On the properties of generalized multiplicative coupled fibonacci sequence of \( r \)th order

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Abstract: Coupled Fibonacci sequences of lower order have been generalized in number of ways. In this paper the Multiplicative Coupled Fibonacci Sequence has been generalized for \( r \)th order with some new interesting properties.

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1. Introduction

Definition 1.1.
The Multiplicative Coupled Fibonacci Sequence of 2\(^n\)d order is defined as. Let \( \{X_i\}_{i=0}^{\infty} \) and \( \{Y_i\}_{i=0}^{\infty} \) be two infinite sequences and four arbitrary real numbers \( a, b, c, d \) are given. The Multiplicative Coupled Fibonacci Sequence of 2\(^n\)d order is generated by the following four different ways:

First scheme:
\[
X_{n+2} = X_{n+1} \cdot X_n, \quad n \geq 0
\]
\[
Y_{n+2} = Y_{n+1} \cdot Y_n, \quad n \geq 0
\]

Second scheme:
\[
X_{n+2} = Y_{n+1} \cdot X_n, \quad n \geq 0
\]
\[
Y_{n+2} = X_{n+1} \cdot Y_n, \quad n \geq 0
\]

Third scheme:
\[
X_{n+2} = X_{n+1} \cdot Y_n, \quad n \geq 0
\]
\[
Y_{n+2} = Y_{n+1} \cdot X_n, \quad n \geq 0
\]

Fourth scheme:
\[
X_{n+2} = Y_{n+1} \cdot Y_n, \quad n \geq 0
\]
\[
Y_{n+2} = X_{n+1} \cdot X_n, \quad n \geq 0
\]

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Definition 1.2.
The Multiplicative Coupled Fibonacci Sequence of 3\textsuperscript{rd} order is defined as, Let \(\{X_i\}_{i=0}^{\infty}\) and \(\{Y_i\}_{i=0}^{\infty}\) be two infinite sequences and six arbitrary real numbers \(a, b, c, d, e, f\) are given. The Multiplicative Coupled Fibonacci Sequence of 3\textsuperscript{rd} order is generated by the following eight different ways:

First scheme:
\[
X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, n \geq 0 \\
Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, n \geq 0
\]

Second scheme:
\[
X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot X_n, n \geq 0 \\
Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot Y_n, n \geq 0
\]

Third scheme:
\[
X_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, n \geq 0 \\
Y_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, n \geq 0
\]

Fourth scheme:
\[
X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, n \geq 0 \\
Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, n \geq 0
\]

Fifth scheme:
\[
X_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, n \geq 0 \\
Y_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, n \geq 0
\]

Sixth scheme:
\[
X_{n+3} = X_{n+2} \cdot X_{n+1} \cdot Y_n, n \geq 0 \\
Y_{n+3} = Y_{n+2} \cdot Y_{n+1} \cdot X_n, n \geq 0
\]

Seventh scheme:
\[
X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot Y_n, n \geq 0 \\
Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot X_n, n \geq 0
\]

Eighth scheme:
\[
X_{n+3} = X_{n+2} \cdot Y_{n+1} \cdot X_n, n \geq 0 \\
Y_{n+3} = Y_{n+2} \cdot X_{n+1} \cdot Y_n, n \geq 0
\]

In recent years many authors have been generalized Coupled Fibonacci sequences of lower order in number of ways. In this paper the Multiplicative Coupled Fibonacci Sequence has been generalized for \(r\textsuperscript{th}\) order with some new interesting properties.

2. Multiplicative Coupled Fibonacci Sequence of \(r\textsuperscript{th}\) order

Definition 2.1.
The Multiplicative Coupled Fibonacci Sequence of \(r\textsuperscript{rd}\) order is defined as, Let \(\{X_i\}_{i=0}^{\infty}\) and \(\{Y_i\}_{i=0}^{\infty}\) be two infinite sequences and \(2r\) arbitrary real numbers \(x_0, x_1, x_2, x_3, \ldots, x_{r-1}\) and \(y_0, y_1, y_2, y_3, \ldots, y_{r-1}\) are given. The Multiplicative Coupled Fibonacci Sequence of \(r\textsuperscript{th}\) order is generated by the following \(2^r\) different ways:

First scheme:
\[
X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0 \\
Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0
\]

\[
\vdots
\]

\((2^r)^{th}\) scheme:
\[
X_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n, n \geq 0 \\
Y_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n, n \geq 0
\]

are given below:
3. Main Results

In this section many of the fabulous properties of generalized multiplicative coupled Fibonacci sequence of \( r \)th order under \( 2^{rth} \) scheme are established.

**Theorem 3.1.**
For every integer \( n \geq 0, r \geq 0 \)
\[
X_{n(r+1)} \cdot Y_0 = Y_{n(r+1)} \cdot X_0
\] (1)

**Proof.** If \( n = 0 \), then result is true because
\[X_0 \cdot Y_0 = Y_0 \cdot X_0\]

Assume that the result is true for some integer \( n \geq 1 \)

Now,
\[
X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n
\]
\[
Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n,
\]

Using Induction Method,
For, \( n + 1 \)
\[
X_{n+1(r+1)} \cdot Y_0 = \left[ Y_{n(r+1)+1} \cdot Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1} \right] \cdot Y_0
\]
\[
= \left[ Y_{n(r+1)+1} \cdot Y_{n(r+1)+r-2} \cdot X_{n(r+1)+r-3} \cdots X_{n(r+1)+1} \right] \cdot \left[ Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1} \right] \cdot Y_0
\]
\[
= \left[ Y_{n(r+1)+1} \cdot X_{n(r+1)+r-2} \cdot Y_{n(r+1)+r-3} \cdots X_{n(r+1)+1} \right] \cdot \left[ Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1} \right] \cdot X_{n(r+1)} \cdot Y_0
\]

Using induction hypothesis,
\[
= \left[ X_{n(r+1)+1} \cdot X_{n(r+1)+r-2} \cdot Y_{n(r+1)+r-3} \cdots X_{n(r+1)+1} \right] \cdot \left[ Y_{n(r+1)+r-1} \cdot Y_{n(r+1)+r-2} \cdots Y_{n(r+1)+1} \right] \cdot X_{n(r+1)} \cdot Y_0
\]
\[
= \left[ Y_{n(r+1)+1} \cdot X_{n(r+1)+r-2} \cdot Y_{n(r+1)+r-3} \cdots X_{n(r+1)+1} \right] \cdot \left[ X_{n(r+1)+r} \cdot X_0 \right]
\]
\[
= \left[ Y_{n+1(r+1)} \cdot X_0 \right]
\]

Hence result is true for \( n + 1 \).

**Theorem 3.2.**
For every integer \( n \geq 0, r \geq 0 \)
\[
X_{n(r+1)+1} \cdot Y_1 = Y_{n(r+1)+1} \cdot X_1
\] (2)

**Proof.** If \( n = 0 \), then result is true because
\[X_1 \cdot Y_1 = Y_1 \cdot X_1\]

Assume that the result is true for some integer \( n \geq 1 \)

Now,
\[
X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdot Y_{n+r-3} \cdots Y_n
\]
\[
Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdot X_{n+r-3} \cdots X_n,
\]
Using Induction Method,

For, \( n + 1 \)

\[
X_{(n+1)(r+1)+1} \cdot Y_1 = X_{n(r+1)+r+2} \cdot Y_1
\]

\[
= \left[ X_{n(r+1)+r+1} \cdot Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots Y_{n(r+1)+2} \right] \cdot Y_1
\]

\[
= \left[ X_{n(r+1)+r} \cdot X_{n(r+1)+r-1} \cdots X_{n(r+1)+2} \right] \cdot \left[ Y_{n(r+1)+r} \cdot Y_{n(r+1)+r-1} \cdots Y_{n(r+1)+2} \right] \cdot Y_1
\]

Hence result is true for \( n + 1 \).

In the similar way, it could be proved for the following results by induction method.

**Theorem 3.3.**

*For every integer \( n \geq 0, r \geq 0 \)

\[
X_{n(r+1)+r+2} \cdot Y_2 = Y_{n(r+1)+2} \cdot X_2 \quad (3)
\]

**Theorem 3.4.**

*For every integer \( n \geq 0, r \geq 0 \)

\[
X_{n(r+1)+3} \cdot Y_3 = Y_{n(r+1)+3} \cdot X_3 \quad (4)
\]

**Theorem 3.5.**

*For every integer \( n \geq 0, r \geq 0 \) and \( m \geq 0 \)

\[
X_{n(r+1)+m} \cdot Y_m = Y_{n(r+1)+m} \cdot X_m \quad (5)
\]

**Proof.**  

If \( n = 0 \), then result is true because

\[
X_m \cdot Y_m = Y_m \cdot X_m
\]

Assume that the result is true for some integer \( n \geq 1 \)

Now,

\[
X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdots Y_0
\]

\[
Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdots X_{n+r-3} \cdots X_0,
\]

Using Induction Method,

For, \( n + 1 \)

\[
X_{(n+1)(r+1)+m} \cdot Y_m = X_{n(r+1)+m+1} \cdot Y_m
\]

\[
= \left[ X_{n(r+1)+r+m} \cdot Y_{n(r+1)+r+m-1} \cdots Y_{n(r+1)+r+2} \right] \cdot Y_m
\]

\[
= \left[ Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots Y_{n(r+1)+m} \right] \cdot X_m
\]

\[
= \left[ X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+r+m-2} \cdots X_{n(r+1)+m} \right] \cdot Y_m
\]

\[
= \left[ Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots Y_{n(r+1)+m} \right] \cdot X_m
\]

\[
= \left[ X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+r+m-2} \cdots X_{n(r+1)+m} \right] \cdot Y_m
\]

\[
= \left[ Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+r+m-2} \cdots Y_{n(r+1)+m} \right] \cdot X_m
\]

\[
= \left[ X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+r+m-2} \cdots X_{n(r+1)+m} \right] \cdot Y_m
\]
On the properties of generalized multiplicative coupled Fibonacci sequence of \(r\)th order

Using induction hypothesis,

\[
= \left[ X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdot X_{n(r+1)+(r+m-3)} \cdots X_{n(r+1)+m+1} \right] \\
\cdot \left[ Y_{n(r+1)+r+m-1} \cdot Y_{n(r+1)+(r+m-2)} \cdots Y_{n(r+1)+m+1} \right] \\
= \left[ X_{n(r+1)+r+m-1} \cdot X_{n(r+1)+(r+m-2)} \cdots X_{n(r+1)+m+1} \right] \\
\cdot \left[ Y_{n(r+1)+r+m+1} \right] \\
= \left[ Y_{n+1(r+1)+m} \right]
\]

Hence result is true for \(n+1\).

**Theorem 3.6.**

For every integer \(n \geq 0, r \geq 0\)

1. \(\prod_{i=1}^{n} X_{r+1} = \prod_{i=1}^{n} Y_i\)

2. \(\prod_{i=1}^{n} Y_{r+1} = \prod_{i=1}^{n} X_i\)

**Proof.** If \(n = 1\), then result is true because

\[X_{r+1} = Y_r \cdot Y_{r-1} \cdot Y_{r-2} \cdots Y_1\]

Assume that the result is true for some integer \(n \geq 1\)

Now,

\[X_{n+r} = Y_{n+r-1} \cdot Y_{n+r-2} \cdots Y_n\]
\[Y_{n+r} = X_{n+r-1} \cdot X_{n+r-2} \cdots X_n\]

Using induction method,

For, \(n + 1\)

\[\prod_{i=1}^{n+1} X_{r+1} = \prod_{i=1}^{n} X_{r+1} \cdot X_{r(n+1)+1}\]

Using induction hypothesis,

\[= \prod_{i=1}^{n} Y_i \cdot X_{n+r+1}\]
\[= \prod_{i=1}^{n} Y_i \cdot Y_{n+r} \cdot Y_{n+r-1} \cdots Y_{n+1}\]
\[= \prod_{i=1}^{n+r} Y_i\]

Hence result is true for \(n + 1\).

In the similar way (2) can be proved using induction method.

**4. Conclusions**

The identities of generalized multiplicative coupled Fibonacci sequence of \(r\)th order are described here, the ideas can be extended for generalized multiplicative coupled Fibonacci sequence of \(r\)th order with negative integers.

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References