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Existence of solutions for a class of *p(x)*-Kirchhoff type equation via topological methods

Research Article

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Abstract: The aim of this work is to obtain weak solutions for a class of $p(x)$-Kirchhoff type problem under no-flux boundary conditions. Our result is obtained using a Fredholm-type result for a couple of nonlinear operators and the theory of the variable exponent Sobolev spaces.

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Keywords: p-Kirchhoff type equations • Variational methods • No-flux boundary condition.

1. Introduction

In this paper we discuss the existence of weak solutions for the following Kirchhoff type problem

$$\left.\begin{aligned}
&-M\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)[\operatorname{div}(a(x,\nabla u))-|u|^{p(x)-2}u]=f(x,u)|u|^{t(x)}_{s(x)} \quad \text{in } \Omega,\\
&u=\text{constant on } \partial\Omega,\\
&\int_{\partial\Omega} a(x,\nabla u).\nu\, d\Gamma=0.
\end{aligned}\right\} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $n\geq 1$, $p,s,t\in C(\overline{\Omega})$ for any $x\in\overline{\Omega}$; $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function, f is a Caratheodory function and $\operatorname{div}\big(a(x,\nabla u)\big)$ is a $p(x)$-Laplacian type operator. The study of differential and partial differential equations with variable exponent has been received considerable attention in recent years .This importance reflects directly into various range of applications.There are applications concerning elastic mechanics [1], thermorheological and electrorheological fluids [2, 3], image restoration [4] and mathematical biology [5]. Eq. (1) is called a nonlocal problem because of the term M, which implies that the equation in (1) is no longer a pointwise equation. This causes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called Kirchhoff-type equations because Kirchhoff [6] investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature is that the (2) contains a nonlocal coefficient $\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx$ which

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depends on the average $\frac{1}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx$, and hence the equation is no longer a pointwise equation. The parameters in (2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [38] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [38], various equations of Kirchhoff-type have been studied extensively, see e.g. [7, 8] and [9]-[17].The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [9, 10, 12, 15, 17, 18]) and $p(x)$-Laplacian (see [11, 13, 14, 19, 20]).
In [21] , Fan has discussed the nonlocal $p(x)$ -Laplacian Dirichlet problem with non-variational form

$$\begin{aligned}&-\mathscr{A}(u)\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mathscr{B}(u) f(x,u) \quad \text{in } \Omega,\\ &u = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

and nonlocal p(x)-Laplacian Dirichlet problem with variational form

$$\begin{aligned}&-a\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = b\Big(\int_\Omega F(x,u)\,dx\Big) f(x,u) \quad \text{in } \Omega,\\ &u = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

respectively and has obtained the existence of solutions , where $\mathscr{A}$,$\mathscr{B}$ are two functional defined on $W_0^{1,p(x)}$, and $F(x,t) = \int_0^t f(x,s)\,ds$.
More recently, Yucedag et al.[22] , have dealt with Kirchhoff type problem

$$\begin{aligned}&-M\Big(\int_\Omega A(x,\nabla u)\Big)\,\mathrm{div}\big(a(x,\nabla u)\big) = m(x)|u|^{q(x)-2}u \quad \text{in } \Omega,\\ &u = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

by variational methods.

The nonlocal boundary condition in (1) have been studied by Berestycki and Brezis [23], Ortega [24] , Amster et al. [25], Zhao et al.[26], Boureanou et al.[27], Cabanillas L. et al. al.[28] and the references therein.They arise from certain models in plasma physics:specifically,a model describing the equilibrium of a plasma confined in a toroidal cavity, called a Tokamak machine. A detailed description of this model can be found in the Appendix of [29].

Motivated by the above papers and the results in [22], we consider (1) to study the existence of weak solutions.We note that our problem has no variational structure , so the most usual variational techniques can not used to study it. To attack it we will employ a Fredholm type theorem for a couple of nonlinear operators due to Dinca [16].
This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces.In Sections 3, we give some existence results of weak solutions of problem (1) and their proofs.

2. Preliminaries

To discuss problem (1), we need some theory on $W^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space (for details, see [30]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$\begin{aligned}&C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},\\ &h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).\end{aligned}$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| \equiv \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([30]).
The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.2 ([30]).
Set $\rho(u)=\int_\Omega |u(x)|^{p(x)}\,dx$. For any $u\in L^{p(x)}(\Omega)$, then

(1) *for $u\neq 0$, $|u|_{p(x)}=\lambda$ if and only if $\rho(\frac{u}{\lambda})=1$;*

(2) *$|u|_{p(x)}<1\ (=1;>1)$ if and only if $\rho(u)<1\ (=1;>1)$;*

(3) *if $|u|_{p(x)}>1$, then $|u|_{p(x)}^{p^-}\le\rho(u)\le|u|_{p(x)}^{p^+}$;*

(4) *if $|u|_{p(x)}<1$, then $|u|_{p(x)}^{p^+}\le\rho(u)\le|u|_{p(x)}^{p^-}$;*

(5) *$\lim_{k\to+\infty}|u_k|_{p(x)}=0$ if and only if $\lim_{k\to+\infty}\rho(u_k)=0$;*

(6) *$\lim_{k\to+\infty}|u_k|_{p(x)}=+\infty$ if and only if $\lim_{k\to+\infty}\rho(u_k)=+\infty$.*

Proposition 2.3 ([30, 31]).
If $q\in C_+(\overline{\Omega})$ and $q(x)\le p^*(x)$ $(q(x)<p^*(x))$ for $x\in\overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, where

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

Proposition 2.4 ([30, 32]).
The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1$ holds a.e. in Ω. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

Theorem 2.1 ([16]).
Let X and Y be real Banach spaces and two nonlinear operators $T,S:X\to Y$ such that

1. *T is bijective and T^{-1} is continuous.*
2. *S is compact.*
3. *Let $\lambda\neq 0$ be a real number such that: $\|(\lambda T-S)(x)\|\to+\infty$ as $\|x\|\to+\infty$;*
4. *There is a constant $R>0$ such that*
   $\|(\lambda T-S)(x)\|>0$ if $\|x\|\ge R$, $\quad d_{LS}(I-T^{-1}(\frac{S}{\lambda}),B(\theta,R),0)\neq 0$.

Then $\lambda I-S$ is suryective from X onto Y.
Here $d_{LS}(G,B,0)$ denotes the Leray-Schauder degree.

Throughout this paper, let

$$V=\{u\in W^{1,p(x)}(\Omega):u|_{\partial\Omega}=\text{constant}\}.$$

The space V is a closed subspace of the separable and reflexive Banach space $W^{1,p(x)}(\Omega)$ (See [33]) ,so V is also separable and reflexive Banach space with the usual norm of $W^{1,p(x)}(\Omega)$.The space V is the space where we will try to find weak solutions for problem (1).

Definition 2.1.
A function $u\in V$ is said to be a weak solution of (1) if

$$M\Big(\int_\Omega(A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v\,dx+\int_\Omega|u|^{p(x)-2}uv\,dx\Big]=\int_\Omega f(x,u)|u|_{s(x)}^{t(x)}v\,dx \quad ,$$

for all $v\in V$.

We assume that $a(x,\xi):\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$, $A=A(x,\xi)$, i.e. $a(x,\xi)=\nabla_\xi A(x,\xi)$. Suppose that a and A satisfy the following assumptions:

(A1) a satisfies the growth condition $|a(x,\xi)|\le c_0(a_0(x)+|\xi|^{p(x)-1})$ for all $x\in\Omega$, $\xi\in\mathbb{R}^N$, for some constant $c_0>0$; $a_0\in L^{p'(x)}$ is a nonnegative function.

(A2) $A(x,0)=0$ for all $x\in\Omega$;

(A3) $|\xi|^{p(x)}\le a(x,\xi)\xi\le p(x)A(x,\xi)$ for all $x\in\Omega$, $\xi\in\mathbb{R}^N$.

(A4) The monotonicity condition $0\le\left[a(x,\eta_1)-a(x,\eta_2)(\eta_1-\eta_2)\right]$, for all $x\in\Omega$ and all $\eta_1,\eta_2\in\mathbb{R}^N$,with equality if and only if $\eta_1=\eta_2$.

(F1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function satisfying the following conditions

$$|f(x,s)|\le c_1+c_2|s|^{\alpha(x)-1}),\quad \forall x\in\Omega, s\in\mathbb{R},$$

for some $\alpha\in C_+(\Omega)$ such that $1<\alpha(x)<p^*(x)$ for $x\in\overline{\Omega}$ and c_1,c_2 are positive constants.

(M0) $M:[0,+\infty[\to]m_0,+\infty[$ is a continuous and nondecreasing function with $m_0>0$.

3. Existence of solutions

In this section we will discuss the existence of weak solutions of (1). Our main result is as follows.

Theorem 3.1.
Assume that hypotheses (A1)–(A4), (F1) and (M0) hold. Then (1) has a weak solution in V.

Proof. In order to apply theorem (2.1), we take $Y=V'$ and the operators $T,S:V\to V'$ in the following way

$$\langle Tu,v\rangle = M\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\int_\Omega a(x,\nabla u)\nabla v\,dx+\int_\Omega |u|^{p(x)-2}uv\,dx\Big]$$

$$\langle Su,v\rangle = \int_\Omega f(x,u)|u|^{t(x)}_{s(x)}v\,dx$$

for all $u,v\in V$ Then $u\in V$ is a solution of (1) if and only if

$$Tu=Su\quad \text{in } V'.$$

Next, we split the proof in several steps.
Step1.We prove that T is an injection.
First we observe that

$$\Phi(u)=\widehat{M}\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)$$

is a continuously Gâteaux differentiable function whose Gâteaux derivative at the point $u\in V$ is the functional $\Phi'(u)\in V'$ given by

$$\langle\Phi'(u),v\rangle=\langle T(u),v\rangle\quad \text{for all } u\in V.$$

On the other hand, we consider the functional $L:V\to\mathbb{R}$ defined by

$$L(u)=\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\quad \text{for all } u\in V$$

then $L\in C^1(V,\mathbb{R}$ and

$$\langle L'(u),v\rangle=\int_\Omega a(x,\nabla u)\nabla v\,dx+\int_\Omega |u|^{p(x)-2}uv\,dx\quad \text{for all } u,v\in V.$$

Using (A4) and taking into account the inequality [37, (2.2)]

$$\langle |x|^{p-2}x-|y|^{p-2}y,x-y\rangle\ge\begin{cases}C_p|x-y|^p & \text{if } p\ge2\\ C_p\frac{|x-y|^2}{(|x|+|y|)^{p-2}},\ (x,y)\neq(0,0) & \text{if } 1<p<2,\end{cases}\tag{3}$$

for some for $C_p > 0$ and for all $x, y \in \mathbb{R}^N$, we obtain for all $u, v \in V$ with $u \neq v$

$$\langle L'(u) - L'(v), u - v\rangle > 0$$

which means that L' is strictly monotone. So, by [34, Prop. 25.10], L is strictly convex. Moreover, since M is nondecreasing, $\widehat{M}$ is convex in $[0, +\infty[$. Thus, for every $u, v \in X$ with $u \neq v$, and every $s, t \in (0,1)$ with $s + t = 1$, one has

$$\widehat{M}(L(su + tv)) < \widehat{M}(sL(u) + tL(v)) \leq s\widehat{M}(L(u)) + t\widehat{M}(L(v)).$$

This shows that Φ is strictly convex, and as $\Phi'(u) = T(u)$ in V' we infer that T is strictly monotone in V, then T is an injection.

Step2. We prove that the inverse $T^{-1} : V' \to V$ of T is continuous.

For any $u \in V$ with $\|u\| > 1$, one has

$$\frac{\langle T(u), u\rangle}{\|u\|} = \frac{M\Big(\int_\Omega (A(x,\nabla u) + \frac{1}{p(x)}|u|^{p(x)})\, dx\Big)\left[\int_\Omega a(x,\nabla u)\nabla u\, dx + \int_\Omega |u|^{p(x)}\, dx\right]}{\|u\|} \geq c_0\|u\|^{p^- -1},$$

from which we have the coercivity of T.

Since T is the Fréchet derivative of Φ, T is continuous.Thus in view of the well known Minty Browder theorem T is a surjection and so $T^{-1} : V' \to V$ and it is bounded.

Now we prove the continuity of T^{-1}.

First, we verify that T is of type (S_+).In fact, if $u_\nu \rightharpoonup u$ in V (so there exists $R > 0$ such that $\|u_\nu\| \leq R$) and the strict monotonicity of T we have

$$0 = \limsup_{\nu\to\infty} \langle Tu_\nu - Tu, u_\nu - u\rangle = \lim_{\nu\to\infty} \langle Tu_\nu - Tu, u_\nu - u\rangle$$

Then

$$\lim_{\nu\to\infty} \langle Tu_\nu, u_\nu - u\rangle = 0$$

That is

$$\lim_{\nu\to\infty} M\Big(\int_\Omega (A(x,\nabla u_\nu) + \frac{1}{p(x)}|u_\nu|^{p(x)})\, dx\Big)\left[\int_\Omega a(x,\nabla u_\nu)\nabla(u_\nu - u)\, dx + \int_\Omega |u_\nu|^{p(x)-2}u_\nu(u_\nu - u)\, dx\right] = 0 \tag{4}$$

Now, for any $x \in \overline{\Omega}$ and $\zeta \in \mathbb{R}^n$ we have

$$A(x,\zeta) = \int_0^1 \frac{d}{dt}A(x, t\zeta)\, dt = \int_0^1 a(x, t\zeta).\zeta\, dt$$

Then using $(A1)$, we get

$$A(x,\zeta) \leq c_1|\zeta| + c_2|\zeta|^{p(x)} \qquad \text{for all } x \in \overline{\Omega},\ \zeta \in \mathbb{R}^n$$

The above inequality implies

$$A(x,\nabla u_\nu) \leq c_1|\nabla u_\nu| + c_2|\nabla u_\nu|^{p(x)} \tag{5}$$

By (5), propositions(2.2), (2.4) and the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}$ we infer that $\Big(\int_\Omega (A(x,\nabla u_\nu)\, dx + \frac{1}{p(x)}|u_\nu|^{p(x)})\, dx\Big)_{\nu\geq 1}$ is bounded .

Then , since M is continuous, up to a subsequence there is $t_0 \geq 0$ such that

$$M\Big(\int_\Omega (A(x,\nabla u_\nu) dx + \frac{1}{p(x)}|u_\nu|^{p(x)})\, dx\Big) \to M(t_0) \geq m_0 \qquad \text{as } \nu \to \infty$$

This and (4) imply

$$\lim_{\nu\to\infty} \left[\int_\Omega a(x,\nabla u_\nu)\nabla(u_\nu - u)\, dx + \int_\Omega |u_\nu|^{p(x)-2}u_\nu(u_\nu - u)\, dx\right] = 0$$

Using, again, the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}$, we have

$$\lim_{\nu\to\infty} \int_\Omega |u_\nu|^{p(x)-2}u_\nu(u_\nu - u)\, dx = 0$$

Thus

$$\lim_{\nu\to\infty}\int_\Omega a(x,\nabla u_\nu)\nabla(u_\nu-u)\,dx=0.$$

By [35, theorem 4.1] we get that

$$u_\nu\to u \quad \text{strongly in } W^{1,p(x)}(\Omega) \text{ as } \quad \nu\to\infty$$

Since $(u_\nu)\subseteq V$ and V is a closed subspace of $W^{1,p(x)}(\Omega)$, we have $u\in V$, so $u_\nu\to u$ in V.
Let $(g_\nu)_{\nu\geq1}$ be a sequence of V' such that $g_\nu\to g$ in V'.Let $u_\nu=T^{-1}g_\nu$, $u=T^{-1}g$, then $Tu_\nu=g_\nu$, $Tu=g$.
By the coercivity of T , we deduce that $(u_\nu)_{\nu\geq1}$ is bounded in V ;up to a subsequence , we can assume that $u_\nu\rightharpoonup u$ in V. Since $g_n\to g$,

$$\lim_{n\to+\infty}\langle Tu_n-Tu,u_n-u\rangle=\lim_{n\to+\infty}\langle g_n-g,u_n-u\rangle=0.$$

Since T is of type (S_+), $u_n\to u$, so T^{-1} is continuous.
Step3 We prove that S is a compact operator.
1.- S is well defined.Indeed,using (f_1) and $t\in C(\overline{\Omega})$,for all u,v in V we have

$$\begin{aligned}|\langle Su,v\rangle|&\leq\int_\Omega|f(x,u)||u|_{s(x)}^{t(x)}|v|\,dx\\&\leq C|f(x,u)|_{\frac{\alpha(x)}{\alpha(x)-1}}|v|_{\alpha(x)}\leq C|f(x,u)|_{\frac{\alpha(x)}{\alpha(x)-1}}\|v\|<\infty\end{aligned}$$

2.- S is continuous on V.
Let $u_\nu\to u$ in V. Then proposition (2.3) implies that $u_\nu\to u$ in $L^{s(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$
So, up to a subsequence we deduce

$$u_\nu\to u \text{ a.e. in } \quad \Omega \tag{6}$$

$$|u_\nu(x)|^{\alpha(x)}\leq k(x) \quad \text{a.e.} \quad x\in\Omega \quad \text{for some} \quad k\in L^1(\Omega) \tag{7}$$

Since $t\in C(\overline{\Omega})$

$$|u_\nu|_{s(x)}^{t(x)}\to|u|_{s(x)}^{t(x)} \quad \text{a.e.} \quad x\in\Omega.$$

Furthermore

$$f(x,u_\nu)\to f(x,u) \quad \text{a.e.} \quad x\in\Omega,$$

Thus , we have

$$f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}\to f(x,u)|u|_{s(x)}^{t(x)} \quad \text{a.e.} \quad x\in\Omega.$$

But, it follows from (f1) and (7) that

$$\begin{aligned}\left|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\right|^{\alpha'(x)}&\leq C2^{(\alpha')^+}\left[|f(x,u_\nu)|^{(\alpha')^+}+|f(x,u)|^{(\alpha')^+}\right]\\&\leq C(1+k(x)+|u|^{\alpha(x)})\end{aligned}$$

Note that $C(1+k(x)+|u|^{\alpha(.)})\in L^1(\Omega)$. Then, applying the Dominated Convergence Theorem , we obtain

$$\lim_{\nu\to\infty}\int_\Omega\left|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\right|^{\alpha'(x)}dx=0$$

This implies that

$$\lim_{\nu\to\infty}\left|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\right|_{\alpha'(x)}=0 \tag{8}$$

Hence by direct computations we get

$$\begin{aligned}|\langle Su_\nu,v\rangle-\langle Su,v\rangle|&\leq\int_\Omega\left|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\right||v|\,dx\\&\leq C\left|f(x,u_\nu)|u_\nu|_{s(x)}^{t(x)}-f(x,u)|u|_{s(x)}^{t(x)}\right|_{\alpha'(x)}\|v\|\end{aligned}$$

therefore,from (8)

$$|Su_\nu - Su| \leq C\Big|f(x,u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x,u)|u|^{t(x)}_{s(x)}\Big|_{\alpha'(x)} \to 0 \tag{9}$$

So, $Su_\nu \to Su$ in V'.
3.- Every bounded sequence $(u_\nu)_\nu$ has a subsequence (still denoted by $(u_\nu)_\nu$) for which $(Su_\nu)_\nu$ converges.
Let $(u_\nu)_\nu$ be a bounded sequence of V, there exists a subsequence again denoted by $(u_\nu)_\nu$ and u in V , such that

$$u_\nu \rightharpoonup u \quad \text{weakly in } W^{1,p(x)}(\Omega)$$

and by the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, we have

$$u_\nu \to u \quad \text{in } L^{\alpha(x)}(\Omega).$$

Hence,similarly to the proof of (9)we get

$$|Su_\nu - Su| \leq C\Big|f(x,u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x,u)|u|^{t(x)}_{s(x)}\Big|_{\alpha'(x)} \to 0$$

So $Su_\nu \to Su$.

Step4

$$\|(T-S)(u)\| \to \infty \qquad \text{as } \|u\| \to \infty \quad \text{for } u \in V.$$

In fact, after some computations we get

$$\|Tu\| \geq C_0\|u\|^{p^- -1} \qquad \text{for all } u \in V \qquad \text{with } \|u\| > 1$$

and

$$\|Su\| \leq C_1\|u\|^{\theta} + C_2 \qquad \text{for all } u \in V, \qquad \text{for some } \theta \in [\alpha^- -1, \alpha^+ -1]$$

Combining the above inequalities,we obtain

$$\|(T-S)(u)\| \geq \|Tu\| - \|Su\| \geq C_0\|u\|^{p^- -1} - C_1'\|u\|^{\alpha^+ -1} - C_2 \tag{10}$$

Since

$$\lim_{t\to\infty}(C_0 t^{p^- -1} - C_1' t^{\alpha^+ -1} - C_2) = \infty$$

and from (10) we conclude that $\|(T-S)(u)\| \to \infty$ as $\|u\| \to \infty$.
Moreover, there exists $r_0 > 1$ such that $\|(T-S)(u)\| > 1$ for all $u \in V$, with $\|u\| > r_0$.

Step5 Set

$$W = \{u \in V : \exists t \in [0,1] \qquad \text{such that } u = tT^{-1}(Su)\}$$

Next,we prove that W is bounded in V.
For $u \in W \setminus 0$, i.e. $u = tT^{-1}(Su)$ for some $t \in [0,1]$ we have

$$\|T(\frac{u}{t})\| = \|Su\| \leq C_1\|u\|^{\theta} + C_2 \text{with } t > 0 \tag{11}$$

Then there exist two constants $a, b > 0$ such that

$$m_0\|u\|^{p^+ -1} \leq a\|u\|^{\alpha^- -1} + b \qquad \text{if } 0 < \|u\| < t,$$
$$m_0\|u\|^{p^- -1} \leq a\|u\|^{\alpha^- -1} + b \qquad \text{if } t \leq \|u\| \leq 1,$$
$$m_0\|u\|^{p^- -1} \leq a\|u\|^{\alpha^+ -1} + b \qquad \text{if } 1 < \|u\|$$

Let $g_1, g_2 : [0,1] \to \mathbb{R}$ and $g_3 :]1,\infty[\to \mathbb{R}$ be defined by

$$g_1(t) = m_0 t^{p^+ -1} - at^{\alpha^- -1} - b,\ g_2(t) = m_0 t^{p^- -1} - at^{\alpha^- -1} - b,\ g_3(t) = m_0 t^{p^- -1} - at^{\alpha^+ -1} - b.$$

The sets $\{t \in [0,1] : g_1(t) \leq 0\}, \{t \in [0,1] : g_2(t) \leq 0\}$ and $\{t \in]1,\infty[: g_3(t) \leq 0\}$ are bounded in $\mathbb{R}$.
From the above inequalities and (11) we infer that W is bounded in V, so

$$W \subseteq B(0,r_1) \qquad \text{for some } r_1 > 0$$

Now, taking $r = \max\{r_0, r_1\}$, it follows from [36, theorem 1.8] that

$$d_{LS}(I - tT^{-1}(S), B(0,r), 0) = 1 \qquad \text{for all } t \in [0,1].$$

In particular

$$d_{LS}(I - T^{-1}(S), B(0,r), 0) = 1$$

Thus, the couple of nonlinear operators (T,S) satisfies the hypotheses of theorem (2.1) for $\lambda = 1$.Then $T - S : V \to V'$ is surjective.Therefore, there exists $u \in V$ such that

$$(T-S)u = 0 \qquad \text{in } V'$$

This completes the proof. □

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