac{[m(x) - \sqrt{k^2(x) + 4}]r_2^n}{r_1 - r_2},$$

Since,

$$r_1 - r_2 = \sqrt{k^2(x) + 4},$$

therefore,

$$M_n(x) = \frac{[m(x) + r_1 - r_2]r_1^n - [m(x) + r_2 - r_1]r_2^n}{r_1 - r_2},$$

or,

$$M_n(x) = \frac{X(r_1^n - Y r_2^n)}{r_1 - r_2},$$

where,$X = m(x) + r_1 - r_2$ and $Y = m(x) + r_2 - r_1$. □

Theorem 3.2.

$$M_{n-r}(x).M_{n+r}(x) - M_n(x)^2 = (-1)^{n-r}[k^2(x) - m^2(x) + 4]F_r^2(x), for, n, r \geq 1 \tag{6}$$

Proof. Using Binet's formula ,we have

$$\begin{aligned} M_{n-r}(x).M_{n+r}(x) - M_n{}^2(x) &= \frac{X r_1^{n-r} - Y r_2^{n-r}}{r_1 - r_2} . \frac{X r_1^{n+r} - Y r_2^{n+r}}{r_1 - r_2} - [\frac{X r_1^n - Y r_2^n}{r_1 - r_2})]^2 \\ &= \frac{XY}{(r_1 - r_2)^2}.[2(r_1 r_2)^n - r_1^{n-r} r_2^{n+r} - r_2^{n-r} r_1^{n+r}], \\ &= \frac{-[m^2(x) - k^2(x) - 4]}{(r_1 - r_2)^2}[(r_1 r_2)^n(\frac{r_2}{r_1})^r + (r_1 r_2)^n(\frac{r_1}{r_2})^r - 2((r_1 r_2)^n)], \\ &= \frac{-[m^2(x) - k^2(x) - 4]}{(r_1 - r_2)^2}(r_1 r_2)^{n-r}[r_1^{2r} - 2(r_1 r_2)^r + r_2^{2r}], \\ &= [k^2(x) - m^2(x) + 4](-1)^{n-r} F_r^2(x). \end{aligned}$$

Hence proof. □

Theorem 3.3.

$$M_{n-1}(x).M_{n+1}(x) - M_n{}^2(x) = (-1)^{n+1}[k^2(x) - m^2(x) + 4], \tag{7}$$

$for, n \geq 1$

Proof. (1) Put $r = 1$ in Catalan's identity. □

Proof. (2)
Consider the 2×2 linear system:

$$M_n(x)u(x) + M_{n-1}(x)v(x) = M_{n+1}(x),$$
$$M_{n+1}(x)u(x) + M_n(x)v(x) = M_{n+2}(x)$$

Since,

$$M_n^2(x) - M_{n-1}(x).M_{n+1}(x) \neq 0, \tag{8}$$

this system has unique solution such that,

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} k(x) \\ 1 \end{pmatrix}$$

Therefore by Cramer's rule,we have

$$v(x) = \frac{\det\begin{pmatrix} M_n(x) & M_{n+1}(x) \\ M_{n+1}(x) & M_{n+2}(x) \end{pmatrix}}{\det\begin{pmatrix} M_n(x) & M_{n-1}(x) \\ M_{n+1}(x) & M_n(x) \end{pmatrix}} = 1$$

thus,

$$M_{n+2}(x).M_n(x) - M_{n+1}^2(x) = M_n^2(x) - M_{n-1}(x).M_{n+1}(x),$$

Now let,

$$P_n(x) = M_{n-1}(x).M_{n+1}(x) - M_n^2(x),$$
$$P_{n+1}(x) = M_n(x).M_{n+2}(x) - M_{n+1}^2(x),$$

Gives,

$$P_{n+1}(x) = -P_n(x), n \geq 1,$$

and

$$P_1(x) = k^2(x) - m^2(x) + 4,$$

Solving the recurrence relation, $P_{n+1}(x) + P_n(x) = 0$, with,$P_1(x) = k^2(x) - m^2(x) + 4$, we get,

$$P_n(x) = (-1)^{n+1}[k^2(x) - m^2(x) + 4],$$

Thus,

$$M_{n-1}(x).M_{n+1}(x) - M_n{}^2(x) = (-1)^{n+1}[k^2(x) - m^2(x) + 4],$$

for $n \geq 1$ □

Theorem 3.4.
For any integer n,Number $[k^2(x)+4]M_n^2(x) + 4(-1)^n[m^2(x) - k^2(x) - 4]$, *is a perfect square.*

Proof. If n is even,

$$[k^2(x)+4]M_n^2(x) + 4[m^2(x) - k^2(x) - 4] = [(m(x) + r_1 - r_2)r_1^n + (m(x) + r_2 - r_1)r_2^n],$$

If n is odd,

$$[k^2(x)+4]M_n^2(x) - 4[m^2(x) - k^2(x) - 4] = [(m(x) + r_1 - r_2)r_1^n - (m(x) + r_2 - r_1)r_2^n],$$

Therefore the number $[k^2(x)+4]M_n^2 + 4(-1)^n[m^2(x) - k^2(x) - 4]$ is always a perfect square. □

Theorem 3.5.
For $n, r \geq 1$,

$$M_r(x).M_{n+1}(x) - M_{r+1}(x)M_n(x) = [k^2(x) - m^2(x) + 4](-1)^n F_{r-n}(x). \tag{9}$$

Proof. Using Binet's formula,

$$\begin{aligned} M_r(x).M_{n+1}(x) - M_{r+1}(x)M_n(x) &= \frac{X r_1^r - Y r_2^r}{r_1 - r_2}.\frac{X r_1^{n+1} - Y r_2^{n+1}}{r_1 - r_2} - \frac{X r_1^{r+1} - Y r_2^{r+1}}{r_1 - r_2}.\frac{X r_1^n - Y r_2^n}{r_1 - r_2}, \\ &= \frac{XY(r_1^r r_2^n)(r_1 - r_2) - XY(r_1^n r_2^r)(r_1 - r_2)}{(r_1 - r_2)^2}, \\ &= XY(-1)^n F_{r-n}(x), \\ &= [k^2(x) - m^2(x) + 4](-1)^n F_{r-n}(x). \end{aligned}$$

Hence proof. □

Theorem 3.6.
For $r \geq 1$,

$$M_r(x).M_{n+1}(x) + M_{r-1}(x)M_n(x) = M_{n+r}(x) + [k^2(x) - \sqrt{k^2(x)+4} + m^2(x) + 4]F_{n+r} + m(x)[2 + \frac{1}{\sqrt{k^2(x)+4}}]L_{n+r}(x). \tag{10}$$

Proof. Using Binet's formula,

$$M_r(x).M_{n+1}(x) + M_{r-1}(x)M_n(x) = \frac{X^2 r_1^{n+r}(r_1 + \frac{1}{r_1}) - XY[r_1^r r_2^n(r_2 + \frac{1}{r_1}) + r_1^n r_2^r(r_1 + \frac{1}{r_2})] + Y^2 r_2^{n+r}[r_2 + \frac{1}{r_2}]}{(r_1 - r_2)^2},$$

Since, $r_1.r_2 = -1$, gives

$$(r_1 + \frac{1}{r_2}) = 0, (r_2 + \frac{1}{r_1}) = 0$$

and

$$(r_1 + \frac{1}{r_1}) = r_1 - r_2$$

$$(r_2 + \frac{1}{r_2}) = -(r_1 - r_2)$$

$$= \frac{X^2 r_1^{n+r} - Y^2 r_2^{n+r}}{(r_1 - r_2)}$$

gives,

$$M_r(x).M_{n+1}(x) + M_{r-1}(x)M_n(x) = M_{n+r}(x) + [k^2(x) - \sqrt{k^2(x)+4} + m^2(x) + 4]F_{n+r} + m(x)[2 + \frac{1}{\sqrt{k^2(x)+4}}]L_{n+r}(x)$$

□

Theorem 3.7.

$$\lim_{n\to\infty} \frac{M_n(x)}{M_{n-r}(x)} = r_1^r. \tag{11}$$

Proof. (1) Consider,

$$\begin{aligned} \lim_{n\to\infty} \frac{M_n(x)}{M_{n-r}(x)} &= \lim_{n\to\infty} \frac{X r_1^n - Y r_2^n}{X r_2^{n-r} - Y r_2^{n-r}}, \\ &= \lim_{n\to\infty} \frac{r_1^n[X - Y(\frac{r_2}{r_1})^n]}{r_1^{n-r}[X - Y(\frac{r_2}{r_1})^{n-r}]} \end{aligned}$$

and taking into account that $\lim_{n\to\infty}(\frac{r_2}{r_1})^n = 0$,and $= \lim_{n\to\infty}(\frac{r_2}{r_1})^{n-r} = 0$, since $|r_2| < r_1$
Gives,

$$lim_{n\to\infty} \frac{M_n(x)}{M_{n-r}(x)} = r_1^r.$$

□

Proof. (2)
Sequence

$$\{t_n(x)\}_{n=1}^{n=\infty} = \{\frac{M_n}{M_{n-1}}\}_{n=1}^{n=\infty}$$

is convergent
Let, $\lim_{n\to\infty} t_n(x) = t(x)$, a polynomial with real coefficients,
Since,

$$\frac{M_{n+1}(x)}{M_n(x)} = k(x) + \frac{M_{n-1}(x)}{M_n(x)}$$

and $k(x) > 0$, for $x > 0$, Then we have,

$$\lim_{n\to\infty} \frac{M_{n+1}(x)}{M_n(x)} = k(x) + \lim_{n\to\infty} \frac{M_{n-1}(x)}{M_n(x)}$$

$$= k(x) + \frac{1}{\lim_{n\to\infty} \frac{M_n(x)}{M_{n-1}(x)}}$$

$$t(x) = k(x) + \frac{1}{t(x)}$$

This gives us the equation, $t^2(x) - k(x)t(x) - 1 = 0$, for $x > 0$,
This equation has single positive root r_1

$$\lim_{n\to\infty} \frac{M_n(x)}{M_{n-1}(x)} = r_1.$$

$$\frac{M_n(x)}{M_{n-r}(x)} = \frac{M_n(x)}{M_{n-1}(x)} \frac{M_{n-1}(x)}{M_{n-2}(x)} \cdots\cdots \frac{M_{n-r+1}(x)}{M_{n-r}(x)}$$

$$\lim_{n\to\infty} \frac{M_n(x)}{M_{n-r}(x)} = \lim_{n\to\infty} \frac{M_n(x)}{M_{n-1}(x)} \lim_{n\to\infty} \frac{M_{n-1}(x)}{M_{n-2}(x)} \cdots\cdots \lim_{n\to\infty} \frac{M_{n-r(x)+1}}{M_{n-r}(x)}$$

$$\lim_{n\to\infty} \frac{M_n(x)}{M_{n-r}(x)} = r_1^r$$

□

Theorem 3.8.
The following equalities are valid for $a, b, c = 1, 2, 3, \ldots..$

$$M_{a+b-1}(x) = \frac{1}{X}[M_a(x)M_b(x) + M_{a-1}(x)M_{b-1}(x) - 2Y r_2^{a+b-1}]$$

$$M_{a+b-2}(x) = \frac{1}{k(x)X}[M_a(x)M_b(x) - M_{a-2}(x)M_{b-2}(x) - 2k(x)Y r_2^{a+b-2}]$$

$$M_{a+b+c-3}(x) = \frac{1}{k(x)X^2}[M_a(x)M_b(x)M_c(x) + k(x)M_{a-1}(x)M_{b-1}(x)M_{c-1}(x) - M_{a-2}(x)M_{b-2}(x)M_{c-2}(x)] - \frac{2}{X^2}[Y r_2^{a+b-3}(M_a(x)r_2 + M_{a-1}(x) + X r_2^a)]$$

Proof. Using Binet's formula we have,

$$M_a(x)M_b(x) + M_{a-1}(x)M_{b-1}(x) = \frac{X r_1^a - Y r_2^a}{r_1 - r_2} . \frac{X r_1^b - Y r_2^b}{r_1 - r_2} + \frac{X r_1^{a-1} - Y r_2^{a-1}}{r_1 - r_2} . \frac{X r_1^{b-1} - Y r_2^{b-1}}{r_1 - r_2},$$

$$= \frac{X^2[r_1^{a+b} + r_1^{a+b-1}] - XY[r_1^{a-1} r_2^{b-1}(1 + r_1 r_2) + r_1^{b-1} r_2^{a-1}(1 + r_1 r_2)] + Y^2(r_2^{a+b} + r_2^{a+b-2})}{(r_1 - r_2)^2}$$

Since, $r_1 r_2 = -1$, gives

$$= \frac{X^2[r_1^{a+b} + r_2^{a+b-2}] + Y^2[r_2^{a+b} + r_2^{a+b-2}]}{(r_1 - r_2)^2},$$

$$= \frac{X^2 r_1^{a+b-1} - Y^2 r_2^{a+b-1}}{r_1 - r_2},$$

Since, $X - Y = 2(r_1 - r_2)$, gives

$$M_{a+b-1}(x) = \frac{1}{X}[M_a(x)M_b(x) + M_{a-1}(x)M_{b-1}(x) - 2Y r_2^{a+b-1}]$$

(2) and (3) can be proved using (1) directly. □

Theorem 3.9.
The generating function for the generalized Fibonacci like polynomial $[M_n(x)]_{n=0}^{\infty}$ *is given by,*

$$G(t)=\frac{2+[m(x)-k(x)]\,t}{1-k(x)t-t^2}$$

Proof. We begin with formal power series representation of the generating function for $[M_n(x)]$,

$$F(t)=M_0(x)+M_1(x)t+M_2(x)t^2+\cdots\cdots\cdots+M_n(x)t^n+\cdots\cdots$$
$$=\sum_{n=0}^{n=\infty} t^n M_n(x)$$

Now,

$$[1-k(x)t-t^2]F(t)=M_0(x)+t[M_1(x)-k(x)M_0(x)]+t^2[M_2(x)-k(x)M_1(x)-M_0(x)]+\cdots\cdots\cdots$$
$$=2+[m(x)-k(x)]t$$
$$F(t)=\frac{2+[m(x)-k(x)]\,t}{1-k(x)t-t^2}$$

□

Theorem 3.10.
If

$$F(t)=\sum_{n=0}^{n=\infty} t^n M_n(x),$$

for $t\in(-\frac{1}{r_1},\frac{1}{r_2})$, $t>0$ *then,*

$$M_n(x)=\frac{F^{(n)}(0)}{n!}$$

Where $F^{(n)}(t)$ *denote the* n^{th} *order derivative of the polynomial* $F(t)$.

Proof. We have,

$$M_0(x)t^0=2$$
$$F^{(1)}(t)=\sum_{n=1}^{n=\infty} nM_n(x)t^{n-1}$$
$$=M_1(x)+\sum_{n=2}^{n=\infty} nM_n(x)t^{n-1}$$
$$F^{(2)}(t)=\sum_{n=2}^{n=\infty} n(n-1)M_n(x)t^{n-1}$$
$$=1\times 2M_2(x)+\sum_{n=3}^{n=\infty} n(n-1)M_n(x)t^{n-2}$$

⋮

$$F^{(r)}(t)=\sum_{n=r}^{n=\infty} n(n-1)(n-2)\cdots\cdots[n-(r-1)]M_n(x)t^{n-r}$$
$$=r(r-1)(r-2)\cdots\cdots 2\times 1M_r(x)+\sum_{n=r+1}^{n=\infty} n(n-1)(n-2)\cdots\cdots[n-(r-1)]M_n(x)t^{n-r}$$
$$=r!M_r(x)+\sum_{n=r+1}^{n=\infty} n(n-1)(n-2)\cdots\cdots[n-(r-1)]M_n(x)t^{n-r}$$

Put, $t=0$, for $n\geq r$, gives

$$M_r(x)=\frac{F^{(r)}(0)}{r!}$$

i.e. for $n\geq 1$, we have

$$M_n(x)=\frac{F^{(n)}(0)}{n!}$$

□

Theorem 3.11.

$$(-1)^n M_{-n}(x) = M_n(x) - 2m(x)F_n(x) \tag{12}$$

Proof. Using (3),

$$\begin{aligned}(-1)^n M_{-n}(x) &= (r_1 r_2)^n \frac{X r_1^{-n} - Y r_2^{-n}}{r_1 - r_2} \\ &= \frac{X r_2^n - Y r_2^n}{r_1 - r_2} \\ &= \frac{X r_1^n - Y r_2^n}{r_1 - r_2} - (X+Y)\frac{r_1^n - r_2^n}{r_1 - r_2} \\ (-1)^n M_{-n}(x) &= M_n(x) - 2m(x)F_n(x)\end{aligned}$$

Hence proof. □

Theorem 3.12.
If $M_n(x)$ is generalized Fibonacci like polynomial,with $r_1 = \frac{k(x)+\sqrt{k^2(x)+4}}{2}$ then,

$$r_1^{n-2} < M_n(x)$$

for $n \geq 3$

Proof. The proof of this theorem can be composed by the induction method.
Let,

$$P(n): M_n(x) > r_1^{n-2}$$

We must prove that $P(n)$ is true when $n \geq 3$,first verify that $P(3)$ and $P(4)$ are true.
Now,

$$r_1 = \frac{k(x)+\sqrt{k^2(x)+4}}{2},$$

and

$$\frac{k(x)}{2} < k^2(x)+1 < m(x)(k^2(x)+1)$$

$$\frac{\sqrt{k^2(x)+4}}{2} < \frac{k^2(x)+4}{2} < k^2(x)+3 < k(x)(k^2(x)+3)$$

Hence,

$$\frac{k(x)+\sqrt{k^2(x)+4}}{2} < k(x)[k^2(x)+3] + m(x)[k^2(x)+1],$$

i.e.

$$M_3(x) > r_1^{3-2}$$

Now,

$$r_1^2 = [\frac{k(x)\sqrt{k^2(x)+4}+k^2(x)+2}{2}],$$

Since,

$$\frac{\sqrt{k^2(x)+4}}{2} < \frac{k^2(x)+4}{2} < \frac{2k^2(x)+4}{2} < k^2(x)+2,$$

$$\frac{k(x)\sqrt{k^2(x)+4}}{2} < k(x)[k^2(x)+2] < m(x)k(x)[k^2(x)+2],$$

and,

$$\frac{k^2(x)+2}{2} < k^2(x)+1 < 2k^2(x)+1 < 2[2k^2(x)+2] < k^4(x)+2[2k^2(x)+2]$$

gives,

$$\frac{k(x)\sqrt{k^2(x)+4}+k^2(x)+2}{2} < k^4(x)+2[2k^2(x)+2]+m(x)k(x)[k^2(x)+2]$$

i.e.

$$r_1^{4-2} < M_4(x)$$

Now assume that $P(t)$ is true i.e.

$$r_1^{t-2} < M_t(x),$$

for all $3 \le t \le n$, where$n \ge 4$
Furthermore we must prove that

$$r_1^{n-1} < M_{n+1}(x)$$

Since r_1 is a solution of $t^2 - k(x)t - 1 = 0$,gives

$$r_1^2 = k(x)r_1 + 1,$$

and

$$r_1^{n-1} = r_1^2 . r_1^{n-3} = [k(x)r_1 + 1]r_1^{n-3} = k(x)r_1^{n-2} + r_1^{n-3}$$

But,

$$r_1^{n-2} < M_n(x),$$

$$r_1^{n-3} < M_{n-1}(x),$$

and,

$$M_{n+1}(x) = k(x)M_n(x) + M_{n-1}(x) > k(x)r_1^{n-2} + r_1^{n-3}$$

i.e.

$$r_1^{n-1} < M_{n+1}(x),$$

Thus,

$$r_1^{n-2} < M_n(x),$$

for $n \ge 3$ □

4. Summations

Theorem 4.1.

(Sum of the first n-terms)

$$\sum_{i=0}^{i=n} M_i(x) = \frac{1}{k(x)}[M_{n+1}(x) + M_n(x) - m(x) + k(x) + 2] \tag{13}$$

Proof. Let,

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{i=n} M_i(x) \\ &= M_0(x) + M_1(x) + M_2(x) + \cdots\cdots + M_n(x) \\ &= [m(x)F_0(x) + L_0(x)] + [m(x)F_1(x) + L_1(x)] + \cdots\cdots + [m(x)F_n(x) + L_n(x)] \\ &= m(x)[F_0(x) + F_1(x) + \cdots\cdots + F_n(x)] + [L_0(x) + L_1(x) + \cdots\cdots + L_n(x)] \\ &= \frac{m(x)}{k(x)}[F_{n+1}(x) + F_n(x) - 1] + \frac{1}{k(x)}[L_{n+1}(x) + L_n(x) - 2] + 1 \end{aligned}$$

gives,

$$\sum_{i=0}^{i=n} M_i(x) = \frac{1}{k(x)}[M_{n+1}(x) + M_n(x) - m(x) + k(x) + 2]$$

□

Theorem 4.2.
(Sum of the first n-terms with odd indices)

$$\sum_{i=0}^{i=n-1} M_{2i+1}(x) = \frac{1}{k(x)}[M_{2n}(x)+2] \tag{14}$$

Proof. Using Eq. (3),

$$\begin{aligned}\sum_{i=0}^{i=n-1} M_{2i+1}(x) &= \sum_{i=0}^{n-1}[\frac{X r_1^{2i+1} - Y r_2^{2i+1}}{r_1 - r_2}] \\ &= \frac{X}{r_1 - r_2}\sum_{i=0}^{i=n-1} r_1^{2i+1} - \frac{Y}{r_1 - r_2}\sum_{i=0}^{i=n-1} r_2^{2i+1} \\ &= \frac{X}{r_1 - r_2}\frac{r_1[1-r_1^{2n}]}{1-r_1^2} - \frac{Y}{r_1 - r_2}\frac{r_2[1-r_2^{2n}]}{1-r_2^2}\end{aligned}$$

Using,

$$\begin{aligned}1-r_1^2 &= -k(x)r_1, \\ 1-r_2^2 &= -k(x)r_2, \\ &= \frac{1}{k(x)}[\frac{X r_1^{2n} - Y r_2^{2n} - X + Y}{r_1 - r_2}] \\ &= \frac{1}{k(x)}[M_{2n}(x)+2]\end{aligned}$$

□

Theorem 4.3.

$$\sum_{i=0}^{i=n-1} M_{2i}(x) = \frac{1}{k(x)}[M_{2n-1}(x)+k(x)-m(x)] \tag{15}$$

Proof. Using Eq. (3),

$$\begin{aligned}\sum_{i=0}^{i=n-1} M_{2i}(x) &= \sum_{i=0}^{n-1}[\frac{X r_1^{2i} - Y r_2^{2i}}{r_1 - r_2}] \\ &= \frac{X}{r_1 - r_2}\sum_{i=0}^{i=n-1} r_1^{2i} - \frac{Y}{r_1 - r_2}\sum_{i=0}^{i=n-1} r_2^{2i} \\ &= \frac{X}{r_1 - r_2}\frac{[1-r_1^{2n}]}{1-r_1^2} - \frac{Y}{r_1 - r_2}\frac{[1-r_2^{2n}]}{1-r_2^2}\end{aligned}$$

Using,

$$\begin{aligned}1-r_1^2 &= -k(x)r_1, \\ 1-r_2^2 &= -k(x)r_2, \\ &= \frac{1}{k(x)}[\frac{X r_1^{2n-1} - Y r_2^{2n-1} - X r_1^{-1} + Y r_2^{-1}}{r_1 - r_2}] \\ &= \frac{1}{k(x)}[M_{2n-1}(x) + \frac{k(x)-m(x)}{k(x)}] \\ \sum_{i=0}^{i=n-1} M_{2i}(x) &= \frac{1}{k(x)}[M_{2n-1}(x)+k(x)-m(x)]\end{aligned}$$

□

Theorem 4.4.
For any integer $n \geq 0$

$$\sum_{i=0}^{i=n} \binom{n}{i} k^i(x) M_{2i}(x) = M_{2n}(x)$$

Proof. Using Eq. (3),

$$\begin{aligned}
\sum_{i=0}^{i=n} \binom{n}{i} k^i(x) M_{2i}(x) \\
&= \sum_{i=0}^{i=n} \binom{n}{i} k^i(x) \frac{X r_1^i - Y r_2^i}{r_1 - r2} \\
&= \frac{X}{r_1 - r_2} \sum_{i=0}^{i=n} \binom{n}{i} k^i(x) r_1^i - \frac{Y}{r_1 - r_2} \sum_{i=0}^{i=n} \binom{n}{i} k^i(x) r_2^i \\
&= \frac{X}{r_1 - r_2} (k(x) r_1 + 1)^n - \frac{Y}{r_1 - r_2} (k(x) r_2 + 1)^n \\
&= \frac{X}{r_1 - r_2} r_1^{2n} - \frac{Y}{r_1 - r_2} r_2^{2n} \\
&= M_{2n}(x)
\end{aligned}$$

$$\sum_{i=0}^{i=n} \binom{n}{i} k^i(x) M_{2i}(x) = M_{2n}(x)$$

□

Theorem 4.5.
For arbitary integers $p, q \geq 1$

$$\sum_{i=1}^{p} M_{qi}(x) = \frac{M_{pq+q}(x) - (-1)^q M_{pq}(x) - M_q(x) + 2(-1)^q}{r_1^q + r_2^q - (-1)^q - 1} \tag{16}$$

Proof. Using Eq. (3),

$$\begin{aligned}
\sum_{i=1}^{p} M_{qi}(x) &= \sum_{i=1}^{p} \frac{X r_1^{qi} - Y r_2^{qi}}{r_1 - r_2} \\
&= \frac{X}{r_1 - r_2} \sum_{i=1}^{p} r_1^{qi} - \frac{Y}{r_1 - r_2} \sum_{i=1}^{p} r_2^{qi} \\
&= \frac{X}{r_1 - r_2} [\frac{r_1^{pq+q} - r_1^q}{r_1^q - 1}] - \frac{Y}{r_1 - r_2} [\frac{r_2^{pq+q} - r_2^q}{r_2^q - 1}] \\
&= \frac{X[r_1^{pq+q} - r_1^q][r_2^q - 1] - Y[r_2^{pq+q} - r_2^q][r_1^q - 1]}{[r_1 - r_2][r_1^q - 1][r_2^q - 1]} \\
&= \frac{X}{r_1 - r_2} [\frac{r_1^{pq+q} r_2^q - (r_1 r_2)^q - r_2^{pq+q} r_1^q}{[r_1 r_2]^q - r_1^q - r_2^q + 1}] \\
&= \frac{[\frac{X r_1^{pq+q} - Y r_2^{pq+q}}{r_1 - r_2}] - [\frac{X r_1^q - Y r_2^q}{r_1 - r_2}] - (-1)^q [\frac{X r_1^{pq} - Y r_2^{pq}}{r_1 - r_2}] + 2(-1)^q}{r_1^q + r_2^q - (-1)^q - 1}
\end{aligned}$$

Gives,

$$\sum_{i=1}^{p} M_{qi}(x) = \frac{M_{pq+q}(x) - (-1)^q M_{pq}(x) - M_q(x) + 2(-1)^q}{r_1^q + r_2^q - (-1)^q - 1}$$

□

Theorem 4.6.
For arbitary integers $p, q, j \geq 1$ with $j \geq q$

$$\sum_{i=1}^{p} M_{qi+j}(x) = \frac{M_{pq+q+j}(x) - (-1)^q M_{pq+j}(x) - M_{q+j}(x) + (-1)^q M_j(x)}{r_1^q + r_2^q - (-1)^q - 1} \tag{17}$$

Proof. Using Eq. (3),

$$\begin{aligned}
\sum_{i=1}^{p} M_{qi+j}(x) &= \sum_{i=1}^{p} \frac{X r_1^{qi+j} - Y r_2^{qi+j}}{r_1 - r_2} \\
&= \frac{X}{r_1 - r_2} \sum_{i=1}^{p} r_1^{qi+j} - \frac{Y}{r_1 - r_2} \sum_{i=1}^{p} r_2^{qi+j} \\
&= \frac{X}{r_1 - r_2} [\frac{r_1^{pq+q+j} r_2^q - (r_1 r_2)^q r_1^j - r_1^{pq+q+j} + r_1^{q+j}}{(r_1 r_2)^q - r_1^q - r_2^q + 1}] \\
&- \frac{Y}{r_1 - r_2} [\frac{r_2^{pq+q+j} r_1^q - (r_1 r_2)^q r_2^j - r_2^{pq+q+j} + r_2^{q+j}}{(r_1 r_2)^q - r_1^q - r_2^q + 1}]
\end{aligned}$$

Gives,

$$\sum_{i=1}^{p} M_{qi+j}(x) = \frac{M_{pq+q+j}(x) - (-1)^q M_{pq+j}(x) - M_{q+j}(x) + (-1)^q M_j(x)}{r_1^q + r_2^q - (-1)^q - 1}$$

□

Theorem 4.7.
For arbitary integers $n, j \geq 1$

$$\sum_{i=1}^{n} M_{i+j}(x) = \frac{1}{k(x)} [M_{n+j+1}(x) + M_{n+j}(x) - M_j(x) - M_{j-1}(x)] \tag{18}$$

Proof. Using Eq. (3),

$$\begin{aligned}
\sum_{i=1}^{n} M_{i+j}(x) &= \sum_{i=1}^{n} [\frac{X r_1^{i+j} - Y r_2^{i+j}}{r_1 - r_2}] \\
&= \frac{X}{r_1 - r_2} \sum_{i=1}^{n} r_1^{i+j} - \frac{Y}{r_1 - r_2} \sum_{i=1}^{n} r_2^{i+j} \\
&= \frac{X}{r_1 - r_2} r_1^j [\frac{r_1^{n+1} - 1}{r_1 - 1}] - \frac{Y}{r_1 - r_2} r_2^j [\frac{r_2^{n+1} - 1}{r_2 - 1}] \\
&= \frac{1}{k} [\frac{X r_1^{n+j+1} - Y r_2^{n+j+1}}{r_1 - r_2} + \frac{X r_1^{n+j} - Y r_2^{n+j}}{r_1 - r_2} - \frac{X r_1^j - Y r_2^j}{r_1 - r_2} - \frac{X r_1^{j-1} - Y r_2^{j-1}}{r_1 - r_2}]
\end{aligned}$$

Gives,

$$\sum_{i=1}^{n} M_{i+j}(x) = \frac{1}{k(x)} [M_{n+j+1}(x) + M_{n+j}(x) - M_j(x) - M_{j-1}(x)]$$

□

Theorem 4.8.

$$\sum_{i=1}^{i=n} M_i^2(x) = \frac{M_n(x) M_{n+1}(x) - 2 M_1(x)}{k(x)} \tag{19}$$

Proof. Let,

$$T(x)=\sum_{i=1}^{i=n} M_i^2(x)$$

Since,

$$M_i(x)=\frac{M_{i+1}(x)-M_{i-1}(x)}{k(x)}$$

Gives,

$$T(x)=\sum_{i=1}^{i=n}[\frac{M_{i+1}(x)-M_{i-1}(x)}{k(x)}]^2$$
$$=\frac{1}{k^2(x)}[\sum_{i=1}^{i=n}M_{i+1}^2(x)-2\sum_{i=1}^{i=n}M_{i+1}(x).M_{i-1}(x)+\sum_{i=1}^{i=n}M_{i-1}^2(x)]$$

Using,(5)

$$=\frac{1}{k^2(x)}[\sum_{i=1}^{i=n}M_{i+1}^2(x)-2\sum_{i=1}^{i=n}M_i^2(x)+\sum_{i=1}^{i=n}M_{i-1}^2(x)-2(m^2(x)-k^2(x)-4)\sum_{i=1}^{i=n}(-1)^i]$$
$$=\frac{1}{k^2(x)}[\sum_{i=2}^{i=n+1}M_i^2(x)-2\sum_{i=1}^{i=n}M_i^2(x)+\sum_{i=0}^{i=n-1}M_i^2(x)-2(m^2(x)-k^2(x)-4)\sum_{i=1}^{i=n}(-1)^i]$$
$$=\frac{1}{k^2(x)}[(T(x)+M_{n+1}^2(x)-M_1^2(x))+(T(x)-M_n^2(x)+M_0^2(x))-2T(x)$$
$$-2(m^2(x)-k^2(x)-4)\sum_{i=1}^{i=n}(-1)^i]$$

Now,

$$M_0(x)=2$$
$$M_1(x)=m(x)+k(x)$$
$$=\frac{1}{k^2(x)}[M_{n+1}^2(x)-M_n^2(x)-(m(x)+k(x))^2+4-2(m^2(x)-k^2(x)-4)\sum_{i=1}^{i=n}(-1)^i]$$

Using,(5)

$$=\frac{1}{k^2(x)}[M_{n+1}(x)(M_{n+1}(x)M_{n-1}(x))-(m(x)+k(x))^2+4-(-1)^n(m^2(x)-k^2(x)-4)$$
$$-2(m^2(x)-k^2(x)-4)\sum_{i=1}^{i=n}(-1)^i]$$
$$=\frac{1}{k^2(x)}[M_{n+1}(x)(k(x)M_n(x))-2k(x)(m(x)+k(x))]$$

Gives,

$$\sum_{i=1}^{i=n}M_i^2(x)=\frac{M_n(x)M_{n+1}(x)-2M_1(x)}{k(x)}$$

□

Theorem 4.9.

$$\sum_{j=1}^{i=n} jM_j(x)=\frac{(nk(x)-1)M_{n+1}(x)+(nk(x)-2)M_n(x)-M_{n-1}(x)-(k^2(x)+k(x)(m(x)-2)+2(m(x)+2))}{k^2(x)} \quad (20)$$

Proof. Using (12),

$$M_1(x)+M_2(x)+\cdots\cdots+M_n=\frac{1}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]$$

$$M_2(x)+M_3(x)+\cdots\cdots+M_n(x)=\frac{1}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-M_1(x)$$

$$M_3(x)+M_4(x)+\cdots\cdots+M_n(x)=\frac{1}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-M_1(x)-M_2(x)$$

$$\vdots$$

$$M_n(x)=\frac{1}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-(M_1(x)-M_2(x)-\cdots\cdots-M_{n-1}(x))$$

Sum of the left hand side gives,

$$\sum_{j=1}^{i=n} jM_j(x),$$

We have for $n=1,2,3,\cdots\cdots,n-1$

$$[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-(M_1(x)-M_2(x)-\cdots-M_{n-1}(x))=[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)] \\ -[M_{t+1}(x)+M_t(x)-(m(x)+k(x)+2)]$$

So,when summing the right hand side we have,

$$\sum_{j=1}^{i=n} jM_j(x)=\frac{n}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-\frac{\sum_{i=1}^{i=n-1}M_i(x)+\sum_{i=2}^{i=n-1}M_i(x)-(n-1)(m(x)+k(x)+2)}{k(x)}$$

$$=\frac{n}{k(x)}[M_{n+1}(x)+M_n(x)-(m(x)+k(x)+2)]-\frac{1}{k(x)}[\frac{1}{k(x)}[M_{n-1}(x)+M_n(x)-(m(x)+k(x)+2)]$$

$$+\frac{1}{k(x)}[M_n(x)+M_{n+1}(x)-(m(x)+k(x)+2)-2]-(n-1)(m(x)+k(x)+2)]$$

$$\sum_{j=1}^{i=n} jM_j(x)=\frac{(nk(x)-1)M_{n+1}(x)+(nk(x)-2)M_n(x)-M_{n-1}(x)-(k^2(x)+k(x)(m(x)-2)+2(m(x)+2))}{k^2(x)}$$

□

Theorem 4.10.

For each real number $p>r_1, x>0$

$$\sum_{i=1}^{i=\infty}\frac{M_i(x)}{p^i}=\frac{p(m(x)+k(x))+2}{p^2-k(x)p-1} \tag{21}$$

Proof. We have,

$$\sum_{i=1}^{i=\infty}\frac{M_i(x)}{p^i}=\lim_{n\to\infty}\sum_{i=1}^{i=n}\frac{M_i(x)}{p^i}$$

Using Eq. (3),

$$\sum_{i=1}^{i=\infty}\frac{M_i(x)}{p^i}=\lim_{n\to\infty}\sum_{i=1}^{i=n}\frac{\frac{Xr_1^i-Yr_2^i}{r_1-r_2}}{p^i}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{i=n}\frac{X(\frac{r_1}{p})^i-Y(\frac{r_2}{p})^i}{r_1-r_2}$$

$$=\frac{1}{r_1-r_2}[\frac{Xr_1}{p-r_1}-\frac{Yr_2}{p-r_2}]$$

$$=\frac{1}{r_1-r_2}[\frac{p(Xr_1-Yr_2)+(X-Y)}{(p-r_1)(p-r_2)}]$$

Since,

$$Xr_1-Yr_2=(r_1-r_2)(m(x)+r_1+r_2), X-Y=2(r_1-r_2)$$

Gives,

$$\sum_{i=1}^{i=\infty}\frac{M_i(x)}{p^i}=\frac{p(m(x)+k(x))+2}{p^2-k(x)p-1}$$

□

5. Properties of generalized Fibonacci like polynomials by matrix methods

Theorem 5.1.

$$\begin{pmatrix} M_{n+1}(x) & M_n(x) \\ M_n(x) & M_{n-1}(x) \end{pmatrix} = P^n \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}, Where, P = \begin{pmatrix} k(x) & 1 \\ 1 & 0 \end{pmatrix} \tag{22}$$

Proof. The proof of this theorem can be composed by induction,
For $n = 1$

$$\begin{pmatrix} M_2(x) & M_1(x) \\ M_1(x) & M_0(x) \end{pmatrix} = \begin{pmatrix} k(x)(m(x)+k(x))+2 & m(x)+k(x) \\ m(x)+k(x) & 2 \end{pmatrix}$$

And,

$$P\begin{pmatrix} (m(x)+k(x)) & 2 \\ 2 & (m(x)-k(x)) \end{pmatrix} = \begin{pmatrix} k(x) & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (m(x)+k(x)) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$
$$= \begin{pmatrix} k(x)(m(x)+k(x))+2 & m(x)+k(x) \\ m(x)+k(x) & 2 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} M_2(x) & M_1(x) \\ M_1(x) & M_0(x) \end{pmatrix} = P^1 \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

Now,assume that result is true for $n-1$
Therefore,we have,

$$\begin{pmatrix} M_n(x) & M_{n-1}(x) \\ M_{n-1}(x) & M_{n-2}(x) \end{pmatrix} = P^{n-1} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

Now,

$$P^n \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = P[P^{n-1} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}]$$
$$= P\begin{pmatrix} M_n(x) & M_{n-1}(x) \\ M_{n-1}(x) & M_{n-2}(x) \end{pmatrix}$$
$$= \begin{pmatrix} k(x) & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} M_n(x) & M_{n-1}(x) \\ M_{n-1}(x) & M_{n-2}(x) \end{pmatrix}$$
$$= \begin{pmatrix} M_{n+1}(x) & M_n(x) \\ M_n(x) & M_{n-1}(x) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} M_{n+1}(x) & M_n(x) \\ M_n(x) & M_{n-1}(x) \end{pmatrix} = P^n \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

□

Theorem 5.2.

$$M_{-n+1}(x).M_{-n-1}(x) - M_{-n}^2(x) = (m^2(x) - k^2(x) - 4)(-1)^n \tag{23}$$

Proof. Using(21)

$$\begin{pmatrix} M_{n+1}(x) & M_n(x) \\ M_n(x) & M_{n-1}(x) \end{pmatrix} = P^n \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix},$$

Where,

$$P = \begin{pmatrix} k(x) & 1 \\ 1 & 0 \end{pmatrix}$$

And,

$$P^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}$$

Now,

$$P^{-n} = (P^n)^{-1} = \frac{1}{F_{n+1}(x)F_{n-1}(x) - F_n^2(x)} \begin{pmatrix} F_{n-1}(x) & -F_n(x) \\ -F_n(x) & F_{n+1}(x) \end{pmatrix}$$

Since,

$$F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n$$

$$P^{-n} = \frac{1}{(-1)^n} \begin{pmatrix} F_{n-1}(x) & -F_n(x) \\ -F_n(x) & F_{n+1}(x) \end{pmatrix}$$

Now,

$$P^{-n} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = (P^n)^{-1} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} M_{-n+1}(x) & M_{-n}(x) \\ M_{-n}(x) & M_{-n-1}(x) \end{pmatrix} = \frac{1}{(-1)^n} \begin{pmatrix} F_{n-1}(x) & -F_n(x) \\ -F_n(x) & F_{n+1}(x) \end{pmatrix} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

Gives,

$$\begin{pmatrix} M_{-n+1}(x) & M_{-n}(x) \\ M_{-n}(x) & M_{-n-1}(x) \end{pmatrix} = \frac{1}{(-1)^n} \begin{pmatrix} (m(x)+k(x))(F_{n-1}(x)-2F_n(x)) & 2F_{n-1}(x)-(m(x)-k(x))F_n(x) \\ 2F_{n+1}(x)-(m(x)+k(x))F_n(x) & (m(x)-k(x))(F_{n+1}(x)-2F_n(x)) \end{pmatrix}$$

Gives,

$$\begin{aligned} M_{-n+1}(x).M_{-n-1}(x) - M_{-n}^2(x) &= F_{n-1}(x)F_{n+1}(x)[(m(x)+k(x))(m(x)-k(x))-4] \\ &\quad - F_n^2(x)[(m(x)+k(x))(m(x)-k(x))-4] \\ M_{-n+1}(x).M_{-n-1}(x) - M_{-n}^2(x) &= [F_{n-1}(x)F_{n+1}(x) - F_n^2(x)][(m(x)+k(x))(m(x)-k(x))-4] \\ M_{-n+1}(x).M_{-n-1}(x) - M_{-n}^2(x) &= (m^2(x)-k^2(x)-4)(-1)^n \end{aligned}$$

□

Theorem 5.3.
For arbitary integer $n, r \geq 0$

$$M_{n-r+1}(x) = (-1)^r [F_{r-1}(x)M_{k,n+1} - F_{k,r}M_{k,n}] \quad (24)$$

$$M_{k,n-r}(x) = (-1)^r [F_{r-1}(x)M_n(x) - F_r(x)M_{n-1}(x)] \quad (25)$$

$$M_{n-r-1}(x) = (-1)^r [F_{r+1}(x)M_{n-1}(x) - F_r(x)M_n(x)] \quad (26)$$

Proof. Since,

$$P^{n-r} = P^n.P^{-r}$$

$$P^{n-r} = (-1)^r \begin{pmatrix} F_{r-1}(x) & -F_r(x) \\ -F_r(x) & F_{r+1}(x) \end{pmatrix} P^n$$

Now,

$$P^{n-r} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = \begin{pmatrix} M_{n-r+1}(x) & M_{n-r}(x) \\ M_{n-r}(x) & M_{n-r-1}(x) \end{pmatrix}$$

and

$$P^{n-r}\begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = (-1)^r \begin{pmatrix} F_{r-1}(x) & -F_r(x) \\ -F_r(x) & F_{r+1}(x) \end{pmatrix} P^n \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$
$$= (-1)^r \begin{pmatrix} F_{r-1}(x) & -F_r(x) \\ -F_r(x) & F_{r+1}(x) \end{pmatrix} \begin{pmatrix} M_{n+1}(x) & M_n(x) \\ M_n(x) & M_{n-1}(x) \end{pmatrix}$$
$$= (-1)^r \begin{pmatrix} F_{r-1}(x)M_{n+1}(x) - F_r(x)M_n(x) & F_{r-1}(x)M_n(x) - F_r(x)M_{n-1}(x) \\ F_{r+1}(x)M_n(x) - F_r(x)M_{n+1}(x) & F_{r+1}(x)M_{n-1}(x) - F_r(x)M_n(x) \end{pmatrix}$$

Gives,

$$M_{n-r+1}(x) = (-1)^r [F_{r-1}(x)M_{n+1}(x) - F_r(x)M_n(x)],$$
$$M_{n-r}(x) = (-1)^r [F_{r-1}(x)M_n(x) - F_r(x)M_{n-1}(x)],$$
$$M_{n-r-1}(x) = (-1)^r [F_{r+1}(x)M_{n-1}(x) - F_r(x)M_n(x)]$$

□

Theorem 5.4.
For $n, r \geq 0$

$$(m(x)+k(x)) - M_{rn+r+1}(x) = (1 - F_{r+1}(x)) \sum_{j=0}^{j=n} M_{rj+1}(x) - F_r(x) \sum_{j=0}^{j=n} M_{rj}(x) \tag{27}$$

$$2 - M_{rn+r}(x) = (1 - F_{r+1}(x)) \sum_{j=0}^{j=n} M_{rj}(x) - F_r(x) \sum_{j=0}^{j=n} M_{rj+1}(x) \tag{28}$$

$$(m(x)-k(x)) - M_{rn+r-1}(x) = (1 - F_{r-1}(x)) \sum_{j=0}^{j=n} M_{rj-1}(x) - F_r(x) \sum_{j=0}^{j=n} M_{rj}(x) \tag{29}$$

Proof. Since,

$$(I - P^r) \sum_{j=0}^{j=n} (P^r)^j = I - (P^r)^{n+1}$$

Now,

$$[I - (P^r)^{n+1}] \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} - P^{rn+r} \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix}$$

$$= \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} - \begin{pmatrix} M_{rn+r+1}(x) & M_{rn+r}(x) \\ M_{rn+r}(x) & M_{rn+r-1}(x) \end{pmatrix}$$
$$= \begin{pmatrix} (m(x)+k(x))M_{rn+r+1}(x) & 2 - M_{rn+r}(x) \\ 2 - M_{rn+r}(x) & (m(x)-k(x))M_{rn+r-1}(x) \end{pmatrix}$$

And,

$$\sum_{j=0}^{j=n} (P^r)^j \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{j=n} M_{rj+1}(x) & \sum_{j=0}^{j=n} M_{rj}(x) \\ \sum_{j=0}^{j=n} M_{rj}(x) & \sum_{j=0}^{j=n} M_{rj-1}(x) \end{pmatrix}$$

Therefore,

$$(I - P^r) \sum_{j=0}^{j=n} (P^r)^j \begin{pmatrix} m(x)+k(x) & 2 \\ 2 & m(x)-k(x) \end{pmatrix} = (I - P^r) \begin{pmatrix} \sum_{j=0}^{j=n} M_{rj+1}(x) & \sum_{j=0}^{j=n} M_{rj}(x) \\ \sum_{j=0}^{j=n} M_{rj}(x) & \sum_{j=0}^{j=n} M_{rj-1}(x) \end{pmatrix}$$

$$= \begin{pmatrix} (1 - F_{r+1}(x)) & -F_r(x) \\ -F_r(x) & (1 - F_{r-1})(x) \end{pmatrix} \begin{pmatrix} \sum_{j=0}^{j=n} M_{rj+1}(x) & \sum_{j=0}^{j=n} M_{rj}(x) \\ \sum_{j=0}^{j=n} M_{rj}(x) & \sum_{j=0}^{j=n} M_{rj-1}(x) \end{pmatrix}$$

$$= \begin{pmatrix} (1-F_{r+1}(x))\sum_{j=0}^{j=n} M_{rj+1}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj}(x) & (1-F_{r+1}(x))\sum_{j=0}^{j=n} M_{rj}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj-1}(x) \\ (1-F_{r-1}(x))\sum_{j=0}^{j=n} M_{rj}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj+1}(x) & (1-F_{r-1}(x))\sum_{j=0}^{j=n} M_{rj-1}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj}(x) \end{pmatrix}$$

Gives,

$$(m(x)+k(x)) - M_{rn+r+1}(x) = (1-F_{r+1}(x))\sum_{j=0}^{j=n} M_{rj+1}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj}(x)$$

$$2 - M_{rn+r}(x) = (1-F_{r+1}(x))\sum_{j=0}^{j=n} M_{rj}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj+1}(x)$$

$$(m(x)-k(x)) - M_{rn+r-1}(x) = (1-F_{r-1}(x))\sum_{j=0}^{j=n} M_{rj-1}(x) - F_r(x)\sum_{j=0}^{j=n} M_{rj}(x)$$

□

6. Conclusion

Some new identities have been obtained for the generalized Fibonacci like polynomials same identities can be derived for negative values of n.

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