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First integrals and exact solutions for path equation describing minimum drag work

Research Article

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Abstract: In this work, we consider the path equation which models the drag forces of object moving in a fluid medium. The drag forces are the major source of energy loss for objects moving in a fluid medium. The Lie symmetry approach with Prelle-Singer method is used to construct first integrals of the path equation. Using the first integrals, the exact solutions are also determined.

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Keywords: Minimum drag work • Lie symmetry • First integrals • Exact solutions

1. Introduction

Conservation laws are the key instruments in the obtaining the physical and mathematical properties of the dynamical models.The conserved quantity is called first integral, which is the analogous of conservation laws for non-linear ordinary differential equations (NLODEs) models [1]. Much effort has been made on the construction of first integrals and exact solutions of the NLODEs. These nonlinear equations have been studied by using various analytical and numerical methods, such as Laplace decompositon method [2], Adomian decomposition method [3], homotopy perturbation method [4], differential transformation method [5], Pade approximanats [6], variational method [7], Lie group analysis [8], and so on.

As stated in the work of Pakdemirli [9], drag forces are the major source of energy loss for objects moving in a fluid medium. Minimization of work due to drag force may reduce fuel consumption. This can be achieved by the selection of the optimum path. The drag force depends on the density of fluid, the drag coefficient, the cross-sectional area and the velocity. These parameters are the combination of the altitude-dependent parameters which can be expressed as a single arbitrary functions.

In [9], using the variational calculus Pakdemirli derived the differential equation describing the path for minimum drag work. With the Taylor series expansion, he obtained approximate analytical solution.

In the literature, we observe also some studies on path equation. In [10], the authors consider the density of fluid as exponentially decaying with altitude. The equation is cast into a dimensionless form and exact solution is given. The authors then analyzed the equation by homotopy analysis method. The first work on Lie group analysis for path equation is done by Pakdemirli and Aksoy [11]. They presented group classification with respect to an altitude-dependent arbitrary function. In addition, using the symmetries they obtained symmetry reductions and group-invariant solutions. Very recently, Gün and Özer [12] obtained integrating factors, first integrals and group invariant solutions using the λ-symmetry and Noether-symmetry methods.

On the other hand, we observe some very important works which incorporates the Lie group analysis with the other approaches existing in the literature. For the detailed reviews of the methods existing literature, please see [1].

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As emphasized in [13] the most versatile and widely used mathematical tools to identify integrable systems belonging to ODEs are Lie symmetry analysis, Darboux polynomials, PS method, λ-symmetries method and JLMs. In that paper the authors establish a relationship between extended PS procedure with all other methods cited above. To achieve this purpose they start their investigations with the extended PS procedure. However in this work our starting point is Lie characteristic function.

The plan of the paper is as follows. In section 2, we describe the PS procedure for solving second order ODEs. However, without solving the determining equations we use one identity which suggested in [13]. Therefore we don't need solving determining equations for getting null forms. Morever, we exhibit the connection between Lie point symmetries and JLMs. Again, we easily obtain integrating factors using the one known identity without solving determining equations. In section 3, we apply these methods and relationships to path equation. In the last section, we summarize and compare our results with those gained by other methods.

2. A short analysis of the Prelle Singer and Lie group methods

We first present notation to be used and recall the definitions that appear in [13, 14].

In this section, first we briefly exhibit the key steps of the modified PS procedure for second order ODEs (for the detailed discussions, please see [14]). Let us consider second-order ODEs of the form

$$\ddot{x} = \phi(t, x, \dot{x}), \tag{1}$$

where over dot denotes differentiation with respect to time. The determining equations for the R and S functions which is integrating factor and null form, respectively is the given in the following:

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{2}$$

$$D[R] = -R\left(S + \phi_{\dot{x}}\right), \tag{3}$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \tag{4}$$

where

$$D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}$$

Once a compatible solution satisfying all the three equations have been found, then functions R and S fix the integral of motion $I\left(t, x, \dot{x}\right)$ by the relation

$$I(t,x,\dot{x}) = \int R\left(\phi + \dot{x}S\right)dt - \int\left(RS + \frac{d}{dx}\int R\left(\phi + \dot{x}S\right)dt\right)dx$$
$$-\int\left\{R + \frac{d}{d\dot{x}}\left[\begin{array}{c}\int R\left(\phi + \dot{x}S\right)dt \\ -\int\left(RS + \frac{d}{dx}\int R\left(\phi + \dot{x}S\right)dt\right)dx\end{array}\right]\right\}d\dot{x}. \tag{5}$$

Let $v = \xi\partial_t + \eta\partial_x$ be the Lie symmetry generator of Eq.(1), when $\xi(t,x)$ and $\eta(t,x)$ are infinitesimals associated with the t and x variables. Then the characteristic of v is given by $Q = \eta - \dot{x}\xi$. Because of the order of Eq.(1) is the second, the infinitesimals operator associated with the vector field v is $v^{(2)} = \xi\partial_t + \eta\partial_x + \eta^{(1)}\partial_{\dot{x}} + \eta^{(2)}\partial_{\ddot{x}}$, where $\eta^{(1)}$ and $\eta^{(2)}$ are the first and second prolongations of the vector field v and are given by $\eta^{(1)} = \dot{\eta} - \dot{x}\dot{\xi}$ and $\eta^{(2)} = \ddot{\eta} - \ddot{x}\dot{\xi}$, where over dot denotes total differentiation with respect to t.

The invariance condition of Eq.(1), $\ddot{x} = \phi(t,x,\dot{x})$, determines the infinitesimal functions ξ and η explicitly, through the condition $v^{(2)}\left[\ddot{x} = \phi(t,x,\dot{x})\right] = 0$. Expanding the latter, one finds the invariance condition in terms of the above evolutionary vector field Q as

$$D^2[Q] = \phi_{\dot{x}}D[Q] + \phi_x Q. \tag{6}$$

In [13], the autors suggested the following identity

$$S = -\frac{D[X]}{X} \tag{7}$$

for demonstrating the relationship between null form and characteristic function of the Lie point generator. Substituting the Eq.(7) to Eq.(2) we obtain

$$D^2[X] = \phi_{\dot{x}}D[X] + \phi_x X. \tag{8}$$

Comparing Eqs.(8) and (6) we find that

$$X = Q. \tag{9}$$

Since $S = -\frac{D[Q]}{Q}$, the null form S can be determined very easily when ξ and η are known.

It is well known that all the nonlinear ODEs do not necessarily admit Lie point symmetries. Under such a situation one may look for λ-symmetries associated with the given equations [15].

Now, if one replace $S = -\lambda$ in Eq.(2) one get

$$D^2[\lambda] = \phi_x + \lambda\phi_{\dot{x}} - \lambda^2, \tag{10}$$

which is nothing but the determining equation for the λ-symmetries for a second order ODE which in turn establishes the connection between λ-symmetries and null forms.

On the other hand one of the important result for studying the second order ODEs is the Jacobi Last Multiplier [16]. The JLM function of $M = \frac{1}{\Delta}$ for the Eq.(1), is defined by the following determinant

$$\Delta = \begin{vmatrix} 1 & \dot{x} & \ddot{x} \\ \xi_1 & \eta_1 & \eta_1^{(1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} \end{vmatrix} \tag{11}$$

where (ξ_1, η_1) and (ξ_2, η_2) are two sets of Lie point symmetries of the second order ODE and $\eta_1^{(1)}$ and $\eta_2^{(1)}$ are the corresponding prolongations. The determinant establishes the connection between the multiplier and Lie point symmetries. It is demonstrated in [13], the relationship between integrating factor R and JLM function M is the following:

$$R = QM \tag{12}$$

3. First integrals and exact solutions of the path equation

The differential equation describing the path of the minimum drag work is given in the form

$$\ddot{x} - \frac{f'(x)}{f(x)}(1 + \dot{x}^2) = 0. \tag{13}$$

where $x = x(t)$ is the altitude function [9, 12].

In this section we examine some different forms of $f(x)$ which obtained in the work of Pakdemirli et al. [11].

3.1. $f(x) = \frac{1}{k_1 x + k_2}$

In this case the equation is given by

$$\ddot{x} = -\frac{k_1}{k_1 x + k_2}\left(1 + \dot{x}^2\right). \tag{14}$$

We try to solve Eq.(8) which is equivalent to solving Eq.(6). We start our analysis by solving the Eq.(6) for the characteristics $Q = \eta - \dot{x}\xi$, using the relation $Q = X$. Substituting in Eq.(6) and equating the various powers of $\dot{x}$, we get a set of partial differential equations for ξ and η. Solving them consistently we find explicit expressions for ξ and η [11]. In our case Eq.(14) admits eight dimensional Lie point symmetries. The corresponding vector fields are:

$$V_1 = \left(\frac{t}{2}(k_1 x^2 + 2k_2 x + k_1 t^2)\right)\partial_t + \left(\frac{(k_1 x^2 + 2k_2 x)^2 - k_1^2 t^4}{4(k_1 x + k_2)}\right)\partial_x,$$

$$V_2 = \left(\frac{1}{2}k_1 x^2 + k_2 x\right)\partial_t + \left(\frac{-k_1 t(3k_1 x^2 + 6k_2 x + k_1 t^2)}{4(k_1 x + k_2)}\right)\partial_x,$$

$$V_3 = \frac{1}{2}t^2\partial_t + \left(\frac{t(k_1 x^2 + 2k_2 x - k_1 t^2)}{4(k_1 x + k_2)}\right)\partial_x,$$

$$V_4 = t\partial_t + \left(\frac{k_1 x^2 + 2k_2 x}{k_1 x + k_2}\right)\partial_x,$$

$$V_5 = \partial_t$$

$$V_6 = \left(\frac{k_1 x^2 + 2k_2 x + k_1 t^2}{2(k_1 x + k_2)}\right)\partial_x,$$

$$V_7 = \left(\frac{t}{k_1 x + k_2}\right)\partial_x,$$

$$V_8 = \frac{1}{k_1 x + k_2}\partial_x.$$

Since we are dealing with a second order ODE, we consider any two vector fields to generate all other factors. For illustration, we consider the vector fields V_3 and V_6 in the following. The results which arise from other pairs of symmetry generators are summarized in Table 1 to Table 3

From the vector fields V_3 and V_8 one can identify two sets of infinitisemals ξ and η as

$$\xi_3 = \frac{1}{2}t^2, \quad \eta_3 = \frac{t(k_1x^2+2k_2x-k_1t^2)}{4(k_1x+k_2)}$$
$$\xi_8 = 0, \quad \eta_8 = \frac{1}{k_1x+k_2}.$$

The associated characteristics $Q_i = \eta_i - \dot{x}\xi_i$, $i = 3,6$, are found to be

$$Q_3 = -\frac{t\left(-k_1x^2-2k_2x+k_1t^2+2k_1tx\dot{x}+2k_2t\dot{x}\right)}{4\left(k_1x+k_2\right)},$$
$$Q_8 = \frac{1}{k_1x+k_2}.$$

Recalling the relation Eqs.(7) and (9) null forms S_3 and S_8 can be readily found. Our analysis shows that

$$S_3 = \frac{k_1\dot{x}}{k_1x+k_2} - \frac{1}{t},$$
$$S_8 = -\frac{k_1\dot{x}}{k_1x+k_2}.$$

One can easily check that S_3 and S_8 are two particular solutions of Eq.(2).

Using the relation $S = -\lambda$ we find

$$\lambda_3 = -S_3 = -\frac{k_1x}{k_1x+k_2} + \frac{1}{t},$$
$$\lambda_8 = -S_8 = \frac{k_1\dot{x}}{k_1x+k_2}.$$

It is readily seen that the λ_i's $i = 3,8$, indeed satisfy Eq.(10).

As we noted earlier the integrating factors can be derived by constructing the JLMs. Since we already derived Lie point symmetries of Eq.(13), we can exploit the connection between Lie point symmetries and JLMs (M). For this purpose let us evaluate the multiplier M which is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where Δ is given by the Eq.(11). Since we need two Lie symmetries to evaluate JLM (see Eq.(11)), we choose again the vector fields V_3 and V_8, to obtain first JLM M_{38}. Evaluating the associated determinant with these two vector fields, we find

$$\Delta_{38} = \begin{vmatrix} 1 & \dot{x} & -\dfrac{k_1(1+\dot{x}^2)}{k_1x+k_2} \\ \dfrac{t^2}{2} & \dfrac{t(k_1x^2+2k_2x-k_1t^2)}{4(k_1x+k_2)} & \dfrac{k_1^2(x^3-3t^2x-3tx^2\dot{x}+t^3\dot{x})+3k_1k_2(x^2-t^2-2tx\dot{x})+2k_2^2(x-t\dot{x})}{4(k_1x+k_2)^2} \\ 0 & \dfrac{1}{k_1x+k_2} & -\dfrac{k_1\dot{x}}{(k_1x+k_2)^2} \end{vmatrix}$$
$$= -\frac{\left(\begin{array}{c} k_1^2x^4-4k_1^2tx^3\dot{x}-2k_1^2t^2x^2+4k_1^2t^2x^2\dot{x}^2+4k_1^2t^3x\dot{x}+k_1^2t^4+4k_1k_2x^3 \\ -12k_1k_2tx^2\dot{x}-8k_1k_2t^2x\dot{x}^2-4k_1k_2t^2x+4k_1k_2t^3\dot{x}+4k_2^2x^2-8k_2^2tx\dot{x}+4k_2^2t^2\dot{x}^2 \end{array}\right)}{8\left(k_1x+k_2\right)^2},$$

from which we obtain

$$M_{38} = \frac{4(k_1x+k_2)^2}{k_1t^2-k_1x^2+2k_1tx\dot{x}+2k_2t\dot{x}-2k_2x}.$$

Now, we construct integrating factors using the Eq.(12). The integrating factor R_{38} which is corresponding the vector fields of V_3 and V_8, is given in the following

$$R_{38} = Q_3M_{38} = -t(k_1x+k_2).$$

Therefore one obtains, the corresponding first integral

$$I_{38} = k_1tx\dot{x} + \frac{k_1t2}{2} + k_2t\dot{x} - \frac{k_1x2}{2} - k_2x \tag{15}$$

by the Eq.(5) for the null form S_3 and integrating factor R_{38}.

Equating the Eq.(15) to C where is constant and after integrating the equation we obtain the following exact solution

$$x(t) = \frac{-k_2 \mp \sqrt{k_2^2-k_1^2t^2-2k_1I_{38}+2Ck_1t}}{k_1} \tag{16}$$

In the sequel, we give the results in the Table 4 to Table 9 corresponding to some special cases of $f(x)$. We note that in the tables, some first integrals does not appear because of the compability condition namely Eq.(4) does not hold. In addition, some exact solution rows demonstrated with empty lines because of the difficulties solving of the first integrals.

Table 1. Vector fields, Charactaristics and the Null Forms for $f(x) = \frac{1}{k_1 x + k_2}$

V	$Q = \eta - \dot{x}\xi$	$S = -\frac{D[Q]}{Q}$
V_1	$Q_1 = -\frac{(k_1 x^2 + 2k_2 x + k_1 t^2)(k_1 t^2 - k_1 x^2 - 2k_2 x + 2k_1 t x\dot{x} + 2k_2 t\dot{x})}{4(k_1 x + k_2)}$	$S_1 = S_6 = \frac{k_1^2(t^2\dot{x} - 2tx - x^2\dot{x}) - 2k_1 k_2(t + x\dot{x}) - 2k_2^2\dot{x}}{(k_1 x^2 + 2k_2 x + k_1 t^2)(k_1 x + k_2)}$
V_6	$Q_6 = \frac{k_1 x^2 + 2k_2 x + k_1 t^2}{2(k_1 x + k_2)}$	
V_2	$Q_2 = -\frac{k_1^2(3tx^2 + t^3 + 2x^3\dot{x}) + 6k_1 k_2 x(t + x\dot{x}) + 4k_2^2 x\dot{x}}{4(k_1 x + k_2)}$	$S_2 = \frac{\begin{pmatrix} k_1^3(x^3 + 3t^2 x + 3tx^2\dot{x} + 2x^3\dot{x}^2 - t^3\dot{x}) \\ +3k_1^2 k_2(x^2 + t^2 + 2tx\dot{x} + 2x^2\dot{x}^2) \\ +2k_1 k_2^2(x + 3t\dot{x} + 4x\dot{x}^2) + 4k_2^3\dot{x}^2 \end{pmatrix}}{\left(k_1^2(3tx^2 + t^3 + 2x\dot{x}) + k_1 k_2(6tx + 6x^2\dot{x}) + 4k_2^2 x\dot{x}\right)(k_1 x + k_2)}$
V_3	$Q_3 = -\frac{t\left(-k_1 x^2 - 2k_2 x + k_1 t^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x}\right)}{4(k_1 x + k_2)}$	$S_3 = S_7 = -\frac{1}{t} + \frac{k_1\dot{x}}{k_1 x + k_2}$
V_7	$Q_7 = \frac{t}{k_1 x + k_2}$	
V_4	$Q_4 = \frac{k_1 x^2 + 2k_2 x}{k_1 x + k_2} - t\dot{x}$	$S_4 = \frac{k_2^2\dot{x} + k_1^2(tx\dot{x}^2 + tx) + k_1 k_2(t\dot{x}^2 + t)}{\left(k_1(-x^2 + tx\dot{x}) + k_2(-2x + t\dot{x})\right)(k_1 x + k_2)}$
V_5	$Q_5 = -\dot{x}$	$S_5 = \frac{k_1(1 + \dot{x}^2)}{\dot{x}(k_1 x + k_2)}$
V_8	$Q_8 = \frac{1}{k_1 x + k_2}$	$S_8 = \frac{k_1\dot{x}}{k_1 x + k_2}$

Table 2. Characteristics, JLMs and Integrating Factors for $f(x) = \frac{1}{k_1 x + k_2}$

Q	$M = \frac{1}{\Delta}$	$R = QM$
Q_1	$M_{12} = \frac{-M_{13}}{k_1} = \frac{16(k_1 x + k_2)^2}{k_1(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)^3}$	$R_{12} = \frac{-R_{13}}{k_1} = \frac{4(k_1 x^2 + 2k_2 x + k_1 t^2)(k_1 x + k_2)}{k_1(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)^2}$
	$M_{14} = -\frac{4(k_1 x + k_2)^2}{(k_1 t + k_1 x\dot{x} + k_2\dot{x})(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)}$	$R_{14} = \frac{(k_1 x^2 + 2k_2 x + k_1 t^2)(k_1 x + k_2)}{(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)(k_1 t + k_1 x\dot{x} + k_2\dot{x})}$
Q_2	$M_{24} = \frac{4(k_1 x + k_2)2}{\left(\frac{(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)}{(k_1^2(2x^2\dot{x}^2 + x^2 + t^2 + 2tx\dot{x}) + 2k_1 k_2(2x\dot{x}^2 + x + t\dot{x}) + 2k_2^2\dot{x}^2)}\right)}$	$R_{24} = \frac{(k_1 x + k_2)\left(k_1^2(3tx^2 + t.33 + 2x^3\dot{x}) + 6k_1 k_2 x(t + x\dot{x}) + 4k_2^2 x\dot{x}\right)}{\left(\frac{(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)}{(k_1^2(2x^2\dot{x}^2 + x^2 + t^2 + 2tx\dot{x}) + 2k_1 k_2(2x\dot{x}^2 + x + t\dot{x}) + 2k_2^2\dot{x}^2)}\right)}$
Q_3	$M_{34} = \frac{M_{36}}{2} = -\frac{4(k_1 x + k_2)^2}{(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)^2}$	$R_{34} = \frac{R_{36}}{2} = \frac{t(k_1 x + k_2)}{k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x}$
Q_7	$M_{38} = \frac{4(k_1 x + k_2)^2}{k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x}$	$R_{38} = R_{78} = -t(k_1 x + k_2)$
	$M_{78} = -(k_1 x + k_2)^2$	
Q_4	$M_{46} = -\frac{2(k_1 x + k_2)^2}{(k_1 t + k_1 x\dot{x} + k_2\dot{x})(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_3 x)}$	$R_{46} = \frac{2(k_1 t x\dot{x} + k_2 t\dot{x} - k_1 x^2 - 2k_2 x)(k_1 x + k_2)}{(k_1 t + k_1 x\dot{x} + k_2\dot{x})(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_3 x)}$
Q_5	$M_{58} = -\frac{(k_1 x + k_2)^2}{k_1}$	$R_{58} = \frac{\dot{x}(k_1 x + k_2)^2}{k_1}$

Table 3. The Null Forms, Integrating Factors, First Integrals and Exact Solutions for $f(x) = \frac{1}{k_1 x + k_2}$

S	R	I^a	$x(t)$
S_1	$R_{12} = -\frac{R_{13}}{k_1}$	$I_{12} = -\frac{I_{13}}{k_1} = -\frac{4(k_1 t + k_1 x\dot{x} + k_2\dot{x})}{k_1(k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x)}$	$x_{12}(t) = \frac{-k_2 I_{12} \mp \sqrt{k_2^2 I_{12}^2 - k_1^2 I_{12}^2 t^2 - 2k_1 I_{12} t - 2C_{12} k_1 I_{12}^2 - 4 - C_{12} k_1^2 I_{12}^3 t}}{k_1 I_{12}}$
	R_{14}	$I_{14} = \ln\lvert\frac{-k_1 I_{12}}{4}\rvert$	
S_2	R_{24}	$I_{24} = \frac{1}{2}\ln\left(\frac{\left(\begin{array}{c} k_1^2(2x^2\dot{x}^2 + x^2 + t^2 + 2tx\dot{x}) \\ +2k_1 k_2(2x\dot{x}^2 + x + t\dot{x}) + 2k_2^2\dot{x}^2 \end{array}\right)}{k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x}\right)$	
$S_3 = S_7$	$R_{34} = \frac{R_{36}}{2}$	$I_{34} = -\frac{1}{2}\ln(2I_{38}) = \frac{I_{36}}{2}$	$x_{38}(t) = \frac{-k_2 \mp \sqrt{k_2^2 - k_1^2 t^2 - 2k_1 I_{38} + 2C_{38} k_1 t}}{k_1}$
	$R_{38} = R_{78}$	$I_{38} = I_{78} = \frac{k_1 t^2}{2} - \frac{k_1 x^2}{2} + k_1 t x\dot{x} + k_2 t\dot{x} - k_2 x$	
S_4	R_{46}	$I_{46} = \frac{1}{2}\ln\left(\frac{k_1 t^2 - k_1 x^2 + 2k_1 t x\dot{x} + 2k_2 t\dot{x} - 2k_2 x}{k_1 t + k_1 x\dot{x} + k_2\dot{x}}\right)$	
S_5	R_{58}	$I_{58} = -\frac{k_1^2(x^2 + x^2\dot{x}^2) + 2k_1 k_2 x(1 + \dot{x}^2) + k_2^2\dot{x}^2}{2k_1}$	$x_{58}(t) = \frac{-k_2 \mp \sqrt{k_2^2 - 2k_1 I_{58}}}{k_1}$, $x_{58}(t) = \frac{-k_2 \mp \sqrt{k_2^2 - 2k_1 I_{58} + 2C_{58} k_1^2 t - C_{58}^2 k_1^2 - k_1^2 t^2}}{k_1}$

[a] The first integrals of I_{ij} where $i, j = 1, \ldots, 8$.

Table 4. Vector fields, Characteristics and The null forms for $f(x)=k$

V	$Q=\eta-\dot{x}\xi$	$S-\frac{D[Q]}{Q}$
$V_1=tx\partial_t+x^2\partial_x$ $V_2=x\partial_t$ $V_6=x\partial_x$	$Q_1=x^2-tx\dot{x}$ $Q_2=-x\dot{x}$ $Q_6=x$	$S_1=S_2=S_6=-\frac{\dot{x}}{x}$
$V_3=t^2\partial_t+tx\partial_x$ $V_4=t\partial_t$ $V_7=t\partial_x$	$Q_3=tx-t^2\dot{x}$ $Q_4=-t\dot{x}$ $Q_7=t$	$S_3=S_4=S_7=-\frac{1}{t}$
$V_5=\partial_t$ $V_8=\partial_x$	$Q_5=-\dot{x}$ $Q_8=1$	$S_5=S_8=0$

Table 5. Characteristics, JLMs and integrating factors for $f(x)=k$

Q	$M=\frac{1}{\Delta}$	$R=QM$
Q_1 Q_2	$M_{13}=-\frac{1}{(-x+t\dot{x})^3}$ $M_{14}=M_{23}=-\frac{1}{(-x+t\dot{x})^2\dot{x}}$ $M_{15}=M_{24}=-\frac{1}{(-x+t\dot{x})\dot{x}^2}$ $M_{18}=\frac{1}{(-x+t\dot{x})\dot{x}}$ $M_{25}=\frac{1}{\dot{x}^3}$ $M_{28}=\frac{1}{\dot{x}^2}$	$R_{13}=R_{23}=\frac{x}{(-x+t\dot{x})^2}$ $R_{14}=R_{24}=\frac{x}{(-x+t\dot{x})\dot{x}}$ $R_{15}=R_{25}=\frac{x}{\dot{x}^2}$ $R_{18}=R_{28}=-\frac{x}{\dot{x}}$
Q_3 Q_4	$M_{35}=M_{18}$ $M_{36}=-\frac{1}{(-x+t\dot{x})^2}$ $M_{38}=\frac{1}{-x+t\dot{x}}$ $M_{45}=-M_{28}$ $M_{46}=-\frac{1}{\dot{x}(-x+t\dot{x})}$ $M_{48}=\frac{1}{\dot{x}}$	$R_{35}=R_{45}=\frac{t}{\dot{x}}$ $R_{36}=R_{46}=\frac{t}{-x+t\dot{x}}$ $R_{38}=R_{48}=-t$
Q_5	$M_{56}=-\frac{1}{\dot{x}^2}$ $M_{57}=-\frac{1}{\dot{x}}$	$R_{56}=\frac{1}{\dot{x}}$ $R_{57}=1$
Q_6	$M_{67}=M_{38}$ $M_{68}=-\frac{1}{\dot{x}}$	$R_{67}=-\frac{x}{-x+t\dot{x}}$ $R_{68}=R_{18}$
Q_7	$M_{78}=-1$	$R_{78}=-t$

Table 6. The null forms, integrating factors, first integrals and exact solutions for $f(x)=k$

S	R	I^a	$x(t)$
$S_1=S_2=S_6$	$R_{13}=R_{23}$ $R_{14}=R_{24}$ $R_{15}=R_{25}$	$I_{13}=I_{23}=\frac{\dot{x}}{-x+t\dot{x}}$ $I_{14}=I_{24}=\ln(I_{13})$ $I_{15}=I_{25}=-\frac{1}{I_{13}}$	$x_{13}(t)=C_{13}t-\frac{C_{13}}{I_{13}}$
$S_3=S_4=S_7$	$R_{36}=R_{37}=R_{46}=R_{47}$ $R_{38}=R_{48}=R_{78}$	$I_{36}=I_{46}=\ln(\frac{1}{-x+t\dot{x}})$ $I_{38}=I_{48}=I_{78}=-x+t\dot{x}$	$x_{36}(t)=-\frac{C_{36}-t}{C_{36}e^{I_{13}}}$
$S_5=S_8$	R_{56} R_{57}	$I_{56}=\ln(\frac{1}{\dot{x}})$ $I_{57}=-\dot{x}$	$x_{56}(t)=e^{-I_{56}}t+C_{56}$

[a]The first integrals of I_{ij} where $i,j=1,\dots,8$.

Table 7. Vector Fields, Characteristics and Null Forms for $f(x) = ke^{\alpha x}$

V	$Q = \eta - \dot{x}\xi$	$S = -\frac{D[Q]}{Q}$
$V_1 = e^{-\alpha x}\cos(\alpha t)\partial_t + e^{-\alpha x}\sin(\alpha t)\partial_x$ $V_2 = e^{-\alpha x}\sin(\alpha t)\partial_t - e^{-\alpha x}\cos(\alpha t)\partial_x$ $V_6 = \partial_x$	$Q_1 = e^{-\alpha x}(\sin(\alpha t) - \dot{x}\cos(\alpha t))$ $Q_2 = -e^{-\alpha x}(\cos(\alpha t) + \dot{x}\sin(\alpha t))$ $Q_6 = 1$	$S_1 = S_2 = S_6 = 0$
$V_3 = \cos(2\alpha t)\partial_t + \sin(2\alpha t)\partial_x$	$Q_3 = \sin(2\alpha t) - \dot{x}\cos(2\alpha t)$	$S_3 = -\frac{\alpha\left(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t)\right)}{-\sin(2\alpha t) + \dot{x}\cos(2\alpha t)}$
$V_4 = \sin(2\alpha t)\partial_t - \cos(2\alpha t)\partial_x$	$Q_4 = -(\cos(2\alpha t) + \dot{x}\sin(2\alpha t))$	$S_4 = -\frac{\alpha\left(-\sin(2\alpha t) - 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t)\right)}{\cos(2\alpha t) + \dot{x}\sin(2\alpha t)}$
$V_5 = \partial_t$	$Q_5 = -\dot{x}$	$S_5 = -\frac{\alpha(1+\dot{x}^2)}{\dot{x}}$
$V_7 = e^{\alpha x}\cos(\alpha t)\partial_x$	$Q_7 = e^{\alpha x}\cos(\alpha t)$	$S_7 = -\frac{\alpha(-\sin(\alpha t) + \dot{x}\cos(\alpha t))}{\cos(\alpha t)}$
$V_8 = e^{\alpha x}\sin(\alpha t)\partial_x$	$Q_8 = e^{\alpha x}\sin(\alpha t)$	$S_8 = -\frac{\alpha(\cos(\alpha t) + \dot{x}\sin(\alpha t))}{\sin(\alpha t)}$

Table 8. Characteristics, JLMs and integrating factors for $f(x) = ke^{\alpha x}$

Q	$M = \frac{1}{\Delta}$	$R = QM$
Q_1	$M_{13} = \frac{e^{\alpha x}}{\alpha(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t))(-\sin(\alpha t) + \dot{x}\cos(\alpha t))}$ $M_{14} = \frac{e^{\alpha x}}{\alpha(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t))(-\sin(\alpha t) + \dot{x}\cos(\alpha t))}$ $M_{15} = \frac{e^{\alpha x}}{\alpha(1+\dot{x}^2)(-\sin(\alpha t) + \dot{x}\cos(\alpha t))}$	$R_{13} = R_{23} = -\frac{1}{\alpha(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t))}$ $R_{14} = R_{24} = -\frac{1}{\alpha(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t))}$ $R_{15} = R_{25} = -\frac{1}{\alpha(1+\dot{x}^2)}$
Q_2	$M_{23} = \frac{e^{\alpha x}}{\alpha(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t))(\cos(\alpha t) + \dot{x}\sin(\alpha t))}$ $M_{24} = \frac{e^{\alpha x}}{\alpha(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t))(\cos(\alpha t) + \dot{x}\sin(\alpha t))}$ $M_{25} = \frac{e^{\alpha x}}{\alpha\left(\dot{x}^3\sin(\alpha t) + \dot{x}^2\cos(\alpha t) + \dot{x}\sin(\alpha t) + \cos(\alpha t)\right)}$	
Q_3	$M_{36} = \frac{1}{\alpha(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t))}$	$R_{36} = -\frac{-\sin(2\alpha t) + \dot{x}\cos(2\alpha t)}{\alpha(-\cos(2\alpha t) - 2\dot{x}\sin(2\alpha t) + \dot{x}^2\cos(2\alpha t))}$
Q_4	$M_{46} = \frac{1}{\alpha(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t))}$	$R_{46} = -\frac{\cos(2\alpha t) + \dot{x}\sin(2\alpha t)}{\alpha(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t))}$
Q_5	$M_{56} = \frac{1}{\alpha(1+\dot{x}^2)}$	$R_{56} = -\frac{\dot{x}}{\alpha(1+\dot{x}^2)}$
Q_7	$M_{78} = \frac{e^{-2\alpha x}}{\alpha}$	$R_{78} = -\frac{e^{-\alpha x}\cos(\alpha t)}{\alpha}$

Table 9. The null forms, integrating factors, first integrals and exact solutions for $f(x) = ke^{\alpha x}$

S	R	I	x(t)
S_1 S_2	$R_{13} = R_{23}$ $R_{14} = R_{24}$ $R_{15} = R_{25}$	$I_{13} = I_{23} = -\frac{arctanh(-\sin(2\alpha t) + \dot{x}\cos(2\alpha t))}{\alpha} - \frac{1}{2\alpha}\ln\left(\frac{1+\sin(2\alpha t)}{\cos(2\alpha t)}\right)$ $I_{14} = I_{24} = -\frac{arctanh(\cos(2\alpha t) + \dot{x}\sin(2\alpha t))}{\alpha} - \frac{1}{2\alpha}\ln\left(\frac{1-\cos(2\alpha t)}{\sin(2\alpha t)}\right)$ $I_{15} = I_{25} = -t + \frac{\arctan(\dot{x})}{\alpha}$	$x_{13}(t) = C_{13} - \frac{\ln(\cos(2\alpha t)) + \tanh(I_{13}\alpha)\ln(\frac{1+\sin(2\alpha t)}{\cos(2\alpha t)})}{2\alpha}$
S_3	R_{36}	$I_{36} = \frac{1}{2\alpha}\ln\Big((\cos(2\alpha t)(\cos(2\alpha t) + 2\dot{x}\sin(2\alpha t) - \dot{x}^2\cos(2\alpha t))\Big) - x$	
S_4	R_{46}	$I_{46} = \frac{1}{2\alpha}\ln\Big(-\sin(2\alpha t) + 2\dot{x}\cos(2\alpha t) + \dot{x}^2\sin(2\alpha t)\Big) - x$	$x_{46}(t) = \frac{1}{2\alpha}\ln\left(\begin{array}{c}\tan(2\alpha t)(\tan^2(\alpha C_{46} - \alpha t) - 1)\\ -2\tan(\alpha C_{46} - \alpha t)\end{array}\right) - \frac{1}{4\alpha}\ln\left(\frac{2}{1+\cos(4\alpha t)}\right) - I_{46}$
S_5	R_{56}	$I_{56} = -x + \frac{\ln(1+\dot{x}^2)}{2\alpha}$	$x_{56}(t) = -I_{56} + \frac{1}{\alpha}\ln\left(\frac{1}{\cos(\alpha C_{56} - \alpha t)}\right)$
S_7	R_{78}	$I_{78} = \frac{(\sin(\alpha t) - \dot{x}\cos(\alpha t))\, e^{-\alpha x}}{\alpha}$	$x_{78}(t) = \frac{1}{\alpha}\ln\left(\frac{1}{\alpha^2 C_{78}\cos(\alpha t) + \alpha I_{78}\sin(\alpha t)}\right)$

4. Conclusion

In this work, we studied null forms, integrating factors, JLMs, first integrals and exact solutions of the path equation which models the drag forces of object moving in a fluid medium. In [13], the autors suggested two new identities which relate the null form with characteristics of the Lie generators and integrating factors with characteristics of Lie generators and JLMs, respectively.

We reversed this method because of to avoid solving the determining equations and ansatzes for null forms and integrating factors, namely Eq.(3)-(5). We applied this method to path equation which the altitude functions have some special cases.

We conclude that in the case of $f(x) = \dfrac{1}{k_1 x + k_2}$, our first integral I_{38} exactly coincides with the first integral I_5 of [12]. In addition, our exact solution x_{38} exactly coincides with the invariant solution (Eq.(33)) of [11].

In the case of $f(x) = k$, our first integrals I_{38} and I_{57} coincide with exactly I_5 and I_4 of [12], respectively.

In the last case, namely $f(x) = ke^{\alpha x}$, our first integral I_{78} coincides with exactly I_4 of [12]. Moreover, our exact solutions x_{78} exactly coincides with the invariant solution (Eq.(22)) of [11].

To the best of our knowledge the other first integrals and exact solutions for above three cases are the new.

As stated in [10], several applications of this study are possible. In [17], the optimum parabolic path for a flying object for which the work done by variable coefficient air friction force is minimum is determined. If a flying object follows the determined path, energy requirement would be less compared with other parabolic or linear paths.Aeroplanes and helicopters can follow the minimum drag path to reduce fuel consumption. Ballistic missiles can also be programmed to follow such an optimum path for reduction of rocket fuel.

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