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On the Superconvergence of h-p finite element method for parabolic equation

Research Article

E. Natarajan*

Department of Mathematics, Indian Institute of Space Science and Technology, Valiamala PO, 695547, Thiruvananthapuram, Kerala, India

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Abstract: In this article we study the superconvergence properties of parabolic partial differential equation for the h-p finite element method. After the finite element solution is obtained postprocessing done elementwise and finite element correction were carried out so that convergence rate can be improved globally. Error estimates for h-p approximation were proved for the heat equation. After h-p approximation the solution u_{hp} was corrected by postprocessing technique using Lobatto polynomials. Finally based on the corrected solution the error estimate shows improved rate of convergence globally.

MSC: 65N30 • 65N12

Keywords: Superconvergence • Lobatto • osteriori correction

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1. Introduction

Superconvergence in the finite element method (FEM) is a phenomenon, where the order of convergence of the finite element error, at certain special points in an element, is higher than the order of convergence of the maximum of the finite element error over that element. These special points are called natural superconvergence points. This phenomenon was first addressed in superconvergence of standard and mixed finite element is well known and practically useful topic in finite element analysis. We call a finite element method to be superconvergent, if at special points the rate of convergence is higher leading to higher global rate of convergence than the usual one Doughlas [1]-[4]. The investigations regarding finite element superconvergence has a long history since the 1970s. For the literature, the reader is referred to books [5]-[10]. Mostly superconvergence analysis are done for the h version of the FEM. There are only few studies done on the p-version and h-p version of the finite element method [14]-[17].

Many postprocessing methods were suggested to achieve higher rate of convergence. Dupont [18],Zhang [19], Zhu and Zhao [20], Chen et. al [21] were some of them to propose certain type of correction methods. But these superconvergence methods are basically based on h version. Zhang [22] proposed superconvergence for spectral collocation methods for p version finite element method. Guo [26] studied the superconvergence of $h-p$ version FEM for two-point boundary value problem. Numerical solutions to unsteady problems were analysed by different numerical techniques by Parag [23], Najat [24] and Fakhrodin [25]. In this paper, $h-p$ version of superconvergence for parabolic problem have been analysed. We generalise the correction procedure proposed by Guo [26].

2. Model problem and h-p finite element discretization

We consider the following 1-D heat equation:

$$\begin{cases} u_t = u_{xx} + f(x,t) \text{ in } (a,b)\times(0,T), \\ u(a) = u(b) = 0, u(x,0) = u_0(x). \end{cases} \tag{1}$$

* *E-mail address:* thanndavam@iist.ac.in

We obtain a variational formulation of the heat equation: Let $I=(a,b)$ given $f\in L^2(I)\times(0,T]$, for any $t>0$, find $u(\cdot,t)\in H_0^1(I), u_t\in L^2(I)$ such that

$$\langle u_t, v\rangle + a(u,v) = \langle f, v\rangle,\ \forall\, v\in H_0^1(I) \tag{2}$$

where $a(u,v)=\int\limits_I u'\,v'\,dx$ and $\langle f,v\rangle=\int\limits_I f\,v\,dx$. We then introduce the Sobolev space for time dependent functions

$$L^q(0,T;H^k(I)) := \{u(x,t)\mid ||u||_{L^q(0,T;H^k(I))} := \left(\int_0^T ||u(\cdot,t)||_k^q\,dt\right)^{\frac{1}{q}} < \infty\}$$

Given $f\in L^2(0,T;H^{-1}(I))$ and $u_0\in H_0^1(I)$, find $u\in L^2(0,T;H_0^1(I))$ such that

$$\begin{cases}\langle u_t, v\rangle + a(u,v) = \langle f,v\rangle,\ \forall\, v\in H_0^1(I)\ \text{and}\ t\in(0,T)\\ u(\cdot,0)=u_0\end{cases} \tag{3}$$

This problem is well-posed. For existence and uniqueness refer to Evans [27].

2.1. Semi-discretization in space

Let $\{\Im_h, h\to 0\}$ be a family of triangulations on I. The semi-discretized finite element method is: given by $f\in V_h'\times(0,T]$, $u_{0,h}\in V_h$, find $u_h\in L^2(0,T;V_h)$ such that

$$\begin{cases}\langle \partial_t u_h, v_h + a(u_h,v_h) = \langle f, v_h\rangle,\ \forall v\in V_h,\ t\in\mathbb{R}^+.\\ u_h(\cdot,0)=u_{0,h}\end{cases} \tag{4}$$

We can expand $u_h=\sum\limits_{i=1}^{N} u_i(t)\,\phi_i(x)$, where ϕ_i be a basis for V_h. The solution u_h can be computed by solving an ODE system

$$\dot{\mathbf{u}}+\mathbf{A}\mathbf{u}=\mathbf{f}$$

where $\mathbf{u}=(u_1,u_2,\dots u_N)^t$, $\mathbf{A}$ is the stiffness matrix, and $\mathbf{f}=(f_1,f_2,\dots,f_N)^t$.

2.2. Legendre and Lobatto polynomials

Let

$$L_n(\xi)=\frac{1}{2^n\,n!}\frac{d^n}{d\xi^n}[(\xi^2-1)^n],\ n\geqslant 0$$

be the Legendre polynomials which forms an orthogonal basis of $L^2[-1,1]$. By the mapping of standard element to an arbitrary element these polynomials can be scaled to $L^2(e)$. The scaled versions of Lobatto polynomials are $\omega_0(x)=1$, $\omega_1(x)=\sqrt{\frac{1}{2}}h_e^{-1}(x-x_e+h_e)$ and

$$\omega_i(x)=h_e^{\frac{1}{2}}\left(\frac{1}{\sqrt{(2i-1)(2i+1)}}\widetilde{L_i}(x)-\frac{1}{\sqrt{(2i-1)(2i-3)}}\widetilde{L_{i-2}}(x)\right),\ i\geqslant 2$$

where $\widetilde{L}_i(x)$ is the scaled version of the Legendre polynomial.

Lemma 2.1.
The Legendre $\{\widetilde{L_i}(x)\}$ and Lobatto polynomials $\{\omega_i(x)\}$ satisfy the following properties:

(1) $\langle\widetilde{L_i},\widetilde{L_j}\rangle_e=\delta_{ij}$, $\omega_{i+1}'(x)=h_e^{-\frac{1}{2}}\widetilde{L_i}(x)$,

(2) $\omega_i(x_e\pm h_e)=0$, $i\geqslant 2$,

(3) $\langle\omega_i,\varphi_{i-3}\rangle_e=0$, $\forall\varphi_{i-3}\in P_{i-3}(e)$, $i\geqslant 3$,

(4) $||\omega_i||^2_{L^2(e)}=\dfrac{2h_e}{(2i-3)(2i+1)}$, $i\geqslant 2$

For the proof refer to paper Guo [26].

2.3. Projection operators

Let $\widetilde{\Pi}_{p_e}$ be the element projection operator such that

$$\widetilde{\Pi}_{p_e} : H^1(e) \to P_{p_e}(e),\ \widetilde{\Pi}_{p_e} u(x) = \sum_{j=0}^{p_e} \beta_j \omega_j(x),\ \forall x \in e$$

and Π_p be the global projection operator such that

$$\Pi_p : H^1(I) \to V,\ \Pi_p u|_e = \widetilde{\Pi}_{p_e} u,\ \forall e \in \Im^h.$$

3. Error estimates

Lemma 3.1.
Let $I = (a,b)$ be an interval and let $\Im_h$ be any mesh in I. Assume that $u \in H^1(I)$ satisfies $u' \in H^k(e)$ with integer $k \geqslant 0$. Then there holds

$$||u - \widetilde{\Pi}_{p_e} u||_{m,e} \leqslant C \frac{h_e^{\mu+1-m}}{p_e^{k+1-m}} ||u||_{k+1,e},\ m = 0,1 \tag{5}$$

where $\mu = min\{p_e, k\}$

For the proof refer to Guo [26].

Theorem 3.1.
Let $\Im_h$ be a quasi-uniform partition over I and let V_h be the corresponding finite element space over this partition with uniform degree p. Then the finite element solution u_{hp} satisfy the following error estimate

$$\begin{aligned} ||u - u_{hp}||_{L^\infty([0,T];H^m(I))} &\leqslant ||u_0 - u_{hp,0}||_{k+1,I} \\ &+ C\frac{h^{\mu+1-m}}{p^{k+1-m}} \left(||u_0||_{k+1,I} + ||u_t||_{L^1((0,T),H^{k+1}(I))}\right) \end{aligned} \tag{6}$$

where $\mu = min\{p,k\}$ and $m = 0,1$

Proof. We can prove for the case $m = 0$. Using Lemma (3.1), we can get the L^2-estimate

$$||u(t) - \Pi_p u(t)||_{L^2(I)} \leqslant C\frac{h^{\mu+1}}{p^{k+1}} ||u(t)||_{k+1,I},\ \forall t \in [0,T] \tag{7}$$

Let us perform the analysis by comparing u_h not directly to u, but rather to an appropriate representative $w_h \in C^1([0,T], V_h)$. For w_h we choose the elliptic projection of u, defined by

$$a(w_h, v) = a(u, v),\ v \in V_h,\ 0 \leqslant t \leqslant T.$$

Using the estimate Eq (7) we have the following estimate

$$||u(t) - w_h(t))||_{L^2(I)} \leqslant C\frac{h^{\mu+1}}{p^{k+1}} ||u(t)||_{k+1,I},\ \forall t \in [0,T] \tag{8}$$

If we differentiate the Eq (8), we see that $\frac{\partial w_h}{\partial t}$ is the elliptic projection of $\frac{\partial u}{\partial t}$, so

$$||\frac{\partial u}{\partial t}(t) - \frac{\partial w_h}{\partial t}(t)|| \leqslant C\frac{h^{\mu+1}}{p^{k+1}} ||\frac{\partial u}{\partial t}(t)||_{k+1},\ t \in [0,T]$$

$$\begin{aligned} \langle\frac{\partial w_h}{\partial t}, v\rangle + a(w_h, v) &= \langle\frac{\partial w_h}{\partial t}, v\rangle + a(u, v) \\ &= \langle\frac{\partial (w_h - u)}{\partial t}, v\rangle + \langle f, v\rangle,\ v \in V_h,\ 0 \leqslant t \leqslant T. \end{aligned} \tag{9}$$

Let $e_h = w_h - u_{hp}$. Substracting Eq (3) from Eq (9), we get

$$\langle\frac{\partial e_h}{\partial t}, v\rangle + a(e_h, v) = \langle\frac{\partial (w_h - u)}{\partial t}, v\rangle,\ v \in V_h, 0 \leqslant t \leqslant T.$$

For each t, let $v = e_h(t) \in V_h$.

$$||e_h||\frac{d}{dt}||e_h|| + a(e_h, e_h) = \langle\frac{\partial(w_h - u)}{\partial t}, e_h\rangle \leqslant ||\frac{\partial(w_h - u)}{\partial t}||\,||e_h||, \tag{10}$$

$$\frac{d}{dt}||e_h|| \leqslant ||\frac{\partial(w_h - u)}{\partial t}|| \leqslant C\frac{h^{\mu+1}}{p^{k+1}}||\frac{\partial u}{\partial t}(t)||_{k+1}$$

Integrating over $[0, t]$ we get

$$||e_h(t)|| \leqslant ||e_h(0)|| + C\frac{h^{\mu+1}}{p^{k+1}}||u_t||_{L^1([0,T];H^{k+1}(I))}$$

$$\begin{aligned}||e_h(0)|| &= ||w_h(0) - u_{hp}(0)||\\ &\leqslant ||w_h(0) - u(0)|| + ||u_0 - u_{hp}(0)||\\ &\leqslant C\frac{h^{\mu+1}}{p^{k+1}}||u_0||_{k+1} + ||u_0 - u_h(0)||.\end{aligned}$$

Assuming sufficient smoothness on exact solution and initial data we get estimate in terms of $w_h - u_{hp}$. Using the estimate Eq (7) and triangle inequality we get the estimate Eq (6). Same analysis can be done for the case $m = 1$. □

4. Finite element correction

We can improve the global convergence rate by correction scheme after computing the finite element solution u_{hp}. For $u(\cdot, t) \in H^{k+1}(I)$, with $k \geqslant 1$, $\forall t$ then $u_{xx} = u_t - f$. We can get

$$\begin{aligned}\beta_l(t) &= h_e^{\frac{1}{2}}\langle u_x, \widetilde{L_{l-1}}\rangle_e\\ &= -h_e\langle u_{xx}, \omega_l\rangle_e = h_e\langle f - u_t, \omega_l\rangle_e\\ &= h_e\langle f - \partial_t u_{hp}, \omega_l\rangle_e + h_e\langle\partial_t u_{hp} - u_t, \omega_l\rangle_e,\ l \geqslant 2, \forall t\end{aligned}$$

Define $\beta_l^*(t) = h_e\langle f - \partial_t u_{hp}, \omega_l\rangle_e$. Then

$$\begin{aligned}|\beta_l(t) - \beta_l^*(t)| &= |h_e\langle\partial_t u_{hp} - u_t, \omega_l\rangle_e|\\ &\leqslant C h_e||\partial_t(u_{hp} - u)(t)||_{0,e}||\omega_l||_{0,e}\end{aligned}$$

Let

$$u_{hp}^*(x, t) = u_{hp}(x, t) + \sum_{l=p+1}^{p*}\beta_l^*(t)\omega_l(x),\ \forall x \in e, \forall e \in \Im_h, \forall t$$

where $p* \geqslant p + 1$, and u_{hp}^* is the corrected value of u_{hp} which can be calculated. This correction requires $u(\cdot, t) \in H^{k+1}(I)$.

Theorem 4.1.
Let $\Im_h$ be a quasi-uniform partition over the interval I and let V_h be the corresponding finite element space over this partition with uniform degree p. Let u_{hp}^* be the finite element correction defined above and $u \in H^{k+1}(I)$, $p* = [p^{1+\sigma}]$ be the smallest integer no less than $p^{1+\sigma}$ with $0 < \sigma < 2$, then

$$||u(t) - u_{hp}^*(t)||_{0,I} \leqslant C\left[\left(\frac{h^{\mu*+1}}{p^{(k+1)(1+\sigma)}} + \frac{h^{\mu+3}}{p^{k+3}}\right)||u(t)||_{k+1,I} + \frac{h^{\mu+3}}{p^{k+2}}||u_t(t)||_{k+1,I}\right] \tag{11}$$

Proof.

$$\begin{aligned}u(x, t) - u_{hp}^*(x, t) &= (u - \widetilde{\Pi}_{p*})(x, t) + (\widetilde{\Pi}_p - u_{hp})(x, t)\\ &\quad + \left(\widetilde{\Pi_{p*}}u - \widetilde{\Pi}_p u - \sum_{l=p+1}^{p*}\beta_l^*(t)\omega_l\right)(x, t), \forall t\end{aligned} \tag{12}$$

Using Lemma (3.1)

$$||u(t) - \widetilde{\Pi}_{p*}u(t)||_{m,e} \leqslant C\frac{h^{\mu*+1-m}}{p*^{k+1-m}}||u(t)||_{k+1,e},\ m = 0, 1$$

where $\mu* = \min\{p*, k\}$.

$$\begin{aligned} ||\widetilde{\Pi_{p*}}u(t) - \widetilde{\Pi}_p u(t) - \sum_{l=p+1}^{p*} \beta_l^*(t)\omega_l||_{0,e} &= ||\sum_{l=p+1}^{p*}(\beta_l(t) - \beta_l^*(t))\omega_l||_{0,e} \\ &\leqslant C h_e \sum_{l=p+1}^{p*} ||\omega_l||_{0,e}^2 ||\partial_t(u_{hp} - u)(t)||_{0,e} \\ &\leqslant C \frac{h_e^2}{p} ||\partial_t(u_{hp} - u)(t)||_{0,e}, \ \forall t \end{aligned}$$

Using this in the expression Eq (12) for the last term and substituting expressions for the other two terms from previous results. We get the desired estimate. □

5. Conclusion

In this paper we studied the h-p version of FEM for parabolic partial differential equations. Postprocessing technique is applied to h-p version of the FEM and the solution is corrected. Error estimates shows the improved order of convergence for u_{hp}^* compared with u_{hp}. Future scope of the work includes generalization of these results to higher dimensional problems.

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