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A note on Hilbert's weak nullstellensatz

Research Article

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Abstract: In this article, through a suitable generalization of the well-known notion of spectrum of an element of an arbitrary normed algebra of Operator Theory, it will be possible to give another simple proof of the Hilbert's Weak Nullstellensatz.

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1. Introduction

In this paper, we have provided a new proof of the Hilbert's weak nullstellensatz by means of some elementary normed algebra theory methods and notions, starting from a very brief suggestion due to V.I. Danilov (see [1]). The result so achieved is therefore placeable into the crossing zone between pure algebra and functional analysis through a suitable application of operator theory. Due to this feature, the treatment pursued in this work concerns applied mathematics and, for the arguments and disciplines herein involved, it has been kindly accepted by the new IJAAMM which has already published many papers drawn up upon the same cross-disciplinary area in which algebra and functional analysis fruitfully intertwine amongst them.See for instance [2–6].

2. The notion of spectrum

The main references for this section, are [7–11]. Let $\mathbb{K}$ be an arbitrary field of characteristic zero (Whence $card\,\mathbb{K} = \infty$.), not necessarily algebraically closed.

Let $A_{\mathbb{K}}$ be an arbitrary linear unitary commutative $\mathbb{K}$-algebra. For each $a \in A_{\mathbb{K}}$, let ($1_{A_{\mathbb{K}}}$ denotes the unit of such $\mathbb{K}$-algebra.)

$$\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \doteq \{\lambda; \lambda \in \mathbb{K} \text{ such that } \nexists (a - \lambda 1_{A_{\mathbb{K}}})^{-1}\},$$

that we call the $(A_{\mathbb{K}},\mathbb{K})$-*spectrum* of a; $r_{A_{\mathbb{K}},\mathbb{K}}(a) \doteq \mathbb{K} \setminus \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ is said to be the $(A_{\mathbb{K}},\mathbb{K})$-*resolvent* of a.

There exist linear unitary commutative $\mathbb{K}$-algebras in which such a spectrum may be empty for certain their elements: for instance, if $A_{\mathbb{K}}$ is a linear unitary commutative integral $\mathbb{K}$-algebra of finite degree (>1) over $\mathbb{K}$, then it follows that it is a field (because $\varphi_a : x \to ax \ \ \forall x \in A_{\mathbb{K}}$ is an automorphism of $A_{\mathbb{K}}$, for each $a \in A_{\mathbb{K}}$ arbitrarily fixed), and so $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) = \emptyset$ for each $a \in A_{\mathbb{K}} \setminus \mathbb{K}\mathbb{A}_{\mathbb{K}} (\neq \emptyset)$ because, being $a - \lambda 1_{A_{\mathbb{K}}} \in A_{\mathbb{K}} \setminus \{0\} \ \ \forall \lambda \in \mathbb{K}$, there always exists $(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$.

Likewise, if $A_{\mathbb{K}} = \mathbb{K}(X)$, then $\sigma_{A_{\mathbb{K}},\mathbb{K}}(X) = \emptyset$.

It therefore follows that the question related to the emptiness, or not, of the spectrum of the generic element of

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a given linear unitary commutative $\mathbb{K}$-algebra, is not trivial.

If $G(A_{\mathbb{K}})$ denotes the group of units of $A_{\mathbb{K}}$ and $\mathscr{M}(A_{\mathbb{K}})$ the set of all the maximal ideals of $A_{\mathbb{K}}$, then we have $A_{\mathbb{K}} \setminus G(A_{\mathbb{K}}) \subseteq \bigcup_{M\in\mathscr{M}(A_{\mathbb{K}})} M$. Clearly, we have that $a \in G(A_{\mathbb{K}})$ if and only if $0_{\mathbb{K}} \notin \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ or $0_{\mathbb{K}} \in r_{A_{\mathbb{K}},\mathbb{K}}(a)$.

For each $a \in A_{\mathbb{K}}$ such that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$, the correspondence

$$R(a,\cdot) : r_{A_{\mathbb{K}}},\mathbb{K}(a) \to G(A_{\mathbb{K}}), \quad \lambda \rightsquigarrow R(a,\lambda) = (a-\lambda 1_{A_{\mathbb{K}}})^{-1} \quad \forall \lambda \in r_{A_{\mathbb{K}},\mathbb{K}}(a)$$

is called the *spectral map* on a; it is an injective map.
For each $a \in A_{\mathbb{K}}$ such that $r_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$, the correspondence

$$R(a,\cdot) : r_{A_{\mathbb{K}},\mathbb{K}}(a) \to G(A_{\mathbb{K}}), \quad \lambda \rightsquigarrow R(a,\lambda) = (a-\lambda 1_{A_{\mathbb{K}}})^{-1} \quad \forall \lambda \in r_{A_{\mathbb{K}},\mathbb{K}}(a)$$

is called the *resolvent map* on a; it is immediate to prove that $R(a,\cdot)$ is an injective map which verifies the following relation

$$aR(a,\lambda) = 1_{A_{\mathbb{K}}} + \lambda R(a,\lambda) \qquad \forall \lambda \in r_{A_{\mathbb{K}},\mathbb{K}}(a),$$

said to be the *Hilbert's identity,* and the relation

$$R(a,\lambda_1) - R(a,\lambda_2) = (\lambda_1-\lambda_2)R(a,\lambda_1)R(a,\lambda_2) \quad \forall \lambda_1,\lambda_2 \in r_{A_{\mathbb{K}},\mathbb{K}}(a),$$

said to be the *resolvent equation.*

If $A_{\mathbb{K}}$ is a linear unitary commutative $\mathbb{K}$-algebra, then $A^*_{\mathbb{K}}$ denotes the (algebraic) dual $\mathbb{K}$-linear space of $A_{\mathbb{K}}$, namely the set of all linear functionals from the $\mathbb{K}$-linear space $A_{\mathbb{K}}$ to $\mathbb{K}$ (this latter meant as a $\mathbb{K}$-linear space on itself), endowed with the usual structure of $\mathbb{K}$-linear space.
Among the elements of $A^*_{\mathbb{K}}$, there exist particular linear functionals, namely the *multiplicative* linear functionals, that are also (nonzero) ring homomorphisms between the ring structure of the $\mathbb{K}$-algebra $A_{\mathbb{K}}$ and the ring structure of $\mathbb{K}$: we denote their set by $\Delta(A_{\mathbb{K}})(\subseteq A^*_{\mathbb{K}} \setminus \{0\})$, and, a priori, it has no particular algebraic structures. More specifically, the elements of $\Delta(A_{\mathbb{K}})$ are said to be the *characters* of $A_{\mathbb{K}}$.
In the follows, it will be important to establish the possible non-emptiness of $\Delta(A_{\mathbb{K}})$ for certain linear unitary commutative $\mathbb{K}$-algebras $A_{\mathbb{K}}$.

If $A_{\mathbb{K}}$ is an arbitrary linear unitary commutative $\mathbb{K}$-algebra, with support A, then an $A_{\mathbb{K}}$*-ideal* I of it, is a subset $I \subseteq A$ such that $\lambda a + \mu b \in I \ \ \forall \lambda,\mu \in \mathbb{K}, \ \ \forall a,b \in I$, and $ab \in I \ \ \forall a \in A, \ \forall b \in I$. If $(A,+,\cdot)$ is the ring structure of $A_{\mathbb{K}}$, then the class of all the ideals of $(A,+,\cdot)$ is equal to the class of all the ideals of $A_{\mathbb{K}}$: in fact, it is evident that an $A_{\mathbb{K}}$-ideal is also an ideal of $(A,+,\cdot)$, whereas, if I is an ideal of $(A,+,\cdot)$, from $ab \in I \ \ \forall a \in A \ \forall b \in I$, since $a = \lambda 1_{A_{\mathbb{K}}} \in A \ \ \forall \lambda \in \mathbb{K}$, it follows that $\lambda b \in I$ for each $\lambda \in \mathbb{K}$.

Remark 2.1.
Let $A_{\mathbb{K}}$ be a linear unitary commutative integral $\mathbb{K}$-algebra, with $\mathbb{K}$ algebraically closed field. Let $\mathscr{L}(A_{\mathbb{K}})$ be the $\mathbb{K}$-linear space of all the linear endomorphisms of the $\mathbb{K}$-linear space $A_{\mathbb{K}}$.
Then, we have the following representation (of $\mathbb{K}$-algebras)

$$T : A_{\mathbb{K}} \to \mathscr{L}(A_{\mathbb{K}}), \quad a \rightsquigarrow \varphi_a \ \ \forall a \in A_{\mathbb{K}}$$

where $\varphi_a : x \to ax \ \ \forall x \in A_{\mathbb{K}}$. We say T to be the *regular representation* of $A_{\mathbb{K}}$.

Let (with abuse of notation) $[A_{\mathbb{K}} : \mathbb{K}] = \dim_{\mathbb{K}} A_{\mathbb{K}} = n < \infty$. If $\mathscr{B} = \{v_1,...,v_n\}$ is a linear base of the $\mathbb{K}$-linear space $A_{\mathbb{K}}$, let $M^{\mathscr{B}}_a$ be the (n,n)-matrix associated to φ_a with respect to $\mathscr{B}$, so that let[1] $\mathfrak{p}^{\mathscr{B}}_a(\lambda) = \det\,(M^{\mathscr{B}}_a - \lambda I)$ be the characteristic polynomial of φ_a with respect to $\mathscr{B}$, where I is the (n,n)-identity matrix, and $\lambda \in \mathbb{K}$. It is $\mathfrak{p}^{\mathscr{B}}_a(\lambda) \in \mathbb{K}[\lambda]$ with $\partial eg\ \mathfrak{p}^{\mathscr{B}}_a(\lambda) = n$, and, if $Eig_{\mathbb{K}}\ (\varphi_a) = \{\lambda; \lambda \in \mathbb{K},\ \mathfrak{p}^{\mathscr{B}}_a(\lambda) = 0\}$ is the set of eigenvalues of φ_a for every $a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$, then we have $Eig_{\mathbb{K}}\ (\varphi_a) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ because, if $\lambda \in Eig_{\mathbb{K}}\ (\varphi_a)$, then $\varphi_a(x) = \lambda x$ for, at least, one eigenvector $x \neq 0_{A_{\mathbb{K}}}$ of λ, so that $ax = \lambda x$, hence $(a-\lambda 1_{A_{\mathbb{K}}})x = 0_{A_{\mathbb{K}}}$, that is $(a-\lambda 1_{A_{\mathbb{K}}}) = 0_{A_{\mathbb{K}}}$, whence $\nexists (a-\lambda 1_{A_{\mathbb{K}}})^{-1}$. Therefore, since $Eig_{\mathbb{K}}\ (\varphi_a) \neq \emptyset \ \ \forall a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$ due to the algebraic closure of $\mathbb{K}$, it follows that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$ for each $a \in A_{\mathbb{K}} \setminus \{0_{A_{\mathbb{K}}}\}$, with $0_{\mathbb{K}} \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(0_{A_{\mathbb{K}}})$, so we conclude that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset \ \ \forall a \in A_{\mathbb{K}}$.

Nevertheless, in infinite dimension, it may be $Eig_{\mathbb{K}}\ (\varphi_a) = \emptyset$ for some $a \in A_{\mathbb{K}}$, and therefore, in such a case, it is necessary to proceed in other manners.

[1] *The determinant which follows, is computed upon the abstract field* $\mathbb{K}$.

Theorem 2.1.
We have the following

1. *If $A_{\mathbb{K}}$ and $A'_{\mathbb{K}}$ are linear unitary commutative $\mathbb{K}$-algebras and $\varphi : A_{\mathbb{K}} \to A'_{\mathbb{K}}$ is a homomorphism of linear unitary commutative $\mathbb{K}$-algebras, then $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(\varphi(a)) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}}$. If φ is an isomorphism, then $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(\varphi(a)) = \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}}$.*

2. *If $A_{\mathbb{K}}, A'_{\mathbb{K}}$ are linear unitary commutative $\mathbb{K}$-algebras such that $A_{\mathbb{K}} \subseteq A'_{\mathbb{K}}$, then $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}}$.*

3. *If $\mathbb{K}'$ is a (over) field such that $\mathbb{K} \subseteq \mathbb{K}'$ and $A_{\mathbb{K}'}$ is a linear unitary commutative $\mathbb{K}'$-algebra, then, by restriction of the scalars, said $A_{\mathbb{K}}$ the corresponding linear unitary commutative $\mathbb{K}$-algebra, we have: i) $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$; ii) $\sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}'}$; iii) $\sigma_{A_{\mathbb{K}'},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$.*

Let us prove *1.* If $a \in A_{\mathbb{K}}$ and $\lambda \in \sigma_{A'_{\mathbb{K}},\mathbb{K}}(\varphi(a))$, then $\nexists(\varphi(a) - \lambda 1_{A'_{\mathbb{K}}})^{-1} = (\varphi(a) - \lambda\varphi(1_{A_{\mathbb{K}}}))^{-1} = \varphi((a - \lambda 1_{A_{\mathbb{K}}})^{-1})$ in $A'_{\mathbb{K}}$, so that $(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ cannot exist in $A_{\mathbb{K}}$, otherwise $\varphi((a - \lambda 1_{A_{\mathbb{K}}})^{-1})$ would exist in $A'_{\mathbb{K}}$; thus $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$. The last part of *1.* immediately follows from the first one because, being $a = \varphi^{-1}(\varphi(a))\ \ \forall a \in A_{\mathbb{K}}$, we have too $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) = \sigma_{A_{\mathbb{K}},\mathbb{K}}(\varphi^{-1}(\varphi(a)) \subseteq \sigma_{A'_{\mathbb{K}},\mathbb{K}}(\varphi(a))$.
Of course, *2.* follows from *1.* with φ the natural immersion.
In the case of *3.*, first of all, it is $1_{A_{\mathbb{K}}} = 1_{A_{\mathbb{K}'}}$ because $A_{\mathbb{K}}$ and $A_{\mathbb{K}'}$ have the same (inner) addition and multiplication laws but different (external) scalar multiplication law. If $a \in A_{\mathbb{K}}$ and $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$, then $\nexists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ in $A_{\mathbb{K}}$, and since $\lambda \in \mathbb{K} \subseteq \mathbb{K}'$, it is also $\nexists(a - \lambda 1_{A_{\mathbb{K}'}})^{-1}$ in $A_{\mathbb{K}'}$, so that $\lambda \in \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$, whence $\sigma_{A_{\mathbb{K}'},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$, that is *i)*. Conversely, if $\lambda \in \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$, then $\nexists(a - \lambda 1_{A_{\mathbb{K}'}})^{-1}$ in $A_{\mathbb{K}'}$, with $\lambda \in \mathbb{K}'$ and not necessarily with $\lambda \in \mathbb{K} \subseteq \mathbb{K}'$; then, if it is $\lambda \in \mathbb{K}$ as well, we have $\lambda 1_{A_{\mathbb{K}'}} = \lambda 1_{A_{\mathbb{K}}} \in A_{\mathbb{K}}$ so that, by $a \in A_{\mathbb{K}}$, it is $\nexists(a - \lambda 1_{A_{\mathbb{K}}})^{-1}$ in $A_{\mathbb{K}}$, namely $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$, whence $\sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a) \cap \mathbb{K} \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$; but, by *i)*, it follows that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$, whereas, by definition of spectrum, it is also $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \subseteq \mathbb{K}$, so that $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a) \cap \mathbb{K}$; in conclusion $\sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}}$.
As the proof of *iii)*, we have, for each $a \in A_{\mathbb{K}'}$

$$\sigma_{A_{\mathbb{K}'},\mathbb{K}}(a) = \{\lambda; \lambda \in \mathbb{K} \text{ such that } \nexists(\mathrm{a} - \lambda 1_{\mathrm{A}_{\mathbb{K}'}})^{-1} \text{ in } \mathrm{A}_{\mathbb{K}'}\} = \sigma_{\mathrm{A}_{\mathbb{K}'},\mathbb{K}'}(\mathrm{a}) \cap \mathbb{K};$$

but $\sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a) \cap \mathbb{K} = \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$ for each $a \in A_{\mathbb{K}'}$, by *ii)*, and $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$, by *i)*, so that $\sigma_{A_{\mathbb{K}'},\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}'},\mathbb{K}'}(a)$ for each $a \in A_{\mathbb{K}}$.

Theorem 2.2.
If $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]$, then $\sigma_{A_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset$ for each $f \in \mathbb{K}[X_1, ..., X_n]$.

First proof. We remember, once again, that the group of units of $A_{\mathbb{K}} = \mathbb{K}[X_1, ..., X_n]$, say $G(A_{\mathbb{K}})$, is $\mathbb{K}$.
If $f = a \in \mathbb{K}$ is a constant, then $\sigma_{A_{\mathbb{K}},\mathbb{K}}(f) = \{a\} \neq \emptyset$. If f is not constant, then $\partial eg\ f > 0$ with $f \notin G(A_{\mathbb{K}})(= \mathbb{K})$, that is, f is not invertible and so it is also $f - a1_{A_{\mathbb{K}}}$ for each $a \in \mathbb{K}$ (because f not constant implies $f - a$ not constant for each $a \in \mathbb{K}$), whence $\sigma_{A_{\mathbb{K}},\mathbb{K}}(f) = \mathbb{K}$. In any case, we have $\sigma_{A_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset$ for each $f \in \mathbb{K}[X_1, ..., X_n]$.

Second proof[2]. Let $A'_{\mathbb{K}} = K[[X_1, ..., X_n]]$ be the linear unitary commutative $\mathbb{K}$-algebra of the formal power series in the variables $X_1, ..., X_n$. Then $A_{\mathbb{K}} \subseteq A'_{\mathbb{K}}$, so that, by Theorem 1.1-2), we have $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(f) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(f)$ for each $f \in A_{\mathbb{K}}$. On the other hand, if $f = 0$, then $\sigma_{A'_{\mathbb{K}},\mathbb{K}} = \{0\} \neq \emptyset$; instead, if $f \neq 0$ is not an invertible polynomial in $A'_{\mathbb{K}}$, then, at least, $0 \in \sigma_{A'_{\mathbb{K}},\mathbb{K}}(f)$ so that $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset$ in this case, whereas, if $f \neq 0$ is an invertible polynomial in $A'_{\mathbb{K}}$, then it follows that the zero-degree term of f, namely a_0, has to be nonzero, whence $f - \lambda 1_{A'_{\mathbb{K}}}$ is a polynomial without constant term if $\lambda = a_0$, so that $f - a_0 1_{A'_{\mathbb{K}}}$ is not invertible in $A'_{\mathbb{K}}$, and thus, at least $a_0 \in \sigma_{A'_{\mathbb{K}},\mathbb{K}}(f)$, so that $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset$ also in this case. In conclusion, in any case we have $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset$ for each $f \in A_{\mathbb{K}}$, and hence $\emptyset \neq \sigma_{A'_{\mathbb{K}},\mathbb{K}}(f) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(f)\ \ \forall f \in A_{\mathbb{K}}$ implies $\sigma_{A_{\mathbb{K}},\mathbb{K}}(f) \neq \emptyset\ \ \forall f \in A_{\mathbb{K}}$, as required.

Note 1. The statement of Theorem 1.2 holds true if $\mathbb{K}$ is any integral domain.

Theorem 2.3.
Let $A_{\mathbb{K}}$ be an arbitrary linear unitary commutative $\mathbb{K}$-algebra, and let $a \in A_{\mathbb{K}}$. If there exists $\phi \in \Delta(A_{\mathbb{K}})$ such that $\phi(a) = \lambda \in \mathbb{K}$, then $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$.

[2] *The principle of the method of this second proof lies on the possible invertibility of a nonconstant polynomial in $A'_{\mathbb{K}}$, whose group of units is larger than $G(A_{\mathbb{K}}) = \mathbb{K}$.*

Supposing, at first, $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$, if it were $\lambda \notin \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$, then it would exist $(a-\lambda 1_{A_{\mathbb{K}}})^{-1}$, hence we would have $\widetilde{\phi}(a-\lambda 1_{A_{\mathbb{K}}}) \neq 0_{\mathbb{K}} \ \ \forall \widetilde{\phi} \in \Delta(A_{\mathbb{K}})$, because (in $\mathbb{K}$, with $char\ \mathbb{K} = 0$), it is

$$(\star) \qquad 1_{\mathbb{K}} = \widetilde{\phi}(1_{A_{\mathbb{K}}}) = \widetilde{\phi}((a-\lambda 1_{A_{\mathbb{K}}})(a-\lambda 1_{A_{\mathbb{K}}})^{-1}) =$$

$$= \widetilde{\phi}(a-\lambda 1_{A_{\mathbb{K}}})\widetilde{\phi}((a-\lambda 1_{A_{\mathbb{K}}})^{-1}) \quad \forall \widetilde{\phi} \in \Delta(A_{\mathbb{K}}).$$

But, $\widetilde{\phi}(a-\lambda 1_{A_{\mathbb{K}}}) \neq 0 \ \ \forall \widetilde{\phi} \in \Delta(A_{\mathbb{K}})$ implies a contradiction for $\widetilde{\phi} = \phi \in \Delta(A_{\mathbb{K}})$ because, for such a ϕ, we have supposed to be $\phi(a) = \lambda$ which, in turn, implies $\phi(a-\lambda 1_{A_{\mathbb{K}}}) = 0_{\mathbb{K}}$. Therefore, it must be $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$. On the other hand, if it were $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) = \emptyset$, then it would exist $(a-\widetilde{\lambda} 1_{A_{\mathbb{K}}})^{-1}$ for each[3] $\widetilde{\lambda} \in \mathbb{K}$, that is (by $(\star)$, wrote for an arbitrary λ, say $\tilde{\lambda}$), $\widetilde{\phi}(a-\widetilde{\lambda} 1_{A_{\mathbb{K}}}) \neq 0$ for any $\widetilde{\phi} \in \Delta(A_{\mathbb{K}})$ and $\widetilde{\lambda} \in \mathbb{K}$, whence, arguing likewise as made above, we obtain again a contradiction for $\widetilde{\phi} = \phi$ and $\widetilde{\lambda} = \lambda$. Thus, it must be $\sigma_{A_{\mathbb{K}},\mathbb{K}}(a) \neq \emptyset$.

Note. A simpler form of the this latter proof, is as follows. Simply, if it were $\lambda \notin \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$, then it would exist $(a-\lambda 1_{A_{\mathbb{K}}})^{-1}$, hence we would have $\widetilde{\phi}(a-\lambda 1_{A_{\mathbb{K}}}) \neq 0 \ \ \forall \widetilde{\phi} \in \Delta(A_{\mathbb{K}})$, otherwise (in $\mathbb{K}$, with $char\ \mathbb{K} = 0$)

$$(\star') \qquad 1_{\mathbb{K}} = \widetilde{\phi}(1_{A_{\mathbb{K}}}) = \widetilde{\phi}((a-\lambda 1_{A_{\mathbb{K}}})(a-\lambda 1_{A_{\mathbb{K}}})^{-1}) =$$

$$= \widetilde{\phi}(a-\lambda 1_{A_{\mathbb{K}}})\widetilde{\phi}((a-\lambda 1_{A_{\mathbb{K}}})^{-1}) = 0_{\mathbb{K}} \quad \forall \widetilde{\phi} \in \Delta(A_{\mathbb{K}}),$$

so that we have $\tilde{\phi}(a) \neq \lambda \ \ \forall \tilde{\phi} \in \Delta(A_{\mathbb{K}})$, obtaining a contradiction for $\tilde{\phi} = \phi$.

Remark 2.2.
If we put $A^*_{\mathbb{K}}(a) \doteq \{\phi(a); \phi \in \Delta(A_{\mathbb{K}})\}$, then, by Theorem 1.3, it follows that $A^*_{\mathbb{K}}(a) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}(a)$. Therefore, Theorem 1.3. may be used as a criterion for the spectrum to be nonempty (as it will be made in the proof of Theorem 1.4.), when $\Delta(A_{\mathbb{K}})$ is a non-void set. For instance, if $A_{\mathbb{K}} = \mathbb{K}[X_1,...,X_n]$, then the (polynomial) evaluation map $f \xrightarrow{\zeta_k} \zeta_k(f) = f(k) \ \ \forall f \in \mathbb{K}[X_1,...,X_n]$, is an element of $\Delta(A_{\mathbb{K}})$, so that, in this case, $\Delta(A_{\mathbb{K}}) \neq \emptyset$. Furthermore, the equation $\phi(f) = \lambda \in \mathbb{K}$ of Theorem 1.3., becomes $\zeta_k(f) = f(k) = \lambda$, that is, $f(k) - \lambda = 0$, hence $g_\lambda(k) = 0$ with $g_\lambda = f - \lambda \in \mathbb{K}[X_1,...,X_n]$; because of the algebraic closure of $\mathbb{K}$, the equation $g_\lambda(k) = 0$ has always, at least, one solution in $k \in \mathbb{K}$ for each $\lambda \in \mathbb{K}$ (see Theorems 1.2. and 1.3.), say k_λ, so that, for any $\lambda \in \mathbb{K}$, there exists, at least, one $k_\lambda \in \mathbb{K}$ such that $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(\zeta_{k_\lambda})$ for $A_{\mathbb{K}} = \mathbb{K}[X_1,...,X_n]$, whence $\mathbb{K} = \bigcup_{\lambda \in \mathbb{K}} \sigma_{A_{\mathbb{K}},\mathbb{K}}(\zeta_{k_\lambda})$.

Remark 2.3.
If I is an arbitrary maximal ideal of $\mathbb{K}[X_1,...,X_n]$ $(n \geq 1)$, then it follows that $\mathbb{A} = \mathbb{K}[X_1,...,X_n]/I = \mathbb{K}[\bar{X}_1,...,\bar{X}_n]$ is a finitely generated field extension of $\mathbb{K}$, with $\bar{X}_i = \pi(X_i) = X_i + I \ \ i = 1,...,n$, where $\pi : \mathbb{K}[X_1,...,X_n] \to \mathbb{K}[X_1,...,X_n]/I$ is the quotient epimorphism. Therefore, it is possible to consider the UFD $\mathbb{A}[X_1,...,X_n] = \mathbb{K}[X_1,...,X_n,\bar{X}_1,...,\bar{X}_n]$, which is a linear unitary commutative $\mathbb{A}$-algebra, and also by restriction of the scalars, a linear unitary commutative $\mathbb{K}$-algebra; thus, it is also possible to consider $\mathbb{K}[X_1,...,X_n]$ as a sub-UFD of $\mathbb{A}[X_1,...,X_n]$.
Let $A'_{\mathbb{K}}$ be the linear unitary commutative $\mathbb{K}$-algebra obtained considering $\mathbb{A}[X_1,...,X_n]$ as a $\mathbb{K}$-algebra, as said above. Let $A'_{\mathbb{A}}$ be the linear unitary commutative $\mathbb{A}$-algebra $\mathbb{A}[X_1,...,X_n]$, and let $A'^*_{\mathbb{A}}$ be the dual $\mathbb{A}$-linear space of $A'_{\mathbb{A}}$. The map

$$\zeta : \mathbb{A} \to \Delta(A'_{\mathbb{A}}), \quad k \rightsquigarrow \zeta_k \quad \forall k \in \mathbb{A},$$

defined via the (polynomial) evaluation map $\zeta_k(f) = f(k) \in \mathbb{A}$ for each $f \in A'_{\mathbb{A}}$ (see *Remark 2*), proves that there exists $\phi \in \Delta(A'_{\mathbb{A}})$, for instance, $\phi = \zeta_k$ with $k \in \mathbb{A}$, by which $\Delta(A'_{\mathbb{A}}) \neq \emptyset$.

Nevertheless, this last argument is not valid for proving that $\Delta(A'_{\mathbb{K}}) \neq \emptyset$ because, for every $k \in \mathbb{K}$, we however have $f(k) \in \mathbb{A} \ \ \forall f \in A'_{\mathbb{K}}$ being all the coefficients of $f \in A'_{\mathbb{K}} = \mathbb{A}[X_1,...,X_n]$ belonging to $\mathbb{A}$, whence it is necessary to argue in another manner, for instance, as follows.

Amongst other things, later on (see proof of next Theorem 1.4.) it will be important above all to verify, in $A'_{\mathbb{K}}$, whether there exists, at least, one $\phi \in \Delta(A'_{\mathbb{K}})$, where

$$\Delta(A'_{\mathbb{K}}) = \{\phi; \phi : A'_{\mathbb{K}} \to \mathbb{K} \ \text{ multiplicative linear functional}\} (\subseteq A'^*_{\mathbb{K}} \setminus \{0\})$$

is the set of characters of $A'_{\mathbb{K}}$. To this end, since I is a maximal ideal of $A_{\mathbb{K}} = \mathbb{K}[X_1,...,X_n]$, we have that I is a proper ideal of it. Therefore, we have two possible cases, namely $\mathbb{K}[X_1,...,X_n] \setminus I = \mathbb{K}$ or $\mathbb{K} \subset \mathbb{K}[X_1,...,X_n] \setminus I$. If $\mathbb{K}[X_1,...,X_n] \setminus I = \mathbb{K}$,

[3] *In particular, for* $\widetilde{\lambda} = 0$, *it necessarily follows that* $\exists a^{-1}$.

then $\mathbb{K}[X_1,...,X_n] = I \oplus \mathbb{K}$ and let $\phi : \mathbb{K}[X_1,...,X_n] \to \mathbb{K}$ such that $\phi(I) = \{0\}$ and $\phi_{|\mathbb{K}} = \mathrm{id}_{\mathbb{K}}$, whence $\phi \in \Delta(A_{\mathbb{K}})$ (see *Remark 2*), whereas, if $\mathbb{K} \subset \mathbb{K}[X_1,...,X_n] \setminus I$, then there exists, at least, one nonconstant polynomial (hence, a noninvertible element of $A_{\mathbb{K}}$) $f \in \mathbb{K}[X_1,...,X_n] \setminus I$ and, therefore, if J is the maximal ideal containing $\langle f \rangle$, then we have that I and J are comaximal because they are two distinct maximal ideals (by $f \notin I$), so that $\mathbb{K}[X_1,...,X_n] = I + J$. Therefore, it is possible to consider the map $\phi : \mathbb{K}[X_1,...,X_n] \to \mathbb{K}$ such that $\phi(I) = \phi(I \cap J) = 0_{\mathbb{K}}$ and $\phi(f) = f(0) = a_0 \in \mathbb{K}$ for each $f \in J \setminus I$, where a_0 is the zero-degree term of the polynomial f. Then, it is immediate to verify that ϕ is a nonzero[4] multiplicative linear functional on $A'_{\mathbb{K}}$ if we extend ϕ from $A_{\mathbb{K}} = \mathbb{K}[X_1,...,X_n]$ to $A'_{\mathbb{K}} = \mathbb{A}[X_1,...,X_n]$ setting ϕ also zero on $\mathbb{A}[X_1,...,X_n] \setminus \mathbb{K}[X_1,...,X_n]$. In short, we have $\phi \in \Delta(A'_{\mathbb{K}})$, that is, $\Delta(A'_{\mathbb{K}}) \neq \emptyset$.

3. On Hilbert's weak nullstellensatz

From what has been established in the previous section, with respect to the usual, well-known proofs of the Hilbert's Weak Nullstellensatz, it is now possible to give another proof of the latter, as follows.

Theorem 3.1 (Weak Nullstellensatz).
If I is any proper ideal of $\mathbb{K}[X_1,...,X_n]$, then $V(I) \neq \emptyset$.

If J is the maximal ideal containing I, then we have $V(J) \subseteq V(I)$, so that $(V(J) \neq \emptyset) \Rightarrow (V(I) \neq \emptyset)$, and therefore it is possible to consider I to be maximal. Then $A = \mathbb{K}[X_1,...,X_n]/I$ is a field (often denoted by $\mathbb{A}$ as a scalar field – see *Remark 3*), hence a linear unitary commutative $\mathbb{A}$-algebra on itself, namely $A_{\mathbb{A}}$. But, according to the *Remark 3*, A is also a linear unitary commutative $\mathbb{K}$-algebra, namely $A_{\mathbb{K}}$. We are mainly interested in $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$, rather than in $\sigma_{A_{\mathbb{A}},\mathbb{A}}([f])$, for $[f] \in A$, and, to this scope, we observe what follows. If $[f] = f + I \in A_{\mathbb{K}} \setminus \{0\}$, then we have[5], for $\lambda \in \mathbb{K}$, that $(f + I) - \lambda 1_{A_{\mathbb{K}}} = (f + I) - \lambda(1_{\mathbb{K}} + I) = (f - \lambda 1_{\mathbb{K}}) + I$ is not invertible if and only if it is zero (in a field, like A), that is, if and only if $f - \lambda 1_{\mathbb{K}} \in I$, with $f \notin I$ because $[f] \neq 0$ in A. Therefore, it would be required to find, at least, one $\lambda \in \mathbb{K} \setminus \{0\}$ such that[6] $f - \lambda 1_{\mathbb{K}} \in I$, if one wished to prove that $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) \neq \emptyset$.

However, to this purpose, here we follow a different way. To be precise, taking into account *Remark 3* (with $n = 1$), we consider $\mathbb{A}[X]$, with support field $\mathbb{A} = \mathbb{K}[X_1,...,X_n]/I$, as a linear unitary commutative $\mathbb{K}$-algebra, say[7] $A'_{\mathbb{K}}$. Therefore, since, in the *Remark 3*, we have proved that $\Delta(A'_{\mathbb{K}}) \neq \emptyset$, it follows, by Theorem 1.3. and *Remark 2*, that $\sigma_{A'_{\mathbb{K}},\mathbb{K}}([f]) \neq \emptyset$ for each[8] $[f] \in A'_{\mathbb{K}}$, whereas, by Theorem 1.1.-2), it follows that $\emptyset \neq \sigma_{A'_{\mathbb{K}},\mathbb{K}}([f]) \subseteq \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ implies $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) \neq \emptyset$ for any $[f] \in A_{\mathbb{K}}$, if we consider $A_{\mathbb{K}}$ as a linear unitary commutative $\mathbb{K}$-algebra, which is a sub-algebra of $A'_{\mathbb{K}}$ ($= \mathbb{A}[X]$ as $\mathbb{K}$-algebra[9]).

Then $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) \neq \emptyset$ for each $[f] \in A_{\mathbb{K}} \setminus \{0\}$, with $0 \notin \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ because, by $0 \neq [f] \in A_{\mathbb{K}}$, with $A(= \mathbb{A})$ field, it follows that $[f]$ is invertible, so that 0 cannot be an element of $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ $\forall [f] \in A_{\mathbb{K}} \setminus \{0\}$; likewise, each $[f] \neq 0$ is invertible in the field A, so that it must be $[f] - \lambda 1_{A_{\mathbb{K}}} = 0$ for every $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ because, if it were nonzero, then it would admit inverse in A while $\nexists ([f] - \lambda 1_{A_{\mathbb{K}}})^{-1}$ since $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$; therefore, for any $[f] \in A_{\mathbb{K}} \setminus \{0\}$, we have $[f] = \lambda 1_{A_{\mathbb{K}}}$ for each $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$. Thus $([f] =) \lambda 1_{A_{\mathbb{K}}} = \lambda' 1_{A_{\mathbb{K}}}$ $\forall \lambda, \lambda' \in \sigma_{A_{\mathbb{K}},\mathbb{K}}$, by which $\lambda = \lambda'$, whence $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$ is a singleton for every $[f] \in A_{\mathbb{K}} \setminus \{0\}$; on the other hand, if $[f] = 0$, then we have $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = \{0\}$. Therefore $card\ \sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = 1$ for all $[f] \in A_{\mathbb{K}}$.

Thus, it is possible to consider the map

$$\psi : A_{\mathbb{K}} \to \mathbb{K}, \quad [f] \rightsquigarrow \lambda \ (\in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])) \quad \forall [f] \in A_{\mathbb{K}},$$

which is well-defined because, for $[f] \neq 0$, we have the singleton $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]) = \{\lambda\} (\neq \{0_{\mathbb{K}}\})$, whereas, if $[f] = 0$, then $\sigma_{A_{\mathbb{K}},\mathbb{K}}(0) = \{0_{\mathbb{K}}\}$.

[4] *$\mathbb{K}[X_1,...,X_n] = I + J$ implies $\mathbb{K} \subset I + J$, so that, for each $k \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$, there exist $h_k \in I, g_k \in J$ such that $k = h_k + g_k$ with $\partial eg\ h_k, \partial eg\ g_k > 0$ since I and J, as maximal ideals, cannot contain invertible elements of $\mathbb{K}[X_1,...,X_n]$, that is to say, elements of $\mathbb{K}$ (which is the group of units of $\mathbb{K}[X_1,...,X_n]$). Hence, if the scalars are elements of $I + J$, we can only obtain them as a sum of nonconstant polynomials (precisely, as a sum, or difference, of their zero-degree terms) belonging to such maximal ideals (I and J), so that, just these latter have to contain nonconstant polynomials with not null zero-degree terms. From this, it follows that such a ϕ is a nonzero multiplicative linear functional on $\mathbb{K}[X_1,...,X_n]$.*

[5] *$1_{A_{\mathbb{K}}}$ is the unit of $A = \mathbb{K}[X_1,...,X_n]/I$ as $\mathbb{K}$-algebra, whereas $1_{\mathbb{K}}$ is the unit of $\mathbb{K}$, the same of $\mathbb{K}[X_1,...,X_n]$. In general, $1_{A_{\mathbb{K}}} \neq 1_{\mathbb{K}}$, for an arbitrary linear unitary $\mathbb{K}$-algebra $A_{\mathbb{K}}$.*

[6] *It is necessarily $\lambda \neq 0$ because $f \notin I$.*

[7] *In the Remark 3, we have denoted by $A'_{\mathbb{A}}$ and $A'_{\mathbb{K}}$ the linear unitary commutative algebra $\mathbb{A}[X_1,...,X_n]$ with respect to the fields $\mathbb{A}$ and $\mathbb{K}$, whereas, in the first part of this proof, we have denoted by $A_{\mathbb{A}}$ and $A_{\mathbb{K}}$ the field $\mathbb{A}(= A)$ considered as a linear unitary commutative algebra with respect to the fields $\mathbb{A}$ and $\mathbb{K}$. Therefore, $A_{\mathbb{A}}$ is a subalgebra of $A'_{\mathbb{A}}$, whereas $A_{\mathbb{K}}$ is a subalgebra of $A'_{\mathbb{K}}$.*

[8] *Whence, by Theorem 1.1.-3i), with $\mathbb{K}' = \mathbb{A}$, it also follows that $\sigma_{A_{\mathbb{A}},\mathbb{A}}([f]) \neq \emptyset$ for each $[f] \in A_{\mathbb{K}}$. Moreover, we observe that both the support of $A_{\mathbb{K}}$ and that of $A_{\mathbb{A}}$ are equal ($= \mathbb{A}$); likewise for $A'_{\mathbb{K}}$ and $A'_{\mathbb{A}}$, with common support $\mathbb{A}[X]$ (see also the previous footnote).*

We prove that ψ is a monomorphism. In fact, we have $\psi([f]+[g]) = \lambda+\mu$, if $\psi([f]) = \lambda$ and $\psi([g]) = \mu$, because, as we will see later, $\nexists(([f]+[g])-(\lambda+\mu)1_{A_{\mathbb{K}}})^{-1}$ so that $\lambda+\mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f]+[g])$, and since $card\,\sigma_{A_{\mathbb{K}},\mathbb{K}}([h]) = 1$ for all $[h] \in A_{\mathbb{K}}$, it follows that $\sigma_{A_{\mathbb{K}},\mathbb{K}}([f]+[g]) = \{\lambda+\mu\}$; on the other hand, $\lambda+\mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f]+[g])$ because $\nexists([h]-\rho 1_{A_{\mathbb{K}}})^{-1}$ if and only if $h-\rho 1_{A_{\mathbb{K}}} \in I$, and since $\nexists([f]-\lambda 1_{A_{\mathbb{K}}})^{-1}$, $\nexists([g]-\mu 1_{A_{\mathbb{K}}})^{-1}$, we have $f-\lambda 1_{A_{\mathbb{K}}}, g-\mu 1_{A_{\mathbb{K}}} \in I$, that is $(f+g)-(\lambda+\mu)1_{A_{\mathbb{K}}} \in I$, so that $\lambda+\mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f+g]) = \sigma_{A_{\mathbb{K}},\mathbb{K}}([f]+[g])$. Thus $\psi([f]+[g]) = \lambda+\mu = \psi([f])+\psi([g])$.
Analogously, we have $\psi([f][g]) = \lambda\mu$ because $f-\lambda 1_{A_{\mathbb{K}}}, g-\mu 1_{A_{\mathbb{K}}} \in I$ imply $(f-\lambda 1_{A_{\mathbb{K}}})(g-\mu 1_{A_{\mathbb{K}}}) \in I$, that is

$$fg-\mu f-\lambda g+\lambda\mu 1_{A_{\mathbb{K}}} = fg-\lambda\mu 1_{A_{\mathbb{K}}}+2\lambda\mu 1_{A_{\mathbb{K}}}-\mu f-\lambda g = (fg-\lambda\mu 1_{A_{\mathbb{K}}})+\mu(f-\lambda 1_{A_{\mathbb{K}}})+\lambda(g-\mu 1_{A_{\mathbb{K}}}) \in I,$$

and since $f-\lambda 1_{A_{\mathbb{K}}}, g-\mu 1_{A_{\mathbb{K}}} \in I$, it follows that $\mu(f-\lambda 1_{A_{\mathbb{K}}}), \lambda(g-\mu 1_{A_{\mathbb{K}}}) \in I$ and thus $fg-\lambda\mu 1_{A_{\mathbb{K}}} \in I$, whence $\lambda\mu \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([fg]) = \sigma_{A_{\mathbb{K}},\mathbb{K}}([f][g])$, so that $\psi([fg]) = \psi([f][g]) = \lambda\mu = \psi([f])\psi([g])$.
Finally, if $\alpha \in \mathbb{K}\setminus\{0\}$, then $\psi(\alpha[f]) = \alpha\lambda$ because it is enough to observe that $(\alpha[f]-\alpha\lambda 1_{A_{\mathbb{K}}})^{-1} = \alpha^{-1}([f]-\lambda 1_{A_{\mathbb{K}}})^{-1}$, and that $(\alpha[f]-\alpha\lambda 1_{A_{\mathbb{K}}})^{-1}$ exists if and only if $([f]-\lambda 1_{A_{\mathbb{K}}})^{-1}$ exists, so that $\psi(\alpha[f]) = \alpha\lambda = \alpha\psi([f])$.
At last, since $Ker\,\psi = \{[f]; [f] \in A_{\mathbb{K}},\ \psi([f]) = 0\} = \{0\}$, it follows that ψ is injective[9].
Therefore, we have the immersion

$$\psi : A_{\mathbb{K}} = \mathbb{K}[X_1,...,X_n]/I \hookrightarrow \mathbb{K}$$

that, combined with the natural immersion[10] (restriction of the scalars)

$$\sigma : \mathbb{K} \hookrightarrow \mathbb{K}[X_1,...,X_n]/I,$$

gives rise to an isomorphism of the type $\mathbb{K} \cong \mathbb{K}[X_1,...,X_n]/I (= A_{\mathbb{K}})$[11].

On the other hand, for every $[f] = f+I \in A_{\mathbb{K}}\setminus\{0\}$, we have seen that $[f]-\lambda 1_{A_{\mathbb{K}}} = 0$ in $A_{\mathbb{K}}$, for all $\lambda \in \sigma_{A_{\mathbb{K}},\mathbb{K}}([f])$, so that $(f+I)-\lambda(1_{\mathbb{K}}+I) = I$ (since $1_{A_{\mathbb{K}}} = 1_{\mathbb{K}}+I$ in $A_{\mathbb{K}}$). Then $(f-\lambda 1_{A_{\mathbb{K}}}) \in I$ for each $f \in \mathbb{K}[X_1,...,X_n]$, and hence $f \in I+\mathbb{K}$ (for $\mathbb{K} = \{\lambda 1_{\mathbb{K}}; \lambda \in \mathbb{K}\} \subset \mathbb{K}[X_1,...,X_n]$). It follows that $\mathbb{K}[X_1,...,X_n] = I+\mathbb{K}$; then $X_i \in I+\mathbb{K}\ \ \forall i \in \{1,...,n\}$, that is $X_i = a_i+m_i$ with $a_i \in \mathbb{K}$ and $m_i \in I$ for all i, by which $X_i-a_i = m_i \in I\ \ \forall i$, and hence $\langle X_1-a_1,...,X_n-a_n\rangle \subseteq I$. But, as well-known, $\langle X_1-a_1,...,X_n-a_n\rangle$ is a maximal ideal, so that $\langle X_1-a_1,...,X_n-a_n\rangle = I$ (see also next *Remark 4*), and thus $V(I) = V(\langle X_1-a_1,...,X_n-a_n\rangle)$, whence $(a_1,...,a_n) \in V(I)$, that is $V(I) \neq \emptyset$.

Remark 3.1.
If we consider the quotient epimorphism

$$\pi : \mathbb{K}[X_1,...,X_n] \to \mathbb{K}[X_1,...,X_n]/I = \mathbb{K}[\bar{X}_1,...,\bar{X}_n] = A'_{\mathbb{K}}$$

with $\bar{X}_i = \pi(X_i)\ \ \forall i \in \{1,...,n\}$, by Theorem 1.4., it follows that $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(\bar{X}_i) \neq \emptyset\ \ \forall i \in \{1,...,n\}$, that is $\nexists((X_i+I)-(a_i+I)1_{A'_{\mathbb{K}}})^{-1}$, for $a_i \in \sigma_{A_{\mathbb{K}},\mathbb{K}}(\bar{X}_i)$, in the field $A'_{\mathbb{K}} (= \mathbb{A} = \mathbb{K}[\bar{X}_1,...,\bar{X}_n])$, so that it has to be $(X_i+I)-(a_i+I)1_{A'_{\mathbb{K}}} = 0_{A_{\mathbb{K}}}$, that is, $(X_i-a_i)+I = 0_{A'_{\mathbb{K}}}$. Then $X_i-a_i \in I$, and hence $\langle X_1-a_1,...,X_n-a_n\rangle \subseteq I$, whence, by the well-known maximality of the ideal $\langle X_1-a_1,...,X_n-a_n\rangle$, it follows that $I = \langle X_1-a_1,...,X_n-a_n\rangle$. From the latter, it is possible to understand the meaning of $\sigma_{A'_{\mathbb{K}},\mathbb{K}}(\bar{X}_i)$ (via $a_i \in \sigma_{A'_{\mathbb{K}},\mathbb{K}}(\bar{X}_i)\ \ \forall i$), which is strictly correlated with the fact that any maximal ideal of $\mathbb{K}[X_1,...,X_n]$ is of the type $\langle X_1-a_1,...,X_n-a_n\rangle$, for certain $(a_1,...,a_n) \in \mathbb{K}^n$ (with $a_i \in \sigma_{A'_{\mathbb{K}},\mathbb{K}}(\bar{X}_i)\ \ \forall i$).

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[9] *Direct proof: if* $\psi([f]) = \psi([g])$ *for* $[f],[g] \in A_{\mathbb{K}}$, *then* $\lambda = \mu$, *whence* $[f] = \lambda 1_{A_{\mathbb{K}}} = \mu 1_{A_{\mathbb{K}}} = [g]$, *as required.*
[10] *It is defined as follows. If*

$$i : \mathbb{K} \hookrightarrow \mathbb{K}[X_1,...,X_n]$$

is the natural monomorphism of immersion, and

$$\pi : \mathbb{K}[X_1,...,X_n] \to \mathbb{K}[X_1,...,X_n]/I$$

is the canonical epimorphism of projection (quotient map), then it follows that

$$\sigma = \pi \circ i : \mathbb{K} \to \mathbb{K}[X_1,...,X_n]/I$$

is a ring homomorphism with functional law

$$\sigma(k) = \pi(i(k)) = i(k)+I, \quad \forall k \in \mathbb{K}.$$

We prove that σ *is injective. If* $\sigma(k) = \sigma(k')$, *then we have* $\sigma(k-k') = 0$, *whence* $k-k' \in Ker\,\sigma$ *where*

$$Ker\,\sigma = \{k; k \in \mathbb{K}, i(k) \in I\}.$$

But, in any event, $i(k)$ *is a scalar constant (in* $\mathbb{K}[X_1,...,X_n]$*), and, if it were* $k-k' = \xi \neq 0$, *then it would be* $0 \neq i(\xi) \in I$, *that is, the ideal* I *would contain an invertible element, which is impossible being it a proper ideal. Therefore* $i(\xi) = 0$, *or* $\xi = 0$, *that is* $k = k'$, *and thus* σ *is 1-1, hence an immersion.*
[11] *From here, it is also possible to get the thesis of Theorem 1.4., continuing in the next Remark 4.*

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