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A curvature-unified equation for a non-Newtonian power-law fluid flow

Research Article

Samer Al-Ashhab *

Department of Mathematics, Al Imam Mohammad Ibn Saud Islamic University, P.O. Box 90950, Riyadh 11623, Saudi Arabia

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Abstract: A unified equation is derived for both positive and negative curvature (similarity) solutions of a power-law problem that has recently been studied, and for which existence and uniqueness have been established for solutions that do not change the sign of curvature. The derivation is based on a Crocco variable formulation, where a value of a parameter in the equation determines the sign of curvature. The asymptotic behavior of solutions is discussed and a power series solution to this unified problem is obtained. Using the derived power series to estimate the shear stress parameter is also discussed.

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1. Introduction

Due to its many applications and interesting mathematical properties, the problem of boundary-layer flow of non-Newtonian fluids has, in recent years, attracted much attention from many researchers in many fields. This so-called power-law problem has established its presence in many applications in industry and engineering. The most commonly used model in the context of non-Newtonian fluid mechanics is the Ostwald-de Waele model with a power-law rheology, which is characterized by a power-law index n. The value $n = 1$ corresponds to a Newtonian fluid, while $n > 1$ describes a dilatant or shear-thickening fluid and $0 < n < 1$ describes a pseudo-plastic or shear-thinning fluid.

To give a brief overview of existing literature a few of those who worked on the problem and contributed to it are mentioned here: Howell et al. [1] studied the problem in the context of momentum and heat transfer on a continuously moving surface in the power law fluid. Nachman and Talliafero [2] established existence and uniqueness for a mass transfer singular boundary value problem using a Crocco variable formulation. More recently, however, existence and uniqueness were studied by Guedda and Hammouch [3, 4] among others. In [5] and [6] solutions and analytic solutions to the problem were studied. Computational approaches and numerical solutions can be found in [7] and [8] to mention a few. Similarity solutions for non-Newtonian fluids have also been studied for different physical applications in porous media, see [9] for example. Background material and more on the mathematical formulation and aspects as well as the physical properties of the problem can be found in [10], [11], and [12].
The power-law problem has many variations with somewhat different settings (different initial/boundary conditions and also slightly different governing equations). In [13] and [14] Wei and Al-Ashhab applied a certain relatively novel approach to the power-law problem with a novel boundary condition, where existence and uniqueness

* Corresponding author.
E-mail address: ssashhab@imamu.edu.sa

of solutions were established for both positive and negative curvature cases. The proof utilized the use of the Picard-Lindelöf theorem applied to a transformed equation with transformed conditions, where in fact, the eventual equation took slightly different forms depending on the sign of the curvature of solutions. In this paper a unified equation utilizing generalized Crocco variables is derived for both positive and negative curvature cases, where the value of a parameter in the equation determines the sign of curvature.

An infinite power series solution to this unified problem is established using the Adomian decomposition method. This series solution turns out to be useful in determining the value of the shear stress parameter (a parameter that corresponds to an initial value, which in fact is determined so that the equation satisfies a boundary condition at the other end of the domain of the Crocco variables, i.e., by a shooting method).

2. The problem

The model used for the problem at hand is the Ostwald-de Waele model with a power-law rheology, where the relationship between the shear stress τ_{xy} and the strain rate $u_y = \frac{\partial u}{\partial y}$ is governed by

$$\tau_{xy} = k(|u_y|^{n-1}u_y). \tag{1}$$

The shear stress τ_{xy} here is a component of the stress tensor, however a discussion of the stress tensor is beyond the scope of this paper. The physical problem itself is defined by a two dimensional incompressible non-Newtonian steady-state laminar fluid flow on a semi-infinite plate. This flow is governed by equation (1). Let the x-direction be parallel to the plate and the y-direction be perpendicular to it. The boundary layer governing equations are the continuity and momentum equations as follows:

$$u_x + v_y = 0, \tag{2}$$

$$u u_x + v u_y = \nu(|u_y|^{n-1}u_y)_y, \tag{3}$$

where u and v are the velocity components in the x and y directions respectively. We note that the right hand side of equation (3) is indeed equivalent to the rate of change of the shear stress with respect to the spatial coordinate y divided by the density of the fluid ρ, and thus can be expressed as $\frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}$, where in this context $\nu = \frac{k}{\rho}$. The boundary conditions are:

$$u(x,0) = U_w(x) = u_w x^m, \quad v(x,0) = 0, \quad u(x,y) \to C y^{\sigma} \text{ as } y \to \infty. \tag{4}$$

Now let $\psi = \psi(x,y)$ be a function that satisfies $u = \psi_y, v = -\psi_x$ (ψ here is referred to as the stream function). This transforms problem (2−4) into:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu(|\psi_{yy}|^{n-1}\psi_{yy})_y, \tag{5}$$

with conditions

$$\psi_y(x,0) = u_w x^m, \quad \psi_x(x,0) = 0, \quad \psi_y(x,0) \to C y^{\sigma} \text{ as } y \to \infty. \tag{6}$$

Introduce a function f and parameter η via:

$$\psi(x,y) = A x^{\alpha} f(\eta), \quad \eta = B\frac{y}{x^{\beta}}. \tag{7}$$

This is called a similarity transformation where f is referred to as the dimensionless stream function and η is the similarity variable. Substituting (7) into (5) yields the following ordinary differential equation (primes denote differentiation with respect to the similarity variable η):

$$(|f''|^{n-1}f'')' + \alpha f f'' = (\alpha - \beta) f'^2, \tag{8}$$

if and only if the following conditions are satisfied

$$\alpha(n-1) + \beta(2n-1) = 1, \quad \alpha - \beta = m,$$

and where in this context one also assumes $\nu A^{n-1}B^{2n-1}=1$. The inverse relationships determining α and β in terms of m and n are:

$$\alpha=\frac{1+m(2n-1)}{n+1},\quad \beta=\frac{1+m(n-2)}{n+1}.$$

The boundary conditions are then transformed to:

$$f(0)=0,\quad f'(0)=\frac{u_w}{A},\quad f'(\infty)=\lim_{\eta\to\infty}f'(\eta)=C\eta^{\sigma}$$

This power-law problem has, in fact, been studied by many (but not in its full generality). Some authors set some boundary conditions to zero. Others fixed the values of some of the parameters a priori, or considered certain values/ranges of the power-law index n. On the other hand however, some authors chose a non-zero condition for $v(x,0)$ (and consequently for $f(0)$) while setting other parameters to zero, ...etc. It is worthwhile noting here that for the special case of Newtonian fluid $n=1$ one obtains the Blasius equation [15] (which was studied with $m=u_w=\sigma=0, C=1$). In [13] and [14] the authors established existence and uniqueness for all values of the power-law index $n>0$ provided the curvature of the solution does not change sign, for the case $m=0,\sigma=0$, and $C=1$. Within that context: $\alpha=\beta=\frac{1}{n+1}$ and by setting $\epsilon=\frac{u_w}{A}$ the power-law problem takes the form:

$$(|f''|^{n-1}f'')'+\frac{1}{n+1}ff''=0 \tag{9}$$

subject to

$$f(0)=0,\quad f'(0)=\epsilon,\quad f'(\infty)=1 \tag{10}$$

By virtue of the results in [13] and [14], it becomes natural to seek to represent solutions as an infinite power series (which is done here in the so-called Crocco variables), touch on some asymptotic behavior of solutions, and propose a method to numerically estimate the shear stress parameter (a physical quantity of crucial importance).

3. Asymptotic behavior and a unified equation

Observe that for the case of negative curvature solutions ($f''<0$) to problem ($9-10$) one must have $\epsilon>1$, and a Crocco variable transformation is applied to this problem with the variables:

$$z=f'(\eta),\quad h(z)=(-f''(\eta))^n,$$

which leads to the problem consisting of the equation

$$h''(z)=-zh^{-1/n}(z),\quad 1<z<\epsilon,$$

subject to

$$h(1)=0,\quad h'(\epsilon)=0.$$

To discuss the asymptotic behavior of f' (and consequently f) as $\eta\to\infty$ for the case of dilatant fluids $0<n<1$: Let $h(z)$ be represented by $h(z)\approx k(z-1)^p$ for z close to 1 ($z>1$), and for some parameters k and p. It can then be established that

$$p=\frac{2n}{n+1},\quad k=(p(1-p))^{\frac{n}{n+1}}.$$

Substituting back the values of z and h in terms of the Crocco variables (derivatives of f as given above) and integrating yields

$$f'-1\approx\left(k^{\frac{1}{n}}\frac{1-n}{1+n}\eta+K\right)^{\frac{n+1}{n-1}}$$

for large η and where K is a constant. This implies the following asymptotic behavior: the deviation of f' from $f'=1$ is of the form

$$f'-1\to c\cdot\eta^{\frac{n+1}{n-1}}\quad \text{as } \eta\to\infty \tag{11}$$

for $0<n<1$ and for some constant $c>0$. In fact, by virtue of (11) we have the following result: The integral $\displaystyle\int_{\eta_0}^{\infty}(f(\eta)-\eta)d\eta$ converges for $1/3<n<1$ while the same integral diverges for $0<n\le 1/3$ (interestingly the proof of existence and uniqueness in [13] was split into two cases: a case for $0<n\le 1/3$ and another one for $1/3<n<1$).

Now apply the following transformation (which consists of a horizontal reflection and axis rescaling):

$$x=\frac{\epsilon-z}{\epsilon-1},\quad y(x)=h(z).$$

Consequently, y will satisfy the equation

$$y''=-\frac{(\epsilon-1)^3}{n(n+1)}\Big(\frac{\epsilon}{\epsilon-1}-x\Big)y^{-1/n},\quad 0<x<1, \tag{12}$$

subject to

$$y'(0)=0,\quad y(1)=0. \tag{13}$$

Next, consider the positive curvature solutions (for which $f''>0$) to problem (9−10), one must have $\epsilon<1$. In this case, the Crocco variables are

$$z=f'(\eta),\quad h(z)=(f''(\eta))^n.$$

With this set of Crocco variables, problem (9—10) is in fact transformed to the *same problem* as for negative curvature, namely $h''(z)=-zh^{-1/n}(z)$ subject to $h(1)=0, h'(\epsilon)=0$ *but now for* $\epsilon<z<1$.
On the other hand, similar analysis to the one above will yield the following asymptotic behavior: $1-f'\to c\cdot\eta^{\frac{n+1}{n-1}}$ for $0<n<1$ and for some constant $c>0$ as $\eta\to\infty$.
Now apply the following transformation (axis rescaling): $x=1-\frac{1-z}{1-\epsilon}=\frac{z-\epsilon}{1-\epsilon},\quad y(x)=h(z)$, and notice that y satisfies $y''=-\frac{(\epsilon-1)^3}{n(n+1)}(\frac{\epsilon}{\epsilon-1}-x)y^{-1/n}$ for $0<x<1$, subject to $y'(0)=0, y(1)=0$. In fact, this is the same problem (12−13) as before. To simplify this problem let

$$k=\frac{\epsilon}{\epsilon-1},\quad c=-\frac{(\epsilon-1)^3}{n(n+1)}. \tag{14}$$

This produces the problem consisting of the equation:

$$y''=c(k-x)y^{-1/n},\quad 0<x<1, \tag{15}$$

subject to

$$y'(0)=0,\quad y(1)=0. \tag{16}$$

Equation (15) is referred to here as a unified equation for both the positive and negative curvature cases. Observe that $1<k<\infty$ for negative curvature, while $-\infty<k<1$ for positive curvature. Finally notice that the case $k=1$ is not possible (it may only be viewed as a limiting case of the parameter k as $\epsilon\to\pm\infty$).

4. The series solution

The solution to problem (15−16) can be represented as an infinite power series. Eq. (15) is singular at $y=0 (x=1)$, therefore it is sought to represent the solution using a power series expanded at $x=0$ (which in fact turns out to be convergent at $x=1$). To this end a shooting method is used where the condition $y(1)=0$ in (16) above, is replaced by a condition $y(0)=\gamma$, which in turn should satisfy $y(1)=0$. This series turns out to be useful in the calculation/estimation of the shear stress parameter: $\gamma=h(\epsilon)=(\pm f''(0))^n$ where the positive sign is for the positive curvature case, and the negative sign is for the negative curvature case.
To find the power series solution to problem (15—16) one can utilize the Adomian decomposition method [16] which turns out to be convenient in this context. First notice that the solution y satisfies

$$y=\gamma+L^{-1}(c(k-x)y^{-1/n}) \tag{17}$$

where $L^{-1}=\int_0^x\int_0^x(\cdot)dxdx$ is an operator. Now expand y in an infinite series of the form $y=\sum_{i=0}^{\infty}y_i$. The decomposition method is based on substituting this series for y into (17) above and extracting the terms out one by one by a linearization process of the operator. In particular, the y_i's are obtained via

$$y_i=L^{-1}(A_{i-1}), i=1,2,3,\cdots \tag{18}$$

where the A_i's are referred to as the Adomian polynomials and are given by (the first four are included here but note that the rest of them can be found using the same process):

$$A_0 = \frac{c(k-x)}{y_0^{1/n}};$$

$$A_1 = -\frac{1}{n}\frac{c(k-x)y_1}{y_0^{(1/n)+1}},$$

$$A_2 = \frac{1}{2}\frac{1}{n}(1+\frac{1}{n})\frac{(k-x)y_1^2}{y_0^{(1/n)+2}} - \frac{1}{n}\frac{c(k-x)y_2}{y_0^{(1/n)+1}},$$

$$A_3 = -\frac{1}{6}\frac{1}{n}(1+\frac{1}{n})(2+\frac{1}{n})\frac{c(k-x)y_1^3}{y_0^{(1/n)+3}} + \frac{1}{n}(1+\frac{1}{n})\frac{c(k-x)y_1y_2}{y_0^{(1/n)+2}} - \frac{1}{n}\frac{c(k-x)y_3}{y_0^{(1/n)+1}}.$$

Solving for the y_i's recursively using (18) and with $y_0 = \gamma$ yields:

$$y_1 = \frac{c}{\gamma^{1/n}}(-\frac{1}{6}x^3 + \frac{k}{2}x^2),$$

$$y_2 = \frac{-c^2}{n\gamma^{1+(2/n)}}(\frac{1}{180}x^6 - \frac{k}{30}x^5 + \frac{k^2}{24}x^4),$$

$$y_3 = \frac{c^3(n+1)}{2n^2\gamma^{2+(3/n)}}(-\frac{1}{2592}x^9 + \frac{k}{288}x^8 - \frac{5k^2}{504}x^7 + \frac{k^3}{120}x^6) + \frac{c^3}{n^2\gamma^{2+(3/n)}}(-\frac{1}{12960}x^9 + \frac{k}{1440}x^8 - \frac{k^2}{560}x^7 + \frac{k^3}{720}x^6),$$

$$\begin{aligned} y_4 = &-\frac{c^4(n+1)(2n+1)}{6n^3\gamma^{3+(4/n)}}(\frac{1}{28512}x^{12} - \frac{k}{2376}x^{11} + \frac{k^2}{540}x^{10} - \frac{k^3}{288}x^9 - \frac{k^4}{448}x^8) \\ &-\frac{c^4(n+1)}{n^3\gamma^{3+(4/n)}}(\frac{1}{142560}x^{12} - \frac{k}{11880}x^{11} + \frac{23k^2}{64800}x^{10} - \frac{k^3}{1620}x^9 + \frac{k^4}{2688}x^8) \\ &-\frac{c^4(n+1)}{2n^3\gamma^{3+(4/n)}}(\frac{1}{370656}x^{12} - \frac{k}{28512}x^{11} + \frac{k^2}{6720}x^{10} - \frac{23k^3}{90720}x^9 + \frac{k^4}{6720}x^8) \\ &-\frac{c^4}{n^3\gamma^{3+(4/n)}}(\frac{1}{1710720}x^{12} - \frac{k}{142560}x^{11} + \frac{k^2}{36288}x^{10} - \frac{k^3}{22680}x^9 + \frac{k^4}{40320}x^8). \end{aligned}$$

The theoretical solution for $h(z)$ is then given by

$$h(z) = \sum_{i=0}^{\infty} y_i(\frac{\epsilon - z}{\epsilon - 1})$$

and this works for both negative and positive curvature cases ($\epsilon > 1$ for negative curvature and $\epsilon < 1$ for positive curvature). On the other hand, if the series solution is determined upto an order l, one can write an approximate solution for $h(z)$ via $h(z) \approx \sum_{i=0}^{l} y_i(\frac{\epsilon - z}{\epsilon - 1})$ where solving the algebraic equation: $(y(1) \approx) \sum_{i=0}^{l} y_i(1) = 0$ yields an estimate for the shear stress parameter γ. (In fact, that the algebraic equation given above has order $2l$ for Newtonian fluids, or more precisely it is an equation of order l in the variable $\gamma^{1+1/n}$.) Observe that using the values of the constants k and c from (14) one should expect better accuracy in the calculation of γ for values of ϵ closer to 1 (the case $\epsilon = 1$ represents the zero curvature solution for which $\gamma = 0$, so this is consistent with the geometry of the problem) and one should also expect better accuracy for larger values of n. In fact, one can anticipate high accuracy in the calculation of γ for a wide range of ϵ and γ when using upto the fourth polynomial in the series.

5. Conclusions

It is rather interesting that the Crocco variable transformation, coupled by some more classical transformations, produced the same equation and boundary conditions for both positive and negative curvature cases for the power-law problem at hand. This produces convenience in studying and analyzing the problem (as was shown for example in generating a series solution of the problem that works for both cases). Negative and positive curvature cases may have different properties and finding a correspondence from one problem to the other may not always be trivial, but a Crocco variable transformation unified the two problems into one problem.

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