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New approaches for Taylor and Padé approximations

Research Article

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Abstract: In the following paper we shall consider the case where we want to represent the functions that expanded by Taylor series. We compared four approximation methods: Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA).

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1. Introduction

The relation between the coefficients of the Taylor series expansion of a function and the values of the function is both a profound mathematical question and an important practical one. It is basic to the study of mathematical analysis, and to the practical calculation of mathematical models of nature throughout much of applied sciences and engineering problems, for example, electrical and dynamic System's transfer function, biology, physics, chemistry, medicine. Padé approximant [1–3] is a type rational function of given order, under this technique, the approximant's power series agrees with the power series of the function it is approximating. The technique was developed around 1890 by Henri Padé, but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series. This technique better than rational function [4]. Padé approximant was introduced by Brezinski [5], Khan [6] used Padé âĂŞHermite approximant, the convergence of multipoint Padé -type approximants was shown by Ysern and Lagomasino [7].

Restrictive Taylor approximation introduced by Ismail and Elbarbary [8] to solve Parabolic Partial Differential Equations, Ismail et al. [9, 10]approximated a solution for Convection âĂŞ Diffusion equation and KdV-Burgers equation and Rageh et al. [11] used RTA to solve Gardner and KdV equations this done by modifications on finite difference operator, the differintegral operator was introduced by Salman and Joudah [12] also Maheshwari and Sharma [13] establish rate of convergence for for some linear positive operators for functions having derivatives of bounded variation.
Ismail et al. [14–16] applied Restrictive Padé approximation to solve Schrodinger equation and Generalized Burger's equation. Ismail [17]study the convergence of (RPA) to the exact solutions of IBVP of parabolic and hyperbolic types. Ismail [18] introduced solvability and uniqueness for both (RTA) and (RPA).

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2. Taylor approximation

Taylor series is a polynomial of infinite terms.
Thus,

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots \tag{1}$$

It is, of course, impossible to evaluate an infinite number of items. The Taylor polynomial of degree n is defined by

$$f(x) = P_n(x) + R_{n+1}(x) \tag{2}$$

where the Taylor polynomial $P_n(x)$ and the remainder term $R_{n+1}(x)$ are given by

$$P_n(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^n(x_0) \tag{3}$$

$$R_{n+1}(x) = \frac{(x-x_0)^{n+1}}{n+1!} f^{n+1}(\xi), \qquad x_0 \le \xi \le x \tag{4}$$

The Taylor polynomial is a truncated Taylor series, with an explicit remainder, or error term. The Taylor polynomial cannot be used as an approximating function for discrete data because the derivatives required in the coefficients cannot be determined.

3. Padé approximation

The Padé approximants are a particular type of rational fraction approximation to the value of the function. The idea is to match the Taylor series expansion as far as possible [1–3]. The Padé approximant often gives better approximation of the function than truncating its Taylor series and it may still work where the Taylor series does not converge. The Electrical and Dynamic System's transfer function have the following structure

$$H(s) = \frac{A(s)}{B(s)} \tag{5}$$

for such kind of function, Padé approximation is simple and often better.
We knew that Taylor at $x_0 = 0$ (Maclanrin) so if we have

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots = \sum_{i=0}^{\infty} c_i x^i \tag{6}$$

The Padé approximation can be written in the form

$$PA[M/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i}{1 + \sum_{i=1}^{N} b_i x^i} \tag{7}$$

where M and N are positive integers . There are $M+1$ independent numerator coefficients and N denominator coefficients making $M+N+1$ unknown coefficients [1].
The $M+N+1$ unknown suggests that normally the $PA[M/N]$ ought to fit the power series $f(x) = \sum_{i=0}^{\infty} c_i x^i$.

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = PA[M/N]_{f(x)}(x) + O(x^{M+N+1}) \tag{8}$$

$$\sum_{i=0}^{\infty} c_i x^i = PA[M/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i}{1 + \sum_{i=1}^{N} b_i x^i} + O(x^{M+N+1}) \tag{9}$$

Returning to Eq. (9) and cross-multiplying we find that

$$(c_0+c_1x+c_2x^2+...)(1+b_1x+b_2x^2+...+b_nx^N)=(a_0+a_1x+a_2x^2+...+b_mx^M)+O(x^{M+N+1}) \tag{10}$$

Equating the coefficients of $x^{M+1}, x^{M+2}, x^{M+3},, x^{M+N}$

$$\begin{aligned} &b_N c_{M-N+1}+b_{N-1}c_{M-N+2}+...+c_{N+1}=0\\ &b_N c_{M-N+2}+b_{N-1}c_{M-N+3}+...+c_{N+2}=0\\ &\quad\vdots\\ &b_N c_N \;+ b_{N-1}c_{M+1}\;+...+c_{N+M}\;=0 \end{aligned} \tag{11}$$

$c_j=0$ if $j<0$ so Eq. (11) become a set of N linear equations for the N unknown denominator coefficients [1, 4]:

$$\begin{bmatrix} c_{M-N+1} & c_{M-N+2} & c_{M-N+3} & \cdots & c_N\\ c_{M-N+2} & c_{M-N+3} & c_{M-N+4} & \cdots & c_{N+1}\\ c_{M-N+3} & c_{M-N+4} & c_{M-N+5} & \cdots & c_{N+2}\\ \vdots & \vdots & \vdots & \dots & \dots\\ c_N & c_{N+1} & c_{N+2} & \cdots & c_{M+N-1} \end{bmatrix} \times \begin{bmatrix} b_N\\ b_{N-1}\\ b_{N-2}\\ \vdots\\ b_1 \end{bmatrix} = \begin{bmatrix} c_{N+1}\\ c_{N+2}\\ c_{N+3}\\ \vdots\\ c_{N+M} \end{bmatrix} \tag{12}$$

By solving the system of linear Eqs. (12) we may find b_i

The numerator coefficients $a_0, a_1, a_2, ...$ follow immediately from Eq. (10) by equating the coefficients of $1, x, x^2, x^3, x^4, ..., x^M$

$$\begin{aligned} &a_0=c_0\\ &a_1=c_1+b_1c_0\\ &a_2=c_2+b_1c_1+b_2c_0\\ &\vdots\\ &a_M=c_M+\sum_{i=1}^{M} b_i\, c_{M-i} \end{aligned} \tag{13}$$

It is common practice to display the approximates in a table; it is called the Padé table [1–3].as shown in Table 1.
If we take $f(x)=e^x$
where its Maclaurin expansion is
$f(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...=c_0+c_1x+c_2x^2+...$

From Eq. (11)

$$\begin{aligned} &c_0=a_0 &&\longrightarrow\quad a_0=1\\ &c_1+c_0+b_1=a_1 &&\longrightarrow\quad 1+b_1=a_1\\ &c_2+c_1+b_1+c_0+b_2=0 &&\longrightarrow\quad 1/2+b_1=0\\ &b_1=-\tfrac{1}{2}\quad a_1=\tfrac{1}{2}\\ &PA[1/1]_{f(x)}(x)=\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x} \end{aligned}$$

Using Mathematica package we can find $PA[M/N]_{f(x)}(x)$ for any +ve values of M/N as constructed in Table 2.

Table 1. Padé Table

M/N	0	1	2	...	N
0	[0/0]	[0/1]	[0/2]	....	[0/N]
1	[1/0]	[1/1]	...	...	[1/N]
2	...	...	...	...	...
⋮	...	...	...	...	...
M	[M/0]	[M/1]	...	...	[M/N]

In Table 2 we can easily note that the first column represents Maclaurin expansion for Exp(x)

4. Restrictive type approximations

In the following pages, we describe two forms for âĂIJRestrictiveâĂİ type approximations in which we find the restrictive parameter that gives the exact solution in some type of problems.

Table 2. Padé Table for $Exp(x)$

M/N	0	1	2	...	N
0	1	$\frac{1}{1-x}$	$\frac{1}{1-x+\frac{1}{2}x^2}$	....	[0/N]
1	$1+x$	$\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}$	...	...	[1/N]
2	$1+x+\frac{1}{2}x^2$	...	...	...	[2/N]
3	$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$	...	...	...	...
...	...	...	...	...	...
M	[M/0]	[M/1]	...	...	[M/N]

4.1. Restrictive taylor approximation (RTA)

Consider a function $f(x)$ defined in a neighborhood of $x=a$ and it has derivatives up to order $(n+1)$ Constructing a function

$$RT_{n,f(x)}(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\ldots+\frac{(x-a)^{n-1}}{n-1!}f^{n-1}(a)+\frac{\varepsilon(x-a)^n}{n!}f^n(a) \tag{14}$$

ε is a parameter to be determined by adding the following condition [9–11]:

$$RT_{n,f}(x_a)=f(x_a)$$

Some points x_a in the domain of the function f. The function $RT_{n,f}(x)$ is called restrictive Taylor approximation of order n of the function $f(x)$ at the point $x=a$. The following theorem gives the value of the reminder term of this approximation.

Assume the function $f(x)$ and its derivatives up to an order $n+1$ are continuous in a certain neighborhood of a point. Suppose, furthermore, that x is any value of the argument from the indicated neighborhood and ε is the restrictive Taylor parameter, then there is a point ξ which lies between the points a and x such that the formula:

$$f(x)=RT_{n,f}(x)+\Re_{n+1}(x,\varepsilon(x)) \tag{15}$$

is true, for which

$$\Re_{n+1}(x,\varepsilon(x))=\frac{\varepsilon(x-a)^{n+1}}{n+1!}f^{(n+1)}(\xi)-\frac{n(\varepsilon-1)^{n+1}(x-a)^{n+1}}{(n+1)!(x-\xi)}f^{(n)}(\xi) \qquad \xi\in[a,x] \tag{16}$$

where $\Re_{n+1}(x,1)$ is the Taylor reminder term.

4.2. Restrictive Padé approximation (RPA)

We construct a restrictive type of Padé approximation[14, 15] of the function $f(x)$ with parameter to be determined, which if it reduces to zero, we will get the classical Padé approximation. The restrictive Padé approximation is a rational function in the form:

$$RPA[M+\alpha/N]_{f(x)}(x)=\frac{\sum_{i=0}^{M}a_i x^i+\sum_{i=1}^{\alpha}\varepsilon_i x^{M+i}}{1+\sum_{i=1}^{N}b_i x^i} \tag{17}$$

where the positive integer α does not exceed the degree of the numerator,

$$\alpha=0(1)n$$

Such that

$$f(x)=RPA[M+\alpha/N]_{f(x)}(x)+O(x^{M+N+1}) \tag{18}$$

and let $f(x)$ has a Maclaurin series expansion.

Table 3. Restrictive Padé table

$[\alpha/0]$	$[\alpha/1]$	$[\alpha/2]$	...	$[\alpha/N]$	...
$[\alpha+1/0]$	$[\alpha+1/1]$	$[\alpha+1/2]$	...	$[\alpha+1/N]$	...
$[\alpha+2/0]$	$[\alpha+2/1]$	$[\alpha+2/2]$	...	$[\alpha+2/N]$	...
$[\alpha+3/0]$	...	...	...	...	...
...	...	...	...	...	...
$[\alpha+M/0]$	$[\alpha+M/1]$	$[\alpha+M/2]$	...	$[\alpha+M/N]$	...
$\vdots$	...	...	...	...	...

In Table 3 we represent the Restrictive Padé, It is called the restrictive Padé table for $f(x)$, the first j_{th} columns disappear when $\alpha > j-1$. the case $\alpha = 0$ gives the classical Padé Table 1.

The case $\alpha = 1$ gives the following selected elements of RPA table:

$$RPA[1/1]_{f(x)}(x) = \frac{a_0 + a_1 x}{1 + b_1 x}$$

where

$a_0 = c_0$
$a_1 = \varepsilon_1$
$b_1 = \frac{\varepsilon_1 - c_1}{c_0}$

$$RPA[2/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x}$$

where

$a_0 = c_0$
$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$
$a_2 = \varepsilon_1$
$b_1 = \frac{\varepsilon_1 - c_2}{c_1}$

$$RPA[3/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x}$$

where

$a_0 = c_0$
$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$
$a_2 = \varepsilon_1 + \frac{c_1(\varepsilon_1 - c_3)}{c_2}$
$a_3 = \varepsilon_1$
$b_1 = \frac{\varepsilon_1 - c_3}{c_2}$

5. Solvability and uniqueness

Ismail [18] introduced two theorems to find the stability and uniqueness conditions for both restrictive Taylor restrictive Padé approximations

Theorem 5.1.
Restrictive Taylor Approximation (14) is uniquely solvable if:
$\forall i, j = 1(1)\alpha; \quad x_i \neq 0, \quad x_i \neq x_j \quad \forall i \neq j$

Theorem 5.2.
The restrictive Padé approximation (17) for the sufficiently smooth function $f(x)$ is uniquely solvable for $\prod_{i=0}^{\alpha} x_i \neq 0$ if the following the $N \times N$ restrictive determinant
$|D| = \Delta \neq 0$

the proof is done in [18].

6. Numerical examples

In this section we introduce three examples solved by Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA).we graph both expansion and error.

Example 1

$$f(x) = \frac{c_1}{1 + c_2 x^2} \tag{19}$$

For $c_1 = c_2 = 1$ and the expansion domain [-1, 1] as shown in Fig. 1. Also the errors are in Fig. 2.

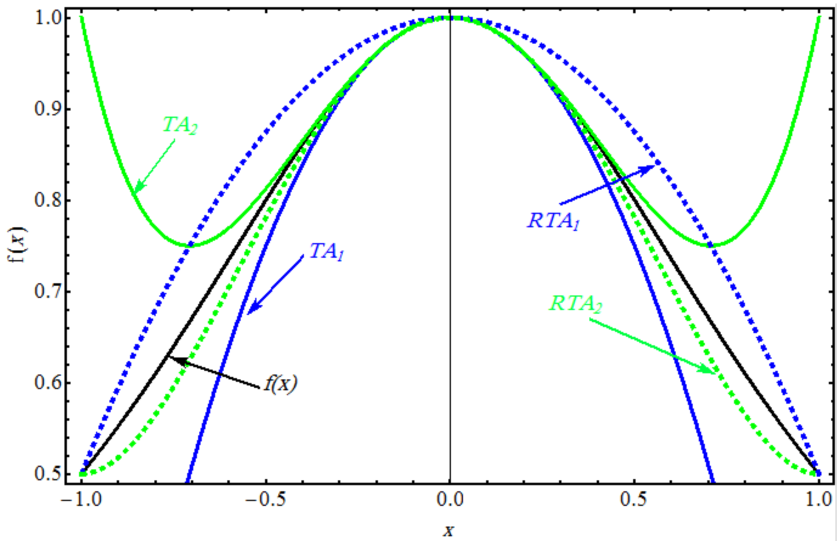

Fig. 1. Expansion of example 1.

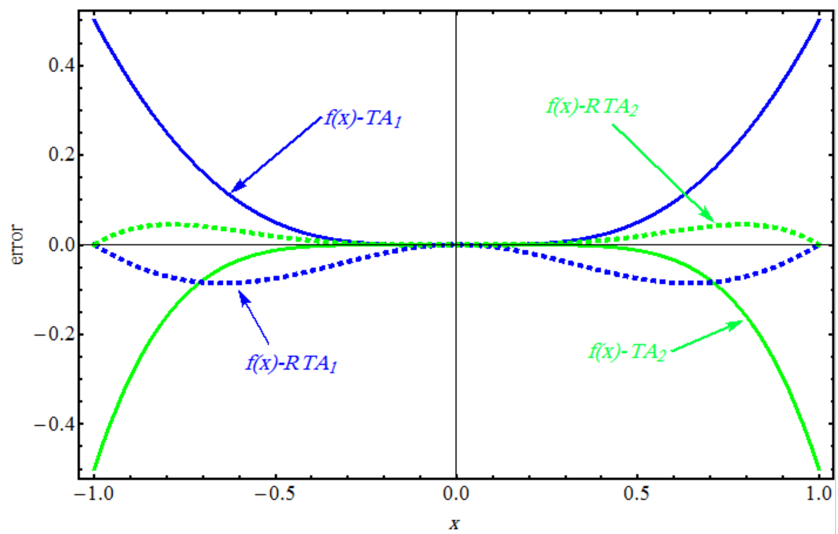

Fig. 2. The errors of example 1.

Example 2

$$f(x) = Exp^{cx} \tag{20}$$

For $c = 0.1$ the rstractive terms are calculated at For $x = 10$ for both RTA and RPA. We expand the expantion domain to [0, 20] and plot TA, PA, RTA, and RPA in Fig. 3. Also the errors are in Fig. 4 .

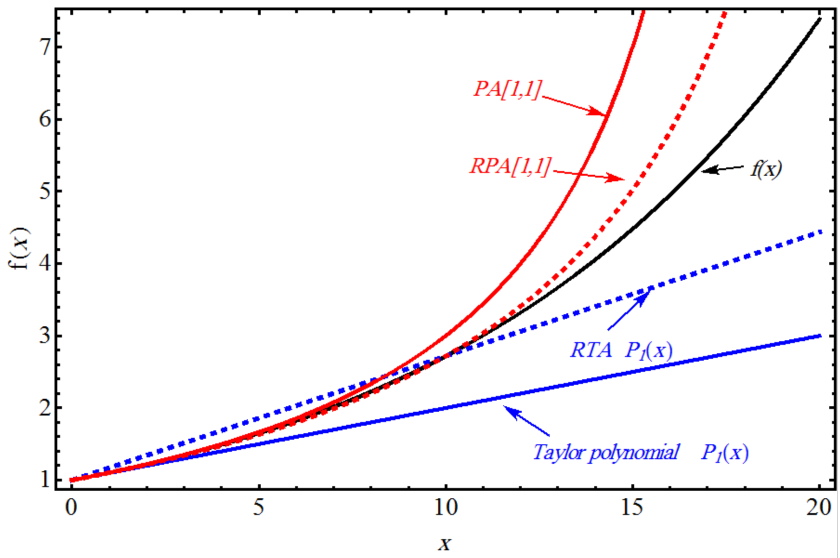

Fig. 3. Expansion of example 1.

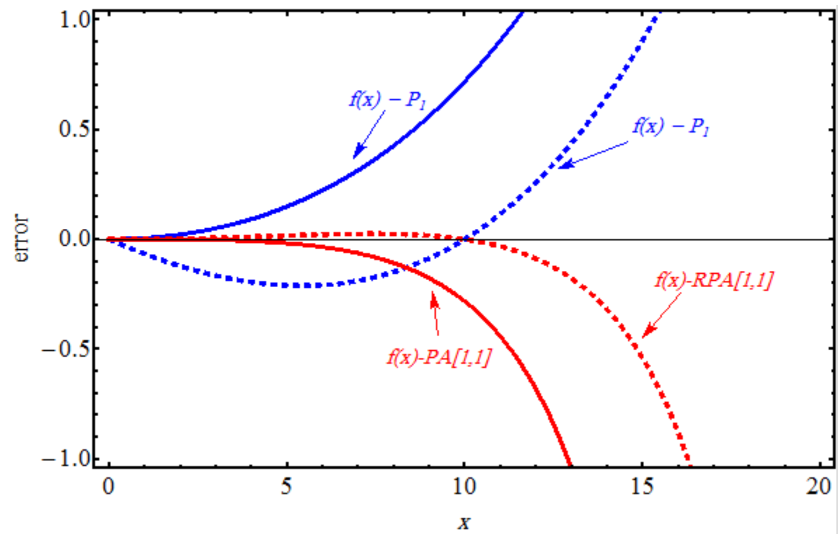

Fig. 4. The errors of example 2.

Example 3

$$f(X) = \sqrt{(\frac{1+\frac{1}{2}x}{1+2x})} \tag{21}$$

Baker and Morris [1] show the accurate of PA via TA in this example we introduce a batter methods (RTA and RPA) as shown in Fig. 5 and the error in Fig. 6.

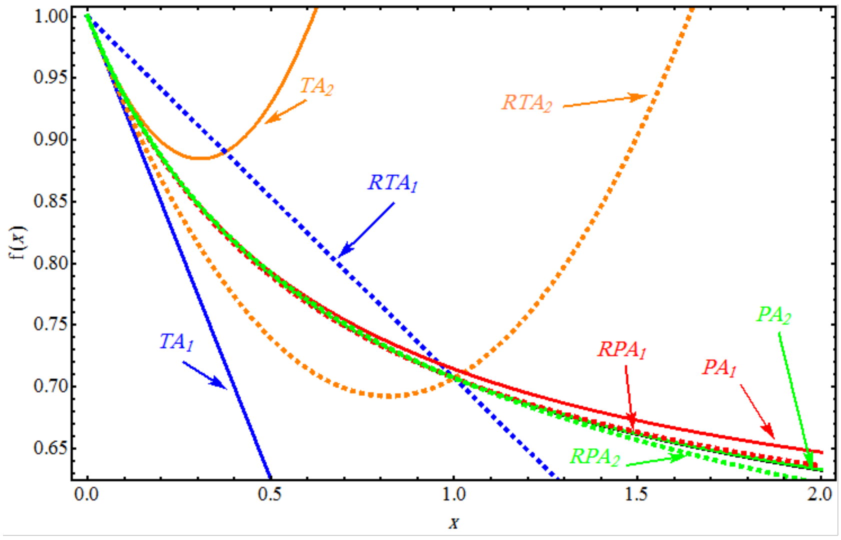

Fig. 5. Expansion of example 3.

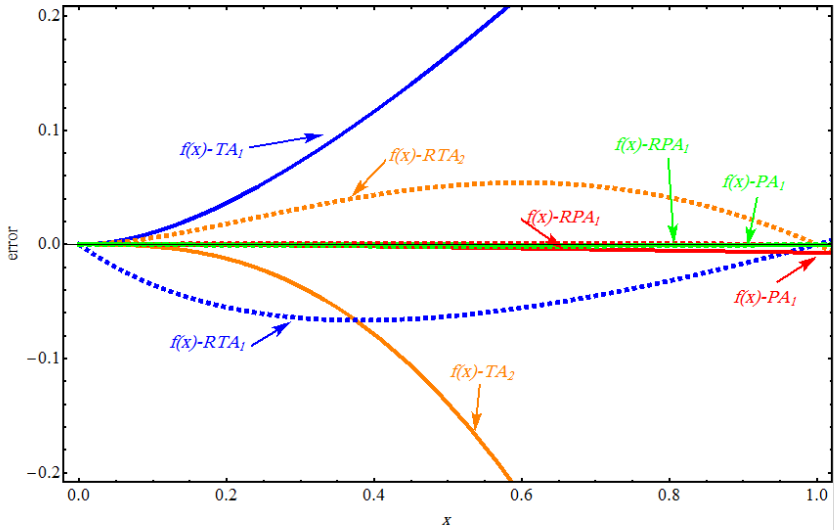

Fig. 6. The errors of example 3.

7. Results and conclusion

The numerical examples show the high accurate of Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA) than Taylor approximation (TA) and Padé approximation (PA).We know that TA and PA gives the exact solution at $x = 0$ also at the point of calculate the restrictive term as shown in example 1, example 2 and example 3 at $x = 1, 10, 1$. In example 2 we plot the solution after the point of calculate restrictive term so the error began to increase, to overcome this error we shall use the piecewise approximation by divide the interval of solution to subintervals and calculate the restrictive term at the end of each subinterval.

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