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# On some properties of (k,h)-Pell sequence and (k,h)-Pell-Lucass sequence

Research Article

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**Abstract:** In this note we first define $(k,h)$-Pell sequence and $(k,h)$-Pell-Lucas sequence. Then we derive some formulas for $n^{th}$ term and sum of the first n terms of these sequences. Finally other properties of these sequences are represented.

**MSC:** 15A15 • 15A60

**Keywords:** Pell-Lucass sequence • Recursion relation • Sum



## 1. Introduction

In [1] Bueno studied $(k,h)$-Jacobsthal sequence of the form

$$T_n = kT_{n-1} + 2hT_{n-2}.$$

He found a formula of $n^{th}$ term and sum of the first n terms of this sequence. In this note we first define $(k,h)$-Pell sequence and $(k,h)$-Pell-Lucas sequence. Then we derive some formulas for $n^{th}$ term and sum of the first n terms of these sequences. Finally other properties of these sequences are represented. For more information about $(k,h)$-Jacobsthal sequence, Fibonacci sequence and some generalizations of this sequence see [1] - [8].

Pell sequence $\{P_n\}$ has the recursive relation

$$P_n = 2P_{n-1} + P_{n-2},$$

where $P_0 = 0, P_1 = 1$. Now we define a generalization of this sequence which we call it $(k,h)$-Pell sequence and denote it by $\Phi_n$. This sequence has the recursive relation

$$\Phi_n = 2k\Phi_{n-1} + h\Phi_{n-2}, \tag{1}$$

where $\Phi_0 = 0$ and $\Phi_1 = 2k$ and $k,h \in Z$ and $k^2 + h > 0$. Also we define (k,h)-Pell-Lucas sequence $\{\Lambda_n\}$ which has the recursive relation

$$\Lambda_n = 2k\Lambda_{n-1} + h\Lambda_{n-2}, \tag{2}$$

where $\Lambda_0 = 2$ and $\Lambda_1 = 2k$. It is known that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}, \tag{3}$$

$$\sum_{k=1}^{n-1} kx^k = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}. \tag{4}$$

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## 2. Main result

**Theorem 2.1.**
*Let $\Phi_n$ be as in (1) then we have*

$$\Phi_n = \frac{2k}{p}\left(\alpha^n - \beta^n\right),$$

*where $\alpha = k + \sqrt{k^2+h}, \beta = k - \sqrt{k^2+h}$ and $p = \alpha - \beta$.*

*Proof.* The recursive relation (1) has the characteristic equation

$$r^2 - 2kr - h = 0.$$

The roots of this equation are $\alpha = k + \sqrt{k^2+h}, \beta = k - \sqrt{k^2+h}$. Also we have $\alpha+\beta = 2k, \alpha-\beta = \sqrt{k^2+h} = p$ and $\alpha\beta = -h$. So the solution of the recursion relation (1) is

$$\Phi_n = c_1\alpha^n + c_2\beta^n. \tag{5}$$

If we use the initial values $\Phi_0 = 0$ and $\Phi_1 = 2k$ we get a linear system with two equations $c_1 + c_2 = 0$ and $c_1\alpha + c_2\beta = 2k$. This linear system has the solution $c_1 = \frac{2k}{p}$ and $c_2 = \frac{-2k}{p}$. By substituting these values in (5) we get

$$\Phi_n = \frac{2k}{p}\left(\alpha^n - \beta^n\right).$$

□

**Theorem 2.2.**
*Let $\Lambda_n$ be as in (2) then we have*

$$\Lambda_n = \alpha^n + \beta^n,$$

*where $\alpha = k + \sqrt{k^2+h}, \beta = k - \sqrt{k^2+h}$ and $\Lambda_0 = 2, \Lambda_1 = 2k$.*

*Proof.* The proof is similar to Theorem 2.1. □

**Theorem 2.3.**
*Let $\Phi_n$ be as in (1) then we have*

$$\sum_{m=0}^{n-1}\Phi_m = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k+h-1}.$$

*Proof.* By Theorem 2.1 we have

$$\sum_{m=0}^{n-1}\Phi_m = \frac{2k}{p}\sum_{m=0}^{n-1}\left(\alpha^m - \beta^m\right) = \frac{2k}{p}\left[\sum_{m=0}^{n-1}\alpha^m - \sum_{m=0}^{n-1}\beta^m\right]$$

According to (3) we get

$$\sum_{m=0}^{n-1}\Phi_m = \frac{2k}{p}\left[\frac{1-\alpha^n}{1-\alpha} - \frac{1-\beta^n}{1-\beta}\right] = \frac{2k}{p}\left[\frac{(1-\alpha^n)(1-\beta)-(1-\beta^n)(1-\alpha)}{(1-\alpha)(1-\beta)}\right].$$

After some calculations we get

$$\sum_{m=0}^{n-1}\Phi_m = \frac{2k}{p}\left(\frac{(\alpha-\beta)-(\alpha^n-\beta^n)+\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{1-(\alpha+\beta)+\alpha\beta}\right)$$

$$= \frac{2k}{p}\left[\frac{\frac{p\Phi_1}{2k} - \frac{p\Phi_n}{2k} + (-h)\frac{p\Phi_{n-1}}{2k}}{1-2k-h}\right] = \frac{\Phi_1 - \Phi_n - h\Phi_{n-1}}{1-2k-h} = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k+h-1}.$$

So we have

$$\sum_{m=0}^{n-1}\Phi_m = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k+h-1}.$$

□

**Theorem 2.4.**
*Let $\Phi_n$ be as in (1) then we have*

$$\sum_{m=0}^{n-1}\Phi_m\Phi_{m-1}=\frac{4k^2}{p^2}\left[\frac{2k-\Lambda_3+h\Lambda_{2n-1}-h^3\Lambda_{2n-3}}{(-h)^3+(-h)-(-h)\Lambda_2}+\frac{2k}{h}\left(\frac{1-(-h)^n}{1+h}\right)\right].$$

*Proof.* By Theorem 2.1 we have

$$\sum_{m=0}^{n-1}\Phi_m\Phi_{m-1}=\left(\frac{2k}{p}\right)^2\sum_{m=0}^{n-1}\left(\alpha^m-\beta^m\right)\left(\alpha^{m-1}-\beta^{m-1}\right)$$

$$=\frac{4k^2}{p^2}\left[\sum_{m=0}^{n-1}\alpha^{2m-1}+\sum_{m=0}^{n-1}\beta^{2m-1}-(\alpha+\beta)\sum_{m=0}^{n-1}(\alpha\beta)^{m-1}\right]$$

$$=\frac{4k^2}{p^2}\left[\sum_{m=0}^{n-1}\alpha^{-1}(\alpha^2)^m+\sum_{m=0}^{n-1}\beta^{-1}(\beta^2)^m-\frac{(\alpha+\beta)}{\alpha\beta}\sum_{m=0}^{n-1}(\alpha\beta)^m\right]$$

According to (3) we get

$$\sum_{m=0}^{n-1}\Phi_m\Phi_{m-1}=\frac{4k^2}{p^2}\left[(\frac{1}{\alpha})\frac{1-\alpha^{2n}}{1-\alpha^2}+(\frac{1}{\beta})\frac{1-\beta^{2n}}{1-\beta^2}-(\frac{2k}{-h})\frac{1-(\alpha\beta)^n}{1-\alpha\beta}\right].$$

$$=\frac{4k^2}{p^2}\left[\frac{1-\alpha^{2n}}{\alpha-\alpha^3}+\frac{1-\beta^{2n}}{\beta-\beta^3}+\frac{2k}{h}\left(\frac{1-(-h)^n}{1+h}\right)\right].$$

After some calculations we get

$$\sum_{m=0}^{n-1}\Phi_m\Phi_{m-1}=\frac{4k^2}{p^2}\left[\frac{(\alpha+\beta)-(\alpha^3+\beta^3)-\alpha\beta(\alpha^{2n-1}+\beta^{2n-1})+\alpha^3\beta^3(\alpha^{2n-3}+\beta^{2n-3})}{(\alpha\beta)^3+\alpha\beta-(\alpha\beta)(\alpha^2+\beta^2)}\right]$$

$$+\frac{4k^2}{p^2}\left[\frac{2k}{h}\left(\frac{1-(-h)^n}{1+h}\right)\right].$$

So by Theorem 2.1 and Theorem 2.2 we conclude that

$$\sum_{m=0}^{n-1}\Phi_m\Phi_{m-1}=\frac{4k^2}{p^2}\left[\frac{2k-\Lambda_3+h\Lambda_{2n-1}-h^3\Lambda_{2n-3}}{(-h)^3+(-h)-(-h)\Lambda_2}+\frac{2k}{h}\left(\frac{1-(-h)^n}{1+h}\right)\right].$$

□

**Theorem 2.5.**
*Let $\Phi_n$ be as in (1) then we have*

$$\sum_{m=0}^{n-1}\Phi_m^2=\frac{4k^2}{p^2}\left[\frac{h^2\Lambda_{2n-2}-\Lambda_{2n}-\Lambda_2+2}{h^2-\Lambda_2+1}+2\frac{(-h)^n-1}{h+1}\right].$$

*Proof.* By Theorem 2.1 we have

$$\sum_{m=0}^{n-1}\Phi_m^2=\sum_{m=0}^{n-1}\left[\frac{2k}{p}(\alpha^m-\beta^m)\right]^2=\frac{4k^2}{p^2}\sum_{m=0}^{n-1}\left(\alpha^{2m}+\beta^{2m}-2(\alpha\beta)^m\right)$$

$$=\frac{4k^2}{p^2}\left[\sum_{m=0}^{n-1}(\alpha^2)^m+\sum_{m=0}^{n-1}(\beta^2)^m-2\sum_{m=0}^{n-1}(\alpha\beta)^m\right].$$

According to (3) we get

$$\sum_{m=0}^{n-1}\Phi_m^2=\frac{4k^2}{p^2}\left[\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(\alpha\beta)^n-1}{\alpha\beta-1}\right]$$

$$=\frac{4k^2}{p^2}\left[\frac{\alpha^2\beta^2(\alpha^{2n-2}+\beta^{2n-2})-(\alpha^{2n}+\beta^{2n})-(\alpha^2+\beta^2)+2}{(\alpha\beta)^2-(\alpha^2+\beta^2)+1}\right]+\frac{4k^2}{p^2}\left[2\frac{(-h)^n-1}{h+1}\right].$$

So by Theorem 2.1 and Theorem 2.2 we deduce that

$$\sum_{m=0}^{n-1}\Phi_m^2=\frac{4k^2}{p^2}\left[\frac{h^2\Lambda_{2n-2}-\Lambda_{2n}-\Lambda_2+2}{h^2-\Lambda_2+1}+2\frac{(-h)^n-1}{h+1}\right].$$

□

**Theorem 2.6.**
*Let $\Lambda_n$ be as in (2) then we have*

$$\sum_{m=0}^{n-1} \Lambda_m = \frac{\Lambda_n + h\Lambda_{n-1} + 2k - 2}{2k + h - 1}.$$

*Proof.* The proof is similar to Theorem 2.3. □

**Theorem 2.7.**
*Let $\Lambda_n$ be as in (2) then we have*

$$\sum_{m=0}^{n-1} \Lambda_m \Lambda_{m-1} = \frac{2k - \Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}}{-h(1 + \Lambda_2 + h^2)} + \frac{2k}{h}\left[\frac{(-h)^n - 1}{h+1}\right].$$

*Proof.* The proof is similar to Theorem 2.4. □

**Theorem 2.8.**
*Let $\Lambda_n$ be as in (2) then we have*

$$\sum_{m=0}^{n-1} \Lambda_m^2 = \frac{2 - \Lambda_2 + h^2\Lambda_{2n-2} - \Lambda_{2n}}{1 - \Lambda_2 + h^2} + 2\frac{1 - (-h)^n}{1+h}.$$

*Proof.* The proof is similar to Theorem 2.5. □

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