On some properties of \((k,h)\)-Pell sequence and \((k,h)\)-Pell-Lucas sequence

Seyyed Hossein Jafari-Petroudi\(^a\) *, Behzad Pirouz\(^b\)

\(^a\) Department of Mathematics, Payame Noor University, P. O. Box 1935-3697, Tehran, Iran
\(^b\) Department of Mathematics, Azad University of Karaj, Karaj, Iran

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Abstract: In this note we first define \((k,h)\)-Pell sequence and \((k,h)\)-Pell-Lucas sequence. Then we derive some formulas for \(n^{th}\) term and sum of the first \(n\) terms of these sequences. Finally other properties of these sequences are represented.

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1. Introduction

In [1] Bueno studied \((k,h)\)-Jacobsthal sequence of the form

\[ T_n = kT_{n-1} + 2hT_{n-2}. \]

He found a formula of \(n^{th}\) term and sum of the first \(n\) terms of this sequence. In this note we first define \((k,h)\)-Pell sequence and \((k,h)\)-Pell-Lucas sequence. Then we derive some formulas for \(n^{th}\) term and sum of the first \(n\) terms of these sequences. Finally other properties of these sequences are represented. For more information about \((k,h)\)-Jacobsthal sequence, Fibonacci sequence and some generalizations of this sequence see [1] - [8].

Pell sequence \(\{P_n\}\) has the recursive relation

\[ P_n = 2P_{n-1} + P_{n-2}, \]

where \(P_0 = 0, P_1 = 1\). Now we define a generalization of this sequence which we call it \((k,h)\)-Pell sequence and denote it by \(\Phi_n\). This sequence has the recursive relation

\[ \Phi_n = 2k\Phi_{n-1} + h\Phi_{n-2}, \] (1)

where \(\Phi_0 = 0\) and \(\Phi_1 = 2k\) and \(k, h \in \mathbb{Z}\) and \(k^2 + h > 0\). Also we define \((k,h)\)-Pell-Lucas sequence \(\{\Lambda_n\}\) which has the recursive relation

\[ \Lambda_n = 2k\Lambda_{n-1} + h\Lambda_{n-2}, \] (2)

where \(\Lambda_0 = 2\) and \(\Lambda_1 = 2k\). It is known that

\[ \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}, \] (3)

\[ \sum_{k=1}^{n-1} kx^k = \frac{(n-1)x^n - nx^{n-1} + 1}{(x - 1)^2}. \] (4)

* Corresponding author.

E-mail addresses: hossein_5798@yahoo.com (Seyyed Hossein Jafari-Petroudi), behzadpirouz@gmail.com (Behzad Pirouz)
2. Main result

**Theorem 2.1.**
Let $\Phi_n$ be as in (1) then we have
\[ \Phi_n = \frac{2k}{p}(a^n - \beta^n), \]
where $\alpha = k + \sqrt{k^2 + h}, \beta = k - \sqrt{k^2 + h}$ and $p = \alpha - \beta$.

**Proof.** The recursive relation (1) has the characteristic equation
\[ r^2 - 2kr - h = 0. \]
The roots of this equation are $\alpha = k + \sqrt{k^2 + h}, \beta = k - \sqrt{k^2 + h}$. Also we have $\alpha + \beta = 2k, \alpha - \beta = \sqrt{k^2 + h} = p$ and $\alpha \beta = -h$. So the solution of the recursion relation (1) is
\[ \Phi_n = c_1 a^n + c_2 \beta^n. \]
If we use the initial values $\Phi_0 = 0$ and $\Phi_1 = 2k$ we get a linear system with two equations $c_1 + c_2 = 0$ and $c_1 \alpha + c_2 \beta = 2k$. This linear system has the solution $c_1 = \frac{2k}{p}$ and $c_2 = -\frac{2k}{p}$. By substituting these values in (5) we get
\[ \Phi_n = \frac{2k}{p}(a^n - \beta^n). \]

**Theorem 2.2.**
Let $\Lambda_n$ be as in (2) then we have
\[ \Lambda_n = \alpha^n + \beta^n, \]
where $\alpha = k + \sqrt{k^2 + h}, \beta = k - \sqrt{k^2 + h}$ and $\Lambda_0 = 2, \Lambda_1 = 2k$.

**Proof.** The proof is similar to Theorem 2.1.

**Theorem 2.3.**
Let $\Phi_n$ be as in (1) then we have
\[ \sum_{m=0}^{n-1} \Phi_m = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k + h - 1}. \]

**Proof.** By Theorem 2.1 we have
\[ \sum_{m=0}^{n-1} \Phi_m = \frac{2k}{p} \sum_{m=0}^{n-1} (a^m - \beta^m) = \frac{2k}{p} \left[ \sum_{m=0}^{n-1} a^m - \sum_{m=0}^{n-1} \beta^m \right] \]
According to (3) we get
\[ \sum_{m=0}^{n-1} \Phi_m = \frac{2k}{p} \left[ \frac{1 - a^n}{1 - \alpha} - \frac{1 - \beta^n}{1 - \beta} \right] = \frac{2k}{p} \left[ (1 - \alpha^n) - (1 - \beta^n) \right] \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha)(1 - \beta)}. \]
After some calculations we get
\[ \sum_{m=0}^{n-1} \Phi_m = \frac{2k}{p} \left[ \frac{(\alpha - \beta) - (a^n - \beta^n) + \alpha \beta (a^{n-1} - \beta^{n-1})}{1 - (\alpha + \beta) + \alpha \beta} \right] \]
\[ = \frac{2k}{p} \left[ \frac{\Phi_1 - \Phi_0 - h\Phi_{n-1} - k \Phi_{n-1} + \beta \Phi_{n-1}}{1 - 2k - h} \right] = \frac{\Phi_1 - \Phi_0 - h\Phi_{n-1} - 2k}{1 - 2k - h} = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k + h - 1}. \]
So we have
\[ \sum_{m=0}^{n-1} \Phi_m = \frac{\Phi_n + h\Phi_{n-1} - 2k}{2k + h - 1}. \]
Theorem 2.4.
Let $\Phi_n$ be as in (1) then we have
\[\sum_{m=0}^{n-1} \Phi_m \Phi_{m-1} = \frac{4k^2}{p^2} \left[ \frac{2k}{1-h} \left( \frac{\Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}}{(-h)^3 + (-h) - (-h)\Lambda_2} + \frac{2k}{h} \left( \frac{1-(-h)^n}{1+h} \right) \right) \right].\]

Proof. By Theorem 2.1 we have
\[\sum_{m=0}^{n-1} \Phi_m \Phi_{m-1} = \left( \frac{2k}{p} \right)^2 \left[ \sum_{m=0}^{n-1} (a^m - \beta^m)(a^{m-1} - \beta^{m-1}) \right]
\]
\[= \frac{4k^2}{p^2} \left[ \sum_{m=0}^{n-1} a^{2m-1} + \sum_{m=0}^{n-1} \beta^{2m-1} - (a + \beta) \sum_{m=0}^{n-1} (a\beta)^{m-1} \right]
\]
\[= \frac{4k^2}{p^2} \left[ \sum_{m=0}^{n-1} a^{-1}(a^2)^m + \sum_{m=0}^{n-1} \beta^{-1}(\beta^2)^m - (a + \beta) \sum_{m=0}^{n-1} (a\beta)^m \right]
\]
According to (3) we get
\[\sum_{m=0}^{n-1} \Phi_m \Phi_{m-1} = \frac{4k^2}{p^2} \left[ \frac{1-a^2n}{1-a^2} + \frac{1-\beta^{2n}}{1-\beta^2} - \frac{2k}{h} \frac{1-(a\beta)^n}{1-a\beta} \right].\]
\[= \frac{4k^2}{p^2} \left[ \frac{1-a^2n}{a-a^2} + \frac{1-\beta^{2n}}{\beta-\beta^2} + \frac{2k}{h} \left( \frac{1-(-h)^n}{1+h} \right) \right].\]
After some calculations we get
\[\sum_{m=0}^{n-1} \Phi_m \Phi_{m-1} = \frac{4k^2}{p^2} \left[ \frac{(a + \beta) - (a^3 + \beta^3) - a\beta(a^{2n-1} + \beta^{2n-1}) + a^3\beta^3(a^{2n-3} + \beta^{2n-3})}{(a\beta)^3 + a\beta - (a\beta)(a^2 + \beta^2)} \right]
\]
\[+ \frac{4k^2}{p^2} \left[ \frac{2k}{h} \left( \frac{1-(-h)^n}{1+h} \right) \right].\]
So by Theorem 2.1 and Theorem 2.2 we conclude that
\[\sum_{m=0}^{n-1} \Phi_m \Phi_{m-1} = \frac{4k^2}{p^2} \left[ \frac{2k - \Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}}{(-h)^3 + (-h) - (-h)\Lambda_2} + \frac{2k}{h} \left( \frac{1-(-h)^n}{1+h} \right) \right].\]

Theorem 2.5.
Let $\Phi_n$ be as in (1) then we have
\[\sum_{m=0}^{n-1} \Phi_m^2 = \frac{4k^2}{p^2} \left[ \frac{h^2\Lambda_{2n-2} - \Lambda_{2n} - \Lambda_2 + 2}{h^2 - \Lambda_2 + 1} + \frac{2(-h)^n - 1}{h + 1} \right].\]

Proof. By Theorem 2.1 we have
\[\sum_{m=0}^{n-1} \Phi_m^2 = \sum_{m=0}^{n-1} \left( \frac{2k}{p}(a^m - \beta^m) \right)^2 = \frac{4k^2}{p^2} \sum_{m=0}^{n-1} (a^2m + \beta^{2m} - 2(a\beta)^m)\]
\[= \frac{4k^2}{p^2} \left[ \sum_{m=0}^{n-1} (a^2)^m + \sum_{m=0}^{n-1} (\beta^2)^m - 2 \sum_{m=0}^{n-1} (a\beta)^m \right].\]
According to (3) we get
\[\sum_{m=0}^{n-1} \Phi_m^2 = \frac{4k^2}{p^2} \left[ \frac{a^{2n}-1}{a^2-1} + \frac{\beta^{2n}-1}{\beta^2-1} - 2 \frac{(a\beta)^n-1}{a\beta-1} \right]
\]
\[= \frac{4k^2}{p^2} \left[ \frac{\alpha^2\beta^2(a^{2n-2} + \beta^{2n-2}) - (a^{2n} + \beta^{2n}) - (a^2 + \beta^2) + 2}{(a\beta)^2 - (a^2 + \beta^2) + 1} \right] + \frac{4k^2}{p^2} \left[ \frac{2(-h)^n - 1}{h + 1} \right].\]
So by Theorem 2.1 and Theorem 2.2 we deduce that
\[\sum_{m=0}^{n-1} \Phi_m^2 = \frac{4k^2}{p^2} \left[ \frac{h^2\Lambda_{2n-2} - \Lambda_{2n} - \Lambda_2 + 2}{h^2 - \Lambda_2 + 1} + \frac{2(-h)^n - 1}{h + 1} \right].\]
**Theorem 2.6.**

Let \( \Lambda_n \) be as in (2) then we have

\[
\sum_{m=0}^{n-1} \Lambda_m = \frac{\Lambda_n + h\Lambda_{n-1} + 2k - 2}{2k + h - 1}.
\]

**Proof.** The proof is similar to Theorem 2.3.

**Theorem 2.7.**

Let \( \Lambda_n \) be as in (2) then we have

\[
\sum_{m=0}^{n-1} \Lambda_m \Lambda_{m-1} = \frac{2k - \Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}}{1 + h + h^2} + \frac{2k}{h} \left[ \frac{(-h)^n - 2}{h + 1} \right] .
\]

**Proof.** The proof is similar to Theorem 2.4.

**Theorem 2.8.**

Let \( \Lambda_n \) be as in (2) then we have

\[
\sum_{m=0}^{n-1} \Lambda_m^2 = \frac{2 - \Lambda_2 + h^2\Lambda_{2n-2} - \Lambda_{2n}}{1 - \Lambda_2 + h^2} + \frac{2}{1 + h} \left[ 1 - (-h)^n \right] .
\]

**Proof.** The proof is similar to Theorem 2.5.

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**References**