Int. J. Adv. Appl. Math. and Mech. 2(4) (2015) 53 - 63 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

The effect of external source of disease on the epidemic model

Research Article

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Received 03 April 2015; accepted (in revised version) 28 May 2015

Abstract: In this paper a mathematical model that describes the flow of infectious disease in a population is proposed and studied. It is assumed that the disease divided the population into three classes: susceptible individuals S, first infected individuals I and second infected individuals I^*. The main objective of this paper is to study the effect of external Source and treatment of behaviors of this model. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global dynamical behaviors of the model are studied. Finally, in order to confirm our obtained results and specify the effects of model's parameters on the dynamical behavior, numerical simulation of this model is performed.

MSC: 92B05 • 92B99

Keywords: Epidemic models • Stability • Treatment • External Source

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1. Introduction

The mathematical models have become important tools in analyzing the spread and control of infectious diseases. The development of such models is aimed at both understand observed epidemiological patterns and predicting the consequences of the introduction of public health interventions to control the spread of diseases. Some diseases not confer immunity against the disease but other diseases confer immunity so recovered individuals gain immunity against disease. These types of disease can be modifications by SI and SIS where S susceptible and I infective respectively. Both epidemic models (SI and SIS) are one of the most basic and most important models in describing of many diseases. Therefore, it attached many authors attention and a number of papers have been published. For example Gao and Hethcote [1], considered an SIS model with a standard disease incidence and density-dependent demographics. Li and Ma [2], studied an SIS model with vaccination and temporary immunity. Kermack and Mckendeick [3], proposed a simple SIS model with infective immigrants. X. Zhou [4], studied HBV infection disease. Muhammad A. [5], proposed $SEIR$ epidemic model with non-linear saturated incidence and temporary immunity. In recent years, many papers found treatment function for example, Li et al [6] and Goodluck [7], proposed the SIS model with a limited resource for treatment. Shurowq k. Shafeeq [8], studied the effect of treatment, immigrants and vaccinated on the dynamic of SIS epidemic model. In this paper we proposed and studied a mathematical model consisting of epidemic model with treatment, in which it is assumed that the disease transmitted by contact as well as external sources in the environment. The local as well as global stability analysis of this model is investigated.

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2. Model development

From a simple epidemiological model in which the total population (say N(t)) at time t is divided into two sub classes the susceptible individuals S(t) and infected individuals $I(t)$. Such model can be represented as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta_1 SI - \mu S \\ \frac{dI}{dt} &= \beta_1 SI - \mu I \end{aligned} \tag{1}$$

Here $\Lambda > 0$ is the recruitment rate of the population, $\mu > 0$ is the natural death rate of the population, $\beta_1 > 0$ is the infected rate (incidence rate) of susceptible individuals due to directed contact with the infected individuals. Now, since there are many infectious disease for example (The flue., tube rculosis and cholera), spread in the environment by different factors including insects, contact or other vectors, therefore, we assumed that the disease in the a above model will transmitted between the population individuals by contact as well as external source of disease in the environment with an external source incidence rate $\beta_\circ \geq 0$. Also it is assumed that the nature recovery rate from infected individuals returns to be susceptible class with a constant rate $\alpha \geq 0$ and $\psi \geq 0$ is the rate of infected individuals from disease I into new disease I^*. Finally $\theta > 0$, $\beta_2 > 0$ the disease related death from second disease and the infected rate by contact between the susceptible individuals and infected individuals of second disease respectively. Then if addition above assumption system (1) can be rewritten in the form:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I \\ \frac{dI}{dt} &= (\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I \\ \frac{dI^*}{dt} &= \beta_2 SI^* + \psi I - (\mu + \theta)I^* \end{aligned} \tag{2}$$

Keeping the above in view, in order to study the effect of treatment on the system (2) let $T(I)$ represented the treatment function which given by [6]:

$$T(I) = \begin{cases} rI^* & if \quad 0 < I^* \leqslant {I_\circ}^*, \\ k & if \quad I^* > {I_\circ}^*. \end{cases} \tag{3}$$

Therefore, system (2) can be modified to:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I + T(I^*) \\ \frac{dI}{dt} &= (\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I \\ \frac{dI^*}{dt} &= \beta_2 SI^* + \psi I - (\mu + \theta)I^* - T(I^*) \end{aligned} \tag{4}$$

her $k = rI_\circ^*$ this means that the treatment rate is proportional to the number of the infected individuals when the capacity of treatment is not reached, and otherwise takes the maximal capacity. Therefore at any point of time t the total number of population becomes $N(t) = S(t) + I(t) + I^*(t)$. Obviously, due to the biological meaning of the variables $S(t), I(t)$ and I^*, system (4) has the domain $R_+^3 = \{(S, I, I^*) \in R_+^3, S \geq 0, I \geq 0, I^* \geq 0\}$ which is positively in variant for system (4). Clearly, the interaction functions on the right hand said of system (4) are continuously differentiable. In fact they are Liptschizan function on R_+^3. Therefore, the solution of system (4) exits and unique. Further, all solutions of system (4) with non-negative initial conditions are uniformly bounded as shown in the following theorem.

Theorem 2.1.
All the solutions of system (1), which are initiate in R_+^3, are uniformly bounded.

Proof. Let $\big(S(t), I(t), I^*(t)\big)$ be any solution of the system (4) with non-negative initial conditions $\big(S(0), I(0), I^*(0)\big)$. Since $N = S(t) + I(t) + I^*(t)$, then:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dI^*}{dt}$$

This gives:

$$\frac{dN}{dt} = \Lambda - \mu\{\big(S, I, I^*\big)\big) - \theta I^*\}$$

So,

$$\frac{dN}{dt} + \mu N \leq \Lambda$$

Now, by using Gronwall Lemma [**?**], it obtains that:

$$N(t) \leq \frac{\Lambda}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}$$

Therefore, $N(t) \leq \frac{\Lambda}{\mu}, as \rightarrow \infty$, hence all the solutions of system (4) that initiate in R^3_+ are confined in the reign:

$$\Gamma = \left\{(S, I, I^*) \in R^3_+ : N \leq \frac{\Lambda}{\mu}\right\}$$

which complete the proof. □

3. Existence of equilibrium point of system (4)

The system (4) has at most three biologically feasible points, namely $E_i = (S_i, I_i, I_i^*), i = 0,1,2$. The existence conditions for each of these equilibrium points are discussed in the following:
1) If $I = 0$ and $I^* = 0$, then the system ((**??**) has an equilibrium point called a disease free equilibrium point and denoted by $E_0 = (S_0, 0, 0)$ where:

$$S_0 = \frac{\Lambda}{\mu} \tag{5}$$

2) If $I^* = 0$, then the system (4) has an equilibrium point called a second disease free equilibrium point and denoted by$E_1 = (S_1, I_1, 0)$ where S_1 and I_1 represented the positive solution of the following set of equations:

$$\begin{aligned} \Lambda - (\beta_\circ + \beta_1 I)S - \mu S + \alpha I = 0 \\ (\beta_\circ + \beta_1 I)S - (\alpha + \mu)I = 0 \end{aligned} \tag{6}$$

From Eq. (1) of above system we get:

$$S_1 = \frac{\Lambda + \alpha I_1}{\beta_\circ + \beta_\circ I_1 + \mu} \tag{7}$$

Substituting S_1 in Eq. (2) of system (6) we get:

$$I_1 = \frac{-D_2}{2D_1} - \frac{1}{2D_1}\sqrt{D_2^2 - 4D_1D_3} \tag{8}$$

here

$$\begin{aligned} D_1 &= -\beta_1\mu \\ D_2 &= \Lambda\{\beta_1 - \mu(\beta_\circ + \alpha + \mu)\} \\ D_3 &= \beta_\circ\Lambda \end{aligned}$$

Clearly, Eq. (8) has a unique positive root by I_1 and then (E_2) exists uniquely in Int. R^3_+ if and only if $D_2 > 0$.
3)If $I \neq 0$ and $I^* \neq 0$ then the system (4) has an equilibrium point called endemic equilibrium point and denoted by $E_2 = (S_2, I_2, I_2^*)$, where, S_2,I_2 and I_2^* represented the positive solution of the following set of equations in case $(0 < I^* < I_\circ^*)$ of Eq. (3) (treatment function):

$$\begin{aligned} \Lambda - (\beta_\circ + \beta_1 I + \beta_2 I^*)S - \mu S + \alpha I + r(I^*) = 0 \\ (\beta_\circ + \beta_1 I)S - (\alpha + \mu + \psi)I = 0 \\ \beta_2 S I^* + \psi I - (\mu + \theta - r)I^* = 0 \end{aligned} \tag{9}$$

Straightforward computation to solve the above system of equations and from Eqs. (2) and 3 of system (9) gives that:

$$S_2 = \frac{(\mu + \alpha + \psi)I_2}{\beta_2 + \beta_1 I_2} \tag{10}$$

$$I_2^* = \frac{-\psi I_2(\beta_\circ + \beta_1 I_2}{\beta_2 I_2(\mu+\alpha+\psi) - (\mu+\theta+r)(\beta_\circ+\beta_1 I_2)} \tag{11}$$

While, I_2^* positive root if and only if

$$\beta_2 I_2(\mu+\alpha+\psi) < (\mu+\theta+r)(\beta_\circ+\beta_1 I_2)$$

Now, substituting S_2 and I_2^* in Eq. (1) of system (9) we get:

$$A_1 I_2^3 + A_2 I_2^2 + A_3 I_2 + A_4 = 0 \tag{12}$$

here

$$A_1 = \beta_1\left\{(\mu+\alpha+\psi)\left[\beta_1(\mu+\theta+r)+\alpha\beta_2\right] + \beta_1\psi r - [\beta_2(\mu+\alpha+\psi)^2 + \beta_2\psi(\mu+\alpha+\psi) + \beta_1\alpha(\mu+\theta+r)]\right\}$$

$$A_2 = \{2\beta_\circ\beta_1\psi r + (\mu+\alpha+\psi)\left[\Lambda\beta_1\beta_2 + \beta_\circ\beta_2\alpha + \beta_\circ\beta_1(\mu+\theta+r)\right] - \left[\beta_2(\mu+\alpha+\psi)\left[\beta_\circ\psi + (\beta_\circ+\mu)(\mu+\alpha+\psi)\right] + \beta_1(\mu+\theta+r)\ \ (\Lambda\beta_1+2\alpha\beta_\circ)\right]\}$$

$$A_3 = \left\{(\mu+\alpha+\psi)\left[\Lambda\beta_\circ\beta_2 + (\mu+\theta+r)\left(\beta_\circ^2+\beta_\circ\beta_1+\beta_\circ\mu+\beta_1\mu\right)\right] \ +\beta_\circ^2\psi r - \beta_\circ(\mu+\theta+r)(2\beta_1+\alpha\beta_\circ)\right\}$$

$$A_4 = -\Lambda\beta_\circ^2(\mu+\theta+r) < 0$$

$$A_4 = -\Lambda\beta_\circ^2(\mu+\theta+r) < 0$$

Clearly, Eq. (12) has a unique positive root by I_2 and then (E_2) exists uniquely in Int R_+^3. if and only if $A_1 > 0$ then we have the following three cases:
Case (1): If the following conditions hold:

$$\begin{aligned} A_2 &> 0 \\ A_3 &> 0 \end{aligned} \tag{13}$$

Case (2): If the following conditions hold:

$$\begin{aligned} A_2 &< 0 \\ A_3 &< 0 \end{aligned} \tag{14}$$

Case (3): If the following conditions hold:

$$\begin{aligned} A_2 &> 0 \\ A_3 &< 0 \end{aligned} \tag{15}$$

4. Local Stability of system (4)

In this section, the local stability analysis of the equilibrium points $E_i, i = 0,1,2$ of the system (4) studied as shown in the following theorems.

Theorem 4.1.
The disease free equilibrium point $E_\circ = (\frac{\Lambda}{\mu}, 0, 0)$ *of system* (4) *is locally asymptotically stable provided that:*

$$\alpha < \frac{\beta_1\Lambda}{\mu} < \mu+\alpha+\psi \tag{16}$$

$$r < \frac{\beta_2\Lambda}{\mu} < \mu+\theta+r \tag{17}$$

$$\left(\frac{\beta_1\Lambda}{\mu} - \alpha\right)\left[\beta_\circ + 2\mu + \alpha + \psi - \frac{\beta_1\Lambda}{\mu}\right] > \psi(r - \frac{\beta_2\Lambda}{\mu}) \tag{18}$$

Proof. The Jacobian matrix of system (4) at $E_\circ$ that denoted by $J(E_\circ)$ and can be written as

$$J(E_\circ) = [a_{ij}]_{3\times 3}$$

where

$$a_{11} = -(\beta_\circ + \mu); a_{12} = \alpha - \frac{\beta_1 \Lambda}{\eta}; a_{13} = r - \frac{\beta_2 \Lambda}{\mu}$$

$$a_{21} = \beta_\circ; a_{22} = \frac{\beta_1 \Lambda}{\mu} - (\mu + \alpha + \psi); a_{23} = a_{31} = 0$$

$$a_{32} = \psi; a_{33} = \frac{\beta_2 \Lambda}{\mu} - (\mu + \theta + r)$$

Then the characteristic equation of $J(E_\circ)$ is given by:

$$\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0 \tag{19}$$

here

$$\Omega_1 = -[a_{11} + a_{22} + a_{33}] = (\beta_\circ + \mu) - \left(\beta_1 S_0 - (\mu + \alpha + \psi)\right) - \left(\beta_2 S_0 - (\mu + \theta + r)\right)$$
$$\Omega_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} + a_{22}a_{33}$$
$$\Omega_3 = [a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{21}a_{32}a_{13}]$$
$$= \left[\beta_\circ\left(\alpha - \beta_1 S_0\right)\left(\beta_2 S_0 - (\mu + \theta + r)\right) + \left(\beta_\circ + \mu\right)\left(\beta_1 S_0 - (\mu + \alpha + \psi)\right)\left(\beta_2 S_0 - (\mu + \theta + r)\right) - \beta_\circ \psi (r - \beta_2 S_0)\right]$$

Further

$$\Delta = \Omega_1\Omega_2 - \Omega_3 = -a_{11}^2(a_{22} + a_{33}) - a_{22}^2(a_{11} + a_{33}) - a_{33}^2(a_{11} + a_{22}) - a_{11}a_{22}a_{33} + a_{21}[a_{12}(a_{11} + a_{22}) + a_{32}a_{13}]$$
$$= -\left(\beta_\circ + \mu\right)^2\left[\beta_1 S_0 - (\mu + \alpha + \psi) + \beta_2 S_\circ - (\mu + \theta + r)\right] - \left(\beta_1 S_0 - (\mu + \alpha + \psi)\right)^2\left[-(\beta_\circ + \mu) + \beta_2 S_\circ - (\mu + \theta + r)\right]$$
$$-\left(\beta_2 S_\circ - (\mu + \theta + r)\right)^2\left[-(\beta_\circ + \mu) + \beta_1 S_0 - (\mu + \alpha + \psi)\right] + (\beta_\circ + \mu)(\beta_1 S_0 - (\mu + \alpha + \psi)(\beta_2 S_\circ - (\mu + \theta + r) + \beta_\circ \times$$
$$\left[(\alpha - \beta_1 S_0)\left(-(\beta_\circ + \mu) + \beta_1 S_0 - (\mu + \alpha + \psi)\right) + \psi(r - \beta_2 S_0\right]$$

Now, according to (Routh-Hurwitz) criterion [10], E_0 will be locally asymptotically stable provided that $\Omega_1 > 0$ and $\Omega_3 > 0$. Clearly, provided that conditions (16)-(17) hold. While, $\Delta = \Omega_1\Omega_2 - \Omega_3 > 0$ Provided that conditions (16)-(18) hold. Hence the proof is complete. □

Theorem 4.2.
The second disease free equilibrium point (E_1) *of system* (4) *is locally asymptotically stable if the following sufficient conditions are satisfied:*

$$\mu > Max.\{2(\beta_1 S_1 - \alpha - \psi), 2(\beta_2 S_1 - r) - \theta\} \tag{20}$$

Proof. The Jacobian matrix of system (4) at (E_1) that denoted by $J(E_1)$ can be written as:

$$J(E_1) = [b_{ij}]_{3\times 3}$$

Where

$$b_{11} = -(\beta_\circ + \beta_1 I_1 + \mu)\ ;\ b_{12} = \alpha - \beta_1 S_1\ ;\ b_{13} = r - \beta_2 S_1$$
$$b_{21} = \beta_\circ + \beta_1 I_1\ ;\ b_{22} = \beta_1 S_1 - (\mu + \alpha + \psi)\ ;\ b_{23} = 0$$
$$b_{31} = 0\ ;\ b_{23} = \psi\ ;\ b_{33} = \beta_2 S_1 - (\mu + \theta + r)$$

Now, according to Gersgorin theorem [11] if the following condition holds:

$$|b_{ii}| > \sum_{i=1, i\neq j}^{3} |b_{ij}|$$

Then all eigenvalues of $J(E_1)$ exists in the region:

$$\wp = \cup\left\{U^* \in C : |U^* - b_{ii}| < \sum_{i=1, i\neq j}^{3} |b_{ij}|\right\}$$

Therefore, according to the given condition (20) all the eigenvalues of $J(E_1)$ exists in the left half plane and hence, E_1 is locally asymptotically stable. □

Theorem 4.3.
The endemic equilibrium point (E_2) *of system* (4) *is locally asymptotically stable if the following sufficient conditions are satisfied:*

$$\mu > Max.\{2(\beta_1 S_2 - \alpha - \psi), 2(\beta_2 S_2 - r) - \theta\} \tag{21}$$

Proof. The Jacobian matrix of system (4) at (E_2) that denoted by $J(E_2)$ can be written as:

$$J(E_2) = [c_{ij}]_{3\times 3}$$

Where:

$$c_{11} = -(\beta_\circ + \beta_1 I_2 + \beta_2 I_2^* + \mu)\ ;\ c_{12} = \alpha - \beta_1 S_2\ ;\ c_{13} = r - \beta_2 S_2$$
$$c_{21} = \beta_\circ + \beta_1 I_2\ ;\ c_{22} = \beta_1 S_2 - (\mu + \alpha + \psi)\ ;\ c_{23} = 0$$
$$c_{31} = \beta_2 I_2^*\ ;\ c_{23} = \psi\ ;\ c_{33} = \beta_2 S_2 - (\mu + \theta + r)$$

Now, according to Gersgorin theorem [8] if the following condition holds:

$$|c_{ii}| > \sum_{i=1, i\neq j}^{3} |c_{ij}|$$

Then all eigenvalues of $J(E_2)$ exists in the region:

$$\varsigma = \cup\left\{U^* \in C : |U^* - c_{ii}| < \sum_{i=1, i\neq j}^{3} |c_{ij}|\right\}$$

Therefore, according to the given condition (21) all the eigenvalues of $J(E_2)$ exists in the left half plane and hence, E_2 is locally asymptotically stable. □

5. Globally stability of all equilibrium point

In this section, the global dynamics of system (4) is studied with the help of Lyapunov function as shown in the following theorems.

Theorem 5.1.
Assume that, the disease free equilibrium point E_0 *of system* (4) *is locally asymptotically stable. Then the basin of attraction of* E_0*, say* $B(E_0) \subset R_+^3$*, it is globally asymptotically stable if satisfy the following condition:*

$$(\beta_\circ + \beta_1 I + \beta_2 I^* S) < (\alpha I + r I^*) \tag{22}$$

Proof. Consider the following positive definite function:

$$V_1 = \left(S - S_0 - S_0 ln\frac{S}{S_0}\right) + I + I^*$$

Clearly, $V_1 : R_+^3 \to R$ is a continuously differentiable function such that $V_1(S_0, 0, 0) = 0$ and $V_1(S, I, I^*) > 0, \forall (S, I, I^*) \neq (S_0, 0, 0)$. Further we have:

$$\frac{dV_1}{dt} = \left(\frac{S - S_0}{S}\right)\frac{dS}{dt} + \frac{dI}{dt} + \frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_1}{dt} = -\frac{\mu}{S}(S - S_0)^2 + \left[\beta_\circ + \beta_1 I + \beta_2 I^* - \frac{(\alpha I + r I^*)}{S}\right] S_0 - \mu(I + I^*) - \theta I^*$$

Obviously, $\frac{dV_1}{dt} < 0$, for every initial points and then V_1 is a Lyapunov function provided that condition (22) hold. Thus E_0 is globally asymptotically stable in the interior of $B(E_0)$ which means that $B(E_0)$ is the basin of attraction and that complete the proof.

□

Theorem 5.2.
Assume that, the second disease free equilibrium point E_1 of system (4) is locally asymptotically stable. Then the basin of attraction of E_1, say $B(E_1) \subset R^3_+$, it is globally asymptotically stable if satisfy the following conditions:

$$\left(\frac{\beta_1 S_1 I - (\alpha I + \beta_\circ S + \beta 1 S I_1}{SI}\right)^2 < 4\left(\frac{\mu+\alpha+\psi-\beta_1 S}{I}\right)\left(\frac{\beta_\circ+\mu+\beta_1 I}{S}\right) \tag{23}$$

$$\left(\beta_2 S_1 I^*\right) < \left(rS_1 + (\mu+\theta)S\right) I^* \tag{24}$$

Proof. Consider the following positive definite function:

$$V_2 = \left(S - S_1 - S_1 ln\frac{S}{S_1}\right) + \left(I - I_1 - I_1 ln\frac{I}{I_1}\right) + I^*$$

Clearly, $V_2 : R^3_+ \to R$ is a continuously differentiable function such that $V_2(S_1, I_1, 0) = 0$ and $V_2(S, I, I^*) > 0, \forall (S, I, I^*) \neq (S_1, I_1, 0)$. Further we have:

$$\frac{dV_2}{dt} = \left(\frac{S-S_1}{S}\right)\frac{dS}{dt} + \left(\frac{I-I_1}{I}\right)\frac{dI}{dt} + \frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_2}{dt} = -q_{11}(S-S_1)^2 - q_{12}(S-S_1)(I-I_1) - q_{22}(I-I_1)^2 - (\beta_2 - \frac{r}{S})(S-S_1)I^* + \beta_2 S I^* + \psi I - (\mu+\theta+r)I^*$$

With:

$$q_{11} = \frac{\beta_\circ+\mu+\beta_1 I}{S}\ ;\ q_{22} = \frac{(\mu+\alpha+\psi)-\beta_1 S}{I}\ ;$$
$$q_{12} = \frac{\beta_1 S_1 I - (\alpha I + \beta_\circ S + \beta_1 S I_1)}{SI}$$

Therefore, according to (23) it is obtaining that:

$$\frac{dV_2}{dt} \leqslant -\left[\sqrt{q_{11}}(S-S_1) + \sqrt{q_{22}}(I-I_1)\right]^2 + \beta_2 S_1 I^* + \psi I - \left(rS_1 + (\mu+\theta)S\right) I^*$$

Obviously, $\frac{dV_2}{dt} < 0$ for every initial points satisfying condition (24) and then V_2 is a Lyapunov function provided that conditions (23)-(24)) hold. Thus E_2 is globally asymptotically stable in the interior of $B(E_2)$, which means that $B(E_2)$ is the basin of attraction and that complete the proof.

□

Theorem 5.3.
Let the endemic equilibrium point E_2 of system (4) is locally asymptotically stable. Then it is globally asymptotically stable provided that:

$$Max.\left\{\frac{\mu+\alpha+\psi}{\beta_1}, \frac{\mu+\theta+r}{\beta_2}\right\} < S_2 \tag{25}$$

$$(\beta_\circ + \beta_1 I + \alpha - \beta_1 S_2)^2 < (\beta_\circ + \mu + \beta_1 I + \beta_2 I^*)(\mu+\alpha+\psi-\beta_1 S_2) \tag{26}$$

$$(\beta_2 I^* + r - \beta_2 S_2)^2 < (\beta_\circ + \mu + \beta_1 I + \beta_2 I^*)(\mu+\theta+r-\beta_2 S_2) \tag{27}$$

$$\psi^2 < (\mu+\alpha+\psi-\beta_1 S_2)(\mu+\theta+r-\beta_2 S_2) \tag{28}$$

Proof. Consider the following positive definite function:

$$V_3 = \frac{(S-S_2)^2}{2} + \frac{(I-I_2)^2}{2} + \frac{(I^*-I_2^*)^2}{2}$$

Clearly, $V_3 : R_+^3 \to R$ is a continuously differentiable function such that $V_3(S_2, I_2, I_2^*) = 0$ and $V_3(S, I, I^*) > 0, \forall (S, I, I^*) \neq (S_2, I_2, I_2^*)$. Further we have:

$$\frac{dV_3}{dt} = (S-S_2)\frac{dS}{dt} + (I-I_2)\frac{dI}{dt} + (I^*-I_2^*)\frac{dI^*}{dt}$$

By simplifying this equation we get:

$$\frac{dV_3}{dt} = -\frac{p_{11}}{2}(S-S_2)^2 + p_{12}(S-S_2)(I-I_2) - \frac{p_{22}}{2}(I-I_2)^2 - \frac{p_{11}}{2}(S-S_2)^2 + p_{13}(S-S_2)\left(I^*-I_2^*\right) - \frac{p_{33}}{2}\left(I^*-I_2^*\right)^2 - \frac{p_{22}}{2}(I-I_2)^2 + p_{23}(I-I_2)\left(I^*-I_2^*\right) - \frac{p_{33}}{2}\left(I^*-I_2^*\right)^2$$

with

$$p_{11} = \left(\beta_\circ + \mu + \beta_1 I + \beta_2 I^*\right) \; ; \; p_{12} = \left(\beta_\circ + \beta_1 I + \alpha - \beta_1 S_2\right)$$
$$p_{22} = (\mu + \alpha + \psi) - \beta_1 S_2 \; ; \; p_{13} = \left(\beta_2 I^* + r - \beta_2 S_2\right)$$
$$p_{33} = (\mu + \theta + r) - \beta_2 S_2 \; ; \; p_{23} = \psi$$

Therefore, according to the conditions (25)-(28) we obtain that:

$$\frac{dV_3}{dt} \leqslant -\left[\sqrt{\frac{p_{11}}{2}}(S-S_2) - \sqrt{\frac{p_{22}}{2}}(I-I_2)\right]^2 - \left[\sqrt{\frac{p_{11}}{2}}(S-S_2) + \sqrt{\frac{p_{33}}{2}}\left(I^*-I_2^*\right)\right]^2 - \left[\sqrt{\frac{p_{22}}{2}}(I-I_2) + \sqrt{\frac{p_{33}}{2}}\left(I^*-I_2^*\right)\right]^2$$

Obviously, $\frac{dV_3}{dt} < 0$, and then V_3 is a Lyapunov function provided that conditions (25)-(28) hold. Thus E_3 is globally asymptotically stable.

□

6. Numerical Simulation

In this section, the system (1) is solved numerically for different sets of hypothesis data and different sets of initial conditions, and then the time series for the trajectories of system (1) are confirm our obtained analytical results. By using (150, 100, 90), (250, 200, 150) and (350, 150, 200) as initial points and the numerical simulations are carried out in the following cases:

Case I For the disease free equilibrium point E_0, we choose the following data:

$$\Lambda = 500, \beta_\circ = 0, \beta_1 = 0.001, \beta_2 = 0.001, \mu = 0.2, \alpha = 2, r = 2, \psi = 0.6, \theta = 0.4 \tag{29}$$

Therefore, the disease free equilibrium point E_0 of system (1) is globally asymptotically stable and then the trajectories of the system (1) is approaches to (2500,0,0) for any time. (See Fig. 1)

Case II For the second equilibrium point E_1, we choose the following data and using (150, 100, 90), (250, 200, 150) and (350, 150, 200) as initial points:

$$\Lambda = 500, \beta_\circ = 0.1, \beta_1 = 0.001, \beta_2 = 0.001, \mu = 0.2, \alpha = 2, r = 2, \psi = 0, \theta = 0.4 \tag{30}$$

Therefore, the second equilibrium point E_1 of system (1) is globally asymptotically stable and then the trajectories of the system (1) is approaches to (1890,609,0) for any time. (See Fig. 2)

Case III For the endemic equilibrium pointE_2, we choose the following data and using (150, 100, 90), (250, 200, 150) and (350, 150, 200) as initial points:

$$\Lambda = 500, \beta_\circ = 0.1, \beta_1 = 0.001, \beta_2 = 0.001, \mu = 0.2, \alpha = 2, r = 2, \psi = 0.6, \theta = 0.4 \tag{31}$$

Therefore, the endemic equilibrium point E_2 of system (1) is globally asymptotically stable and then the trajectories of the system (1) is approaches to (1845,193,153) for any time. (See Fig. 3)

Case IV We choose the incidence rate of disease resulting from external sources $\beta_\circ = 0, 0.3, 0.6$ respectively, keeping other parameters fixed as given in Eq. (31), we get the disease free equilibrium point of system (1) becomes unstable point and the trajectory of system (1) approaches asymptotically to the endemic equilibrium point. but the number of infected in first disease individuals and infected in second disease individuals increases. (See Figs. 4(a)-4(b)), Similar results are obtained, as those shown in case of increasing $\beta_\circ$, in case of increasing the incidence rate of disease resulting by contact between susceptible and infected in first disease, that is means increasing β_1 but increasing the incidence rate of disease resulting by contact between susceptible and infected in second disease β_2 and keeping other parameters fixed as given in (31) we get the number of infected in first disease individuals decrease and infected in second disease individuals increase.

Case V Now we choose treatment rate $r = 0, 2, 5, 9$ respectively, keeping other parameters fixed as given in Eq. (31), we get the trajectories of system (1) still approaches to endemic equilibrium point but the number of infected in first disease individuals increase and infected in second disease individuals decreases. (See Figs. 5(a)-5(b))

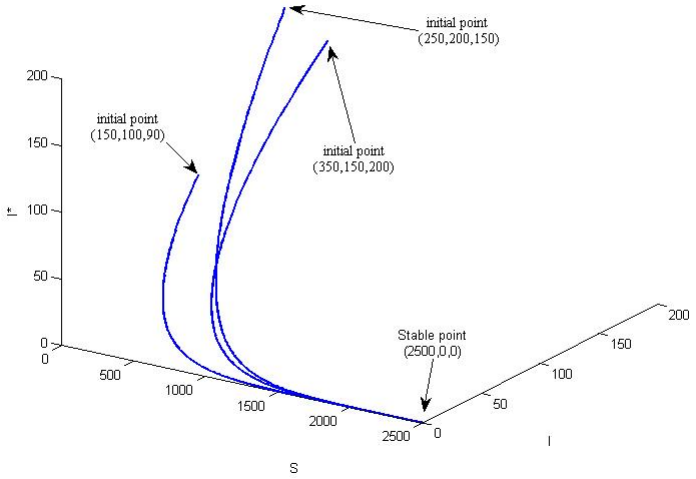


Fig. 1. Phase plot of system (1) starting from different initial points which show that E_0 is globally asymptotically stable.

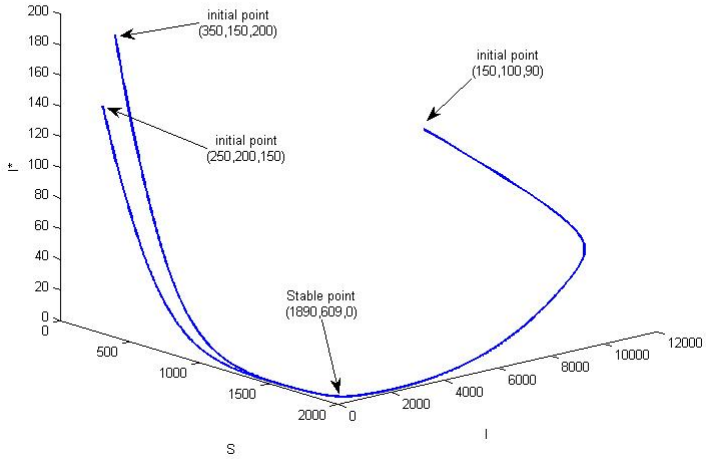


Fig. 2. Phase plot of system (1) starting from different initial points which show that E_1 is globally asymptotically stable.

7. Conclusion and discussion

In this paper, we proposed and analyzed an epidemiological model that described the dynamical behavior of an epidemic model, where the infectious disease transmitted directly from external sources as well as through contact between them. The model included fore non-linear autonomous differential equations that describe the dynamics of three different populations namely susceptible individuals (S) infected individuals for first disease (I) and infected individuals for second disease (evolution of first disease) (I^*). The boundedness of system (1) has been discussed. The conditions for existence, stability for each equilibrium points are obtained. Further, it is observed that the disease free equilibrium point E_0 exists when $I = 0$ and locally stable if the conditions are hold (16)-(18) and it is globally stable if and only if the condition (22) holds. The second disease free equilibrium point E_1 exists if ($D_2 > 0$) holds and locally stable if the conditions (20) are hold while it is globally stable if and only if the conditions (21)-(23)) hold. The endemic equilibrium point E_2 exists if ($A_1 > 0$) and one of three conditions is hold (13)-(15), and locally stable if the conditions (21) hold more than it is globally stable if and only if the conditions (25)-(28) hold. Finally, to understand the effect of varying each parameter on the global system (1) and confirm our above analytical results, the system (1) has been solved numerically for different sets of initial points and different sets of parameters given by Eq. (29), and the following observations are made:

1. The system (1) do not has periodic dynamic, instead it they approach either to the all equilibrium point.

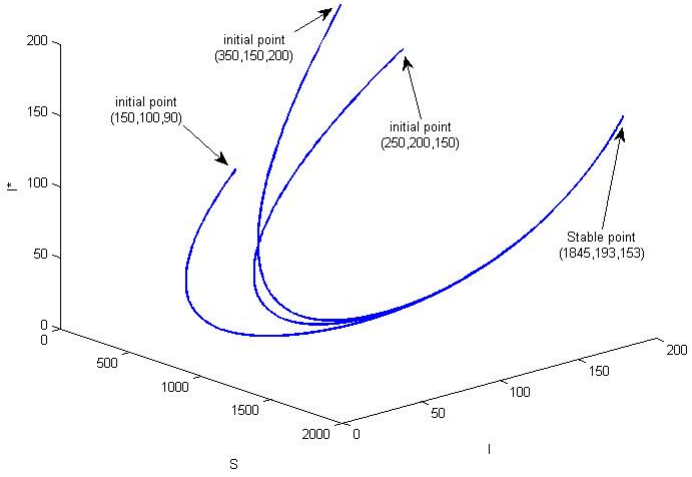
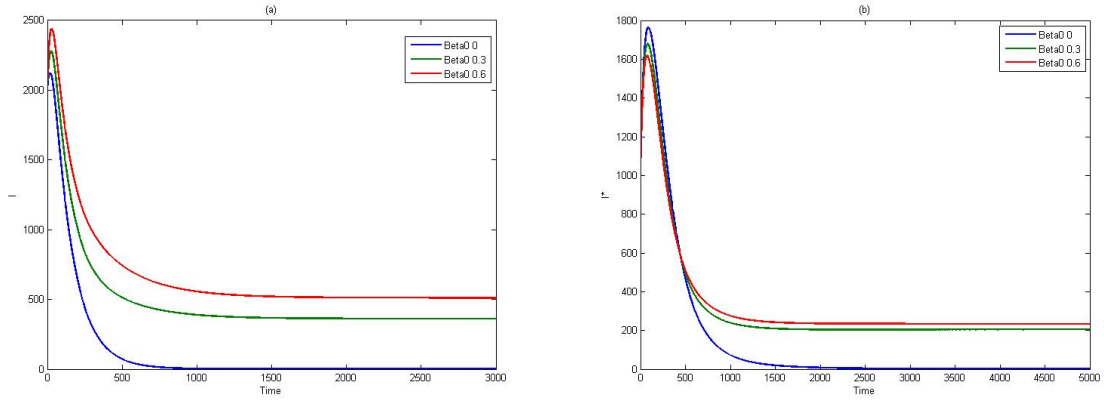


Fig. 3. Phase plot of system (1) starting from different initial points which show that E_2 is globally asymptotically stable.



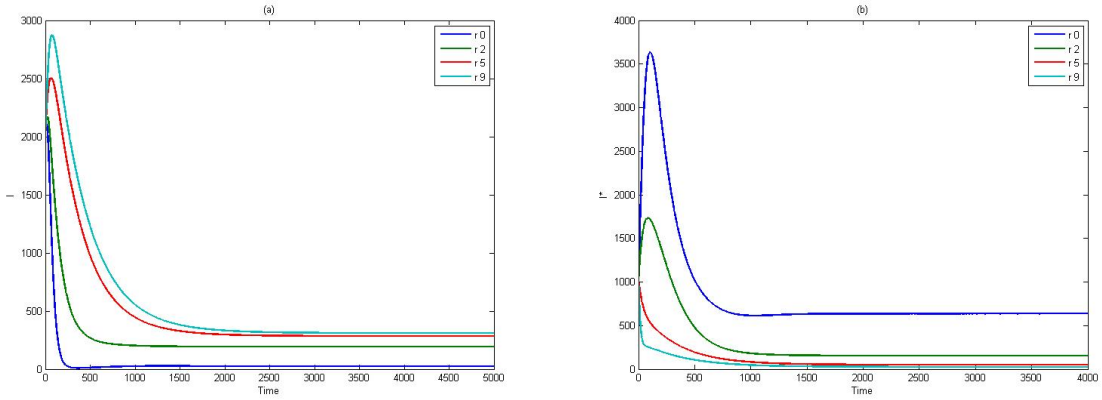
(a) For I.

(b) For I^*.

Fig. 4. Time series of the trajectories of system (1).

2. As the incidence rate of disease (external incidence rate $\beta_\circ$ or β_1 contact incidence rate) increase, the asymptotic behavior of the systems (1) approaching to endemic equilibrium point. In fact are $\beta_i, i = \circ, 1$ increase it are observed that the number of (S) decrease and the number of (I) and I^* increase.

3. As the incidence rate of disease (contact incidence rate β_2) increase, the asymptotic behavior of the systems (1) approaching to endemic equilibrium point. In fact as β_2 increase it is observed that the number of (S) and (I) decrease and the number of I^* increase. As the treatment rate r, the asymptotic behavior of the systems (1) approaching to endemic equilibrium point with increase it is observed that the number of (S) and (I) increase and the number of I^* decrease.



(a) For I. (b) For I^*.

Fig. 5. Time series of the trajectories of system (4).

Acknowledgment

I would like to thank Dr. Raid K. Naji for to help me to finish this my paper

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