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Integro-differential equation of p-Kirchhoff type with no-flux boundary condition and nonlocal source term

Research Article

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Abstract: In our research we will study the existence of weak solutions to the problem

$$[M(\|u\|^p_{1,p})]^{p-1}\left(-\Delta_p u+|u|^{p-2}u\right)=f(x,u)\parallel u\parallel^t_a+\int_\Omega k(x,y)H(u)dy \quad \text{in}\,\Omega.$$

with no-flux boundary condition on a bounded smooth domain of $\mathbb{R}^n$, $1<p<N$; M, f, k and H are given functions. By means of the Galerkin method and using of the Brouwer Fixed Point theorem we get our results.The uniqueness of a weak solution is also considered.

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Keywords: p-Kirchhoff type equations • variational methods • no-flux boundary condition

1. Introduction and Preliminaries

The equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$

presented by Kirchhoff in 1883 [16], is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. A distinguishing feature of equation (1) is that the equation contains a nonlocal coefficient $\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx$ which depends on the average $\frac{1}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx$, and hence the equation is no longer a pointwise identity. Some early classical studies of Kirchhoff equations were Bernstein [7] and Pohožaev [21]. The equation

$$\begin{aligned}-\Big(a+b\int_\Omega|\nabla u|^2dx\Big)\Delta u=f(x,u) \quad &\text{in}\,\Omega,\\ u=0 \quad &\text{on}\,\partial\Omega,\end{aligned} \tag{2}$$

is related to the stationary analogue of the equation (1) and received much attention only after Lions [22] proposed an abstract framework to the problem. Problems like (2) can be used for modelling several physical and biological

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systems where u describes a process which depends on the average of it self,such as the population density, see [13] and its references therein. Some important and interesting results can be found, for example,in [3, 12, 20]. Recently Alves et al. [4] and Ma and Rivera [25] obtained positive solutions of such problems by variational methods.
An interesting generalization of problem (2) is

$$-\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = f(x,u) \quad \text{in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $-\Delta_p u$ is the p-Laplacian: $-\Delta_p u :=$ div$(|\nabla u|^{p-2}\nabla u)$. Correa and Nascimento [9], Liu et al [26], Yang and Chang [30], Correa and Figueiredo [10] and recently Molica Bisci and Radulescu [19] studied questions on the existence of positive solutions.
In [30] Yang and Zhang studied the following problem

$$-\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu}=0, \quad \text{on } \partial\Omega,$$

where $p<N$, they established existence and multiplicity of solutions for the problem under suitable assumptions on M and f.
The question of existence of positive solutions for a class of nonvariational elliptic system with nonlocal source term is studied by Chen and Gao [17], via Galerkin methods.
In this paper we are interested in the following semilinear integro-differential equation of p-Kirchhoff type

$$\begin{aligned}
&\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\left(-\Delta_p u+|u|^{p-2}u\right)=f(x,u)\parallel u\parallel_{\alpha}^{t}+\int_{\Omega}k(x,y)H(u(y))dy \quad \text{in } \Omega,\\
&u(x)=\text{constant}, \quad x\in\partial\Omega,\\
&\int_{\partial\Omega}|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}dS=0,
\end{aligned} \tag{3}$$

with the following conditions:

M) the function $M:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous function and there is a constant $m_0>0$ such that

$$M(t)\geq m_0 \qquad \text{for all} \qquad t\geq 0$$

F) $f(x,t):\overline{\Omega}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and satisfies the subcritical condition

$$|f(x,t)|\leq c_1(|t|^{q-1}+1) \qquad \text{for some} \quad p<q<p^*=\begin{cases}\frac{Np}{N-p} & \text{if } N\geq 3,\\ +\infty & \text{if } N=1,2.\end{cases}$$

H) $H\in C(\mathbb{R})$ satisfying

$$|H(s)|\leq c_2|s|^r, \qquad r\in\langle 1;p-1\rangle.$$

K)

$$k(x,y) \qquad \text{is a non-positive } \; L^p(\Omega\times\Omega)\; function$$

The nonlocal term $\int_\Omega k(x,y)H(u)dy$, with $k=k(x)$ appears in numerous physical models such systems of particles in thermodynamical equilibrium via gravitational (Coulomb) potential, 2-D fully turbulent behavior of real flow, thermal runaway in Ohmic Heating, shear bands in metal deformed under high strain rates, see [24] for references of these applications. The nonlocal boundary conditions in (1) have been studied by Berestycki and Brezis [8] , Ortega [27] and more recently Zhao [31]. They arise from certain models in plasma physics: specifically, a model describing the equilibrium of a plasma confined in a toroidal cavity, called a Tokamak machine. A detailed description of this problem can be found in the appendix of [28]. Semilinear integro-differential equations have become an active area of research, for example in the framework of control theory as well in order to solve noncooperative system,arisen in the classical FitzHugh-Nagumo systems , see e.g. [2, 5, 14, 15, 18, 29]. In case that the kernel k= k(x,y) is symmetric (and $H(s)=s$) , the problem is of variational type and a solution can be found by the Mountain Pass Theorem if the $L^p\times L^p$ norm is sufficiently small,see [6] for $p=2$. Motivated by the above papers and the results in [18, 19, 31], we consider (3) to study the existence of weak solutions, but with non-symmetric kernels, then the problem has no variational structure; so, the most usual variational techniques can not be used to study. To attack problem (3) we will use the Galerkin method through the following version of the Brouwer fixed -point Theorem whose proof may found in Lions (see Lemma 4.3 [23]).

Proposition 1.1.
Suppose that $F:\mathbb{R}^m \to \mathbb{R}^m$ *is a continuous function such that* $\langle F(\xi),\xi\rangle \geq 0$ *on* $|\xi|=r$*, where* $\langle\cdot,\cdot\rangle$ *is the usual inner product in* $\mathbb{R}^m$ *and* $|\cdot|$ *its related norm. Then, there exists* $z_0 \in \overline{B_r}(0)$ *such that* $F(z_0)=0$.

2. Main results and proofs

Let $W^{1,p}(\Omega)=\{u\in L^p(\Omega):\nabla u\in[L^p(\Omega)]^n\}$ be endowed with the norm

$$\|u\|_{1,p}=\|u\|_{W^{1,p}(\Omega)}=\Big(\int_\Omega |\nabla u|^p+|u|^p\,dx\Big)^{1/p}. \tag{3}$$

Let

$$V=\{u\in W^{1,p}(\Omega): u|_{\partial\Omega}=\text{constant}\}.$$

With a straightforward adaptation of Lemma 2.1 [31], it is easy to get the following lemma

Lemma 2.1.
$\big(V,\|.\|_{1,p}\big)$ *is a separable and reflexive Banach space.*

Since we are preoccupied with the existence of weak solutions of problem (3) we begin by giving of such solutions.

Definition 2.1.
A weak solution of problem (3) is any $u\in V$ such that

$$\Big[M\Big(\|u\|^p\Big)\Big]^{p-1}\int_\Omega\Big(|\nabla u|^{p-2}\nabla u.\nabla v+|u|^{p-2}uv\Big)\,dx=\|\,u\,\|_\alpha^t\int_\Omega f(x,u)v\,dx+\int_\Omega\Big(\int_\Omega(k(x,y)H(u(y))\,dy)v\,dx$$

for all $v\in V$.

Our main result is given by the following theorem

Theorem 2.1.
Let us assume that conditions (M)–(F)–(H) and (K) *hold. If* $\|k\|_{L^p(\Omega\times\Omega)}$ *is sufficiently small and the function* f *satisfies*

$$f(x,u)u\leq a|u|^p+b|u|,\qquad ,1<\beta<\min\{p-t,\alpha\}\quad ,1<\alpha<\frac{Np}{N-p} \tag{4}$$

for some constants $a,b>0$ *with* $m_0^{p-1}-a\,c_\alpha^{\beta+t}-|k|_{L^p(\Omega\times\Omega)}c_{rp'}^r c_{p'}c_2>0$, $\frac{1}{p}+\frac{1}{p'}=1$
where c_ξ *is the corresponding embedding Sobolev constant; then problem* (3) *has at least one weak solution. Besides , any solution of* (3) *satisfies the estimate*

$$\|u\|_{1,p}\leq R_1=\max\left\{1,\left(\frac{b|\Omega|^{1/p'}c_\alpha^t}{m_0^{p-1}-a\,c_\alpha^{\beta+t}-|k|_{L^p(\Omega\times\Omega)}c_{rp'}^r c_{p'}c_2}\right)^{1/(p-t-1)}\right\} \tag{5}$$

Proof. Let $\{w_\nu\}_{\nu\geq 1}$ a Schauder's basis for V, which is a separable and reflexive Banach space with the restricted norm of (3) .For each $m\in\mathbb{N}$ consider the finite dimensional space

$$\mathbb{V}_m=\operatorname{span}\{w_1,\dots,w_m\}.$$

Since $(\mathbb{V}_m,\|\cdot\|_{1,p})$ and $(\mathbb{R}^m,|\cdot|)$ are isometric and isomorphic, we make the identification

$$u_m=\sum_{j=1}^m\xi_j w_j\longleftrightarrow\xi=(\xi_1,\dots,\xi_m),\quad \|u\|_{1,p}=|\xi|.$$

We will show that for each m there is $u_m\in\mathbb{V}_m$, an approximate solution of (3), satisfying

$$\Big[M\Big(\|u_m\|_{1,p}^p\Big)\Big]^{p-1}\int_\Omega\Big(|\nabla u_m|^{p-2}\nabla u_m.\nabla w_j+|u_m|^{p-2}u_m w_j\Big)\,dx=\|\,u_m\,\|_\alpha^t\int_\Omega f(x,u_m)w_j\,dx+\int_\Omega\Big(\int_\Omega(k(x,y)H(u_m)dy)w_j\,dx$$

(6)

$j = 1, 2, 3, ..., m$.
To solve this algebraic system we consider the function $F : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$F(\xi) = (F_1(\xi), \ldots, F_m(\xi)),$$
$$F_j(\xi) = \left[M\left(\|u\|^p_{1,p}\right)\right]^{p-1} \int_\Omega \left(|\nabla u|^{p-2}\nabla u.\nabla w_j + |u|^{p-2} u w_j\right) dx - \| u \|^t_\alpha \int_\Omega f(x,u) w_j \, dx -$$
$$\int_\Omega (\int_\Omega (k(x,y) H(u(y)) dy) w_j \, dx$$

where $j = 1, 2, 3, ..., m$. and $u \in V_m$
We note that F is continuous from the continuity of M, $f(x,u)$ with respect to u and $\int_\Omega k(x,y)H(u)dy$.
Therefore, from the hypotheses we have

$$\langle F(u), u\rangle \geq m_0^{p-1}\|u\|^p_{1,p} - a c_\alpha^{\beta+t}\|u\|^{\beta+t}_{1,p} - b c_\alpha^t |\Omega|^{1/q}\|u\|^{t+1}_{1,p} - |k|_{L^p(\Omega\times\Omega)} c^r_{rp'} c_{p'} c_2 \|u\|^{r+1}_{1,p} > 0 \tag{7}$$

if $\|u\|_{1,p} = R$, for R large enough . Thus, because of **Proposition 1** there is $u_m \in \mathbb{V}_m$, $\|u_m\|_{1,p} \leq R$, R does not depend on m such that u_m is solution of (6).
Let us prove that the sequence $(u_m)_{m\geq 1} \subseteq V$ has a convergent subsequence which converges to a solution of (3). Indeed, since (u_m) is bounded there exists a subsequence, still denoted by (u_m), such that

$$\|u\|^p_{1,p} \longrightarrow \gamma . \text{for some } \gamma,$$
$$u_m \rightharpoonup u, \quad \text{in} \quad W^{1,p}(\Omega),$$
$$u_m \longrightarrow u, \quad \text{in} \quad L^q(\Omega),\ 1 \leq q < p^* \tag{8}$$
$$\|u_m\|^t_\alpha \longrightarrow \|u\|^t_\alpha, \tag{9}$$
$$u_m \longrightarrow u, \quad a.e \quad \text{in} \quad \Omega.$$
$$\exists\, g \in L^q(\Omega) \quad : \quad |u_m(x)| \leq g(x) \quad a.e. \quad \text{in} \quad \Omega.$$

In view of continuity of M

$$[M(\|u_m\|^p_{1,p})]^{p-1} \longrightarrow [M(\gamma)]^{p-1} \tag{10}$$

and the continuity of the Nemytskii map

$$f(., u_m) \longrightarrow f(., u) \qquad in \qquad L^q(\Omega). \tag{11}$$

But under the assumption (K) we have

$$|k(x,y)|_p = (\int_\Omega |k(x,y)|^p dy)^{\frac{1}{p}} < +\infty; \qquad i.e, \qquad k(x,y) \in L^p(\Omega)$$

for fixed $x \in \Omega$. Also, noting that $(H(u_m))_{m\geq 1}$ is bounded in $L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain up to subsequence that

$$H(u_m) \rightharpoonup H(u) \text{ in } \quad L^{p'}(\Omega) \tag{12}$$

Therefore, for $x \in \Omega$ we get

$$\int_\Omega k(x,y) H(u_m(y))\, dy \longrightarrow \int_\Omega k(x,y) H(u(y))\, dy, \qquad a.e.$$

Also, we can easily prove that

$$|\int_\Omega k(x,y) H(u_m(y))\, dy|_{L^{p'}(\Omega)} < +\infty.$$

then by [23], Lemma 2.1 we have

$$\int_\Omega k(x,y) H(u_m(y))\, dy \rightharpoonup \int_\Omega k(x,y) H(u(y))\, dy. \tag{13}$$

weakly in $L^{p'}(\Omega)$.
Furthermore, by using the Lebesgue dominated convergence theorem we easily obtain

$$\int_\Omega |u_m|^{p-2} u_m w_j \, dx \longrightarrow \int_\Omega |u|^{p-2} u w_j \, dx \tag{14}$$

From of Theory of Monotone Operators, we get

$$[M(\|u_m\|^p_{1,p})]^{p-1} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m . \nabla w \, dx \longrightarrow [M(\gamma)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla w \, dx \tag{15}$$

$\forall w \in V$.
We now fix $l \leq m$, $V_l \subseteq V_m$, letting $m \longrightarrow +\infty$ in (6) and using (10)–(15), we conclude that

$$[M(\gamma))]^{p-1} \int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla w_l + |u|^{p-2} u w_j \right) dx = \|u\|^t_\alpha \int_\Omega f(x,u) w_l \, dx + \int_\Omega (\int_\Omega k(x,y) H(u) dy) w_l \, dx; , \tag{16}$$

l=1,2,...
From the completeness of $\{w_\nu\}_{\nu \geq 1}$, identity (16) holds with w_l replaced by any $w \in V$.
In particular, when $w = u$ we get

$$[M(\gamma))]^{p-1} \|u\|^p_{1,p} = \|u\|^t_\alpha \int_\Omega f(x,u) u \, dx + \int_\Omega (\int_\Omega k(x,y) H(u) dy) u \, dx \tag{17}$$

On the other hand, taking $w_j = u_m$ in (6) and passing to the limit, we obtain

$$[M(\gamma))]^{p-1} \gamma = \|u\|^t_\alpha \int_\Omega f(x,u) u dx + \int_\Omega (\int_\Omega k(x,y) H(u) dy) u dx \tag{18}$$

and comparing equations (17) and (18) we get

$$[M(\gamma)]^{p-1} \gamma = [M(\gamma)]^{p-1} \|u\|^p_{1,p}$$

Then we conclude $\gamma = \|u\|^p_{1,p}$
This together with (8) , taking into account that V is uniformly convex , yields

$$u_m \longrightarrow u \quad \text{in} \quad W^{1,p}(\Omega)$$

But V is a closed subspace of $W^{1,p}(\Omega)$ and $(u_m) \subseteq V$, hence $u \in V$. Thus, from (16) (with $w_l = w \in V$)

$$M(\|u\|^p_{1,p}))^{p-1} \int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla w + |u|^{p-2} u w \right) dx = \|u\|^t_\alpha \int_\Omega f(x,u) w \, dx + \int_\Omega (\int_\Omega k(x,y) H(u(y)) dy) w \, dx \tag{19}$$

for all $w \in V$, which shows that u is a weak solution of (3).
Finally, if u is any solution of (3),then

$$M(\|u\|^p_{1,p}))^{p-1} \|u\|_{1,p} = \|u\|^t_\alpha \int_\Omega f(x,u) u \, dx + \int_\Omega (\int_\Omega k(x,y) H(u(y)) dy) u \, dx,$$

Therefore, either $\|u\|_{1,p} \leq 1$ or

$$m_0^{p-1} \|u\|^p_{1,p} \leq a c_\alpha^{\beta+t} \|u\|^p_{1,p} + b c^t_\alpha |\Omega|^{1/p'} \|u\|^{t+1}_{1,p} + |k|_{L^p(\Omega \times \Omega)} c^r_{rp'} c_{p'} c_2 \|u\|^{r+1}_{1,p}$$

then

$$\left(m_0^{p-1} - a c_\alpha^{\beta+t} - |k|_{L^p(\Omega \times \Omega)} c^r_{rp'} c_{p'} c_2 \right) \|u\|^p_{1,p} \leq b |\Omega|^{1/p'} c^t_\alpha \|u\|^{t+1}_{1,p}$$

and (5) follows. □

3. Uniqueness of weak solutions

In this section , we are going to consider problem (3) when the exponent p satisfies

$$\frac{2N}{2+N} < p \leq 2, \quad t = 0. \tag{20}$$

To establish the uniqueness of weak solutions,we need the following well known Lemma

Lemma 3.1.
If $p \in]1,2]$,then it hold

i) $||z|^{p-2}z - |y|^{p-2}y| \leq \beta |z-y|^{p-1}$

ii) $\langle |z|^{p-2}z - |y|^{p-2}y|, z-y\rangle \geq (p-1)|z-y|^2\left(|z|^p + |y|^p\right)^{\frac{p-2}{p}}$

for all $y, z \in \mathbb{R}^N$ with β independent of y and z.

Theorem 3.1.
Let the assumptions of Theorem 2.1 hold with (4) replaced by

$$\big(f(x,u) - f(x,v)\big)(u-v) \leq 0 \quad \forall\ x \in \Omega; \forall\ u, v \in \mathbb{R} \tag{21}$$

Let us in addition that M is Lipschitz on $[0, R_1^p]$ where R_1 is defined in (5) and H is a C^1-function such that $|H'(s)| \leq c_3|s|^{r-1}$, $c_3 > 0$. Then if the Lipschitz constant L_M of M is small enough, problem (3) has exactly one solution.

Proof. We will follow some ideas in [1],adapted to our case.
The part of existence follows from Theorem 2.1. Now, let u_1 and u_2 be two solutions to problem . Introduce the function $u = u_1 - u_2$. Taking it for the test-function in the integral identities for u_1 and u_2, we obtain the relation

$$\begin{aligned}
&\left[M\left(\|u_1\|_{1,p}^p\right)\right]^{p-1}\int_\Omega \left[\left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)(\nabla u_1 - \nabla u_2) + \left(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\right)(u_1-u_2)\right]dx = \\
&\int_\Omega (f(x,u_1) - f(x,u_2))(u_1-u_2)\,dx + \int_\Omega\int_\Omega k(x,y)(H(u_1(y)) - H(u_2(y)))(u_1-u_2)\,dy\,dx \\
&+ \left\{\left[M\left(\|u_2\|_{1,p}^p\right)\right]^{p-1} - \left[M\left(\|u_1\|_{1,p}^p\right)\right]^{p-1}\right\}\int_\Omega \left[|\nabla u_2|^{p-2}\nabla u_2(\nabla u_1 - \nabla u_2) + |u_2|^{p-2}u_2(u_1-u_2)\right]dx
\end{aligned}$$

Now, using the hypotheses on M, H, K and Lemma 3.1, after some calculations we have

$$\begin{aligned}
&m_0^{p-1}(p-1)\int_\Omega \left[|\nabla u|^2\left(|\nabla u_1|^p + |\nabla u_2|^p\right)^{\frac{p-2}{p}} + |u|^2\left(|u_1|^p + |u_2|^p\right)^{\frac{p-2}{p}}\right]dx \quad \leq \\
&c_3\int_\Omega\int_\Omega |k(x,y)|\left(|u_1|^{r-1} + |u_2|^{r-1}\right)|u|^2\,dy\,dx + \beta L_M^{p-1} p\left[(\|u_1\|_{1,p} + \|u_2\|_{1,p})\|u_2\|_{1,p}\right]^{p-1}\|u\|_{1,p}^2
\end{aligned} \tag{22}$$

where $\beta = \max_{s \in [0, R_1^p]} M(s)$. It follows from Holder's inequality and the Sobolev immersions that

$$\left|\int_\Omega\int_\Omega |k(x,y)||u_1|^{r-1}|u|^2\,dy\,dx\right| \leq \|k\|_{L^p}|u_1|_{(r-1)p'}^{r-1}|u|_{2p'}^2 \leq c_{(r-1)p'}^{r-1}c_{2p'}^2\|k\|_{L^p}\|u_1\|_{1,p}^{r-1}\|u\|_{1,p}^2$$

with $\frac{1}{2p'} \geq \frac{1}{p} - \frac{1}{N}$. Similar inequality is obtained for u_2.
Let us take a constant $q \in \left(\frac{p}{2}, 1\right) \subseteq \left(\frac{2N}{2+N}, 1\right)$. Using the inverse Holder's inequality in (22) , we obtain

$$\begin{aligned}
&m_0^{p-1}(p-1)\Big[\Big(\int_\Omega |\nabla u|^{2q}\,dx\Big)^{\frac{1}{q}}\Big(\int_\Omega \left(|\nabla u_1|^p + |\nabla u_2|^p\right)^{\frac{2-p}{p}\cdot\frac{q}{1-q}}dx\Big)^{\frac{q-1}{q}} \\
&+\Big(\int_\Omega |u|^{2q}\,dx\Big)^{\frac{1}{q}}\Big(\int_\Omega \left(|u_1|^p + |u_2|^p\right)^{\frac{2-p}{p}\cdot\frac{q}{1-q}}dx\Big)^{\frac{q-1}{q}}\Big] \leq \left(2c_3c_{(r-1)p'}^{r-1}c_{2p'}^2\|k\|_{L^p}R_1^{r-1} + 2^{2p-3}R_1^{2p-2}C_\beta L_M m_1^{p-2}\right)\|u\|_{1,p}^2
\end{aligned} \tag{23}$$

Since $p < 2q < 2$, again using Holder's inequality , the Sobolev embedding $W^{1,2q} \hookrightarrow W^{1,p}$, $L^{2q} \hookrightarrow L^p$ and noting that $\frac{2-p}{p}\cdot\frac{q}{1-q} \leq 1$, we obtain

$$\underbrace{\left(m_0^{p-1}(p-1)C_{2q}^{-2}|\Omega|^{\frac{1}{\theta}}2^{\frac{p-2}{2}}R_1^{p-2}\right)}_{e_0}\|u\|_{1,p}^2 \quad \leq \quad \underbrace{\left(2c_3c_{(r-1)p'}^{r-1}c_{2p'}^2\|k\|_{L^p}R_1^{r-1} + 2^{2p-3}R_1^{2p-2}C_\beta L_M m_1^{p-2}\right)}_{e_1}\|u\|_{1,p}^2 \tag{24}$$

with $\theta = 1 - \frac{(2-p)q}{p(1-q)}$. Hence it follows that

$$(e_0 - e_1)\|u\|_{1,p}^2 = 0 \tag{25}$$

Therefore, if $\|k\|_{L^p}$ and L_M are small enough, we conclude that $\|u\|_{1,p} = 0$, and so $u = 0$. □

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