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Explicit expression for a first integral for some classes of polynomial differential systems

Research Article

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Abstract: In this paper we charecterize the integrability and introduce an explicit expression of first integral then consequently the curves which are formed by the trajectories of the planar differentials systems of the form

$$\begin{cases} x' = P_n(x,y) + xR_m(x,y) \\ y' = Q_n(x,y) + yR_m(x,y) \end{cases},$$

and

$$\begin{cases} x' = x\left(P_n(x,y) + R_m(x,y)\right) \\ y' = y\left(Q_n(x,y) + R_m(x,y)\right) \end{cases}$$

where $P_n(x,y)$, $Q_n(x,y)$, $R_m(x,y)$ homogeneous polynomials of degree n, n, m respectively. Concrete examples exhibiting the applicability of our result is introduced.

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1. Introduction and statement of the main results

The autonomous differential system on the plane given by

$$\begin{cases} x' = \dfrac{dx}{dt} = F(x,y), \\ y' = \dfrac{dy}{dt} = G(x,y). \end{cases} \tag{1}$$

where $F(x,y)$ and $G(x,y)$ are reals functions

In the qualitative theory of planar dynamical systems see [1–7], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem, There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly see [8–13].

There exist three main open problems in the qualitative theory of real planar differential systems, the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals. One of the classical tools in the

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classification of all trajectories of a dynamical system is to find first integrals, for or more details about first integral see for instance [14–23].

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} F(x,y) + \frac{\partial H(x,y)}{\partial y} G(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait see [24].

In this paper we are interested in studying the integrability and the curves which are formed by the trajectories of the 2-dimensional polynomial systems of the form

$$\begin{cases} x' = P_n(x,y) + xR_m(x,y) \\ y' = Q_n(x,y) + yR_m(x,y) \end{cases}, \tag{2}$$

and

$$\begin{cases} x' = x\left(P_n(x,y) + R_m(x,y)\right) \\ y' = y\left(Q_n(x,y) + R_m(x,y)\right) \end{cases} \tag{3}$$

where $P_n(x,y)$, $Q_n(x,y)$, $R_m(x,y)$ homogeneous polynomials of degree n, n, m respectively.

The autonomous differential system (3) on the plane known as Kolmogorov system see [25] . There are many natural phenomena which can be modeled the Kolmogorov systems such as mathematical ecology and population dynamics see [26] chemical reactions, plasma physics see [27], hydrodynamics see [17], economics etc.

We define the trigonometric functions

$$\begin{aligned}
&f_1(\theta) = P_n(\cos\theta, \sin\theta)\cos\theta + Q_n(\cos\theta, \sin\theta)\sin\theta, f_2(\theta) = R_m(\cos\theta, \sin\theta),\\
&f_3(\theta) = Q_n(\cos\theta, \sin\theta)\cos\theta - P_n(\cos\theta, \sin\theta)\sin\theta,\\
&g_1(\theta) = P_n(\cos\theta, \sin\theta)\cos^2\theta + Q_n(\cos\theta, \sin\theta)\sin^2\theta,\\
&g_2(\theta) = Q_n(\cos\theta, \sin\theta)\cos\theta\sin\theta - P_n(\cos\theta, \sin\theta)\cos\theta\sin\theta.
\end{aligned}$$

Our main result on the integrability and the curves which are formed by the trajectories of the planar differentials systems (2) and (3) is the following.

Theorem 1.1.
Consider polynomials systems (2) and (3), then the following statements hold.
(a) If $f_3(\theta) \neq 0$ and $\lambda \neq 0$, then system (2) has the first integral

$$H(x,y) = (x^2+y^2)^{\frac{\lambda}{2}} \exp\left(-\lambda \int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) - \lambda \int^{\arctan\frac{y}{x}} \exp\left(-\lambda \int^{w} A(\omega)\,d\omega\right) B(w)\,dw$$

where $A(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, *and* $\lambda = n - m - 1$
Moreover, the curves which are formed by the trajectories of the differential system (2), written as

$$x^2+y^2 = \left[h\exp\left(\lambda \int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \lambda \exp\left(\lambda \int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \int^{\arctan\frac{y}{x}} \exp\left(-\lambda \int^{w} A(\omega)\,d\omega\right) B(w)\,dw \right]^{\frac{2}{\lambda}}$$

(b) If $f_3(\theta) \neq 0$ and $\lambda = 0$, then system (2) has the first integral

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int^{\arctan\frac{y}{x}} \left(A(\omega) + B(\omega)\right) d\omega\right)$$

Moreover, the curves which are formed by the trajectories of the differential system (2), written as

$$\sqrt{x^2+y^2} = h\exp\left(\int^{\arctan\frac{y}{x}} \left(A(\omega) + B(\omega)\right) d\omega\right)$$

(c) If $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then system (2) has the first integral $H(x,y) = \dfrac{y}{x}$.

Moreover, the curves which are formed by the trajectories of the differential system (2), *written as* $y = hx$ *where* $h \in \mathbb{R}$

(d) *If* $g_2(\theta) \neq 0$ *and* $\lambda + 1 \neq 0$, *then system* (3) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{\lambda+1}{2}} \exp\left(-(\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\,d\omega\right) - (\lambda+1)\int^{\arctan\frac{y}{x}} \exp\left(-(\lambda+1)\int^{w} C(\omega)\,d\omega\right) D(w)\,dw$$

where $C(\theta) = \dfrac{g_1(\theta)}{g_2(\theta)}$, $D(\theta) = \dfrac{f_2(\theta)}{g_2(\theta)}$, *and* $\lambda = n - m - 1$

Moreover, the curves which are formed by the trajectories of the differential system (3), *written as*

$$x^2+y^2 = \left[\begin{array}{c} h\exp\left((\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\,d\omega\right) + \\ (\lambda+1)\exp\left((\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\,d\omega\right)\int^{\arctan\frac{y}{x}} \exp\left(-(\lambda+1)\int^{w} C(\omega)\,d\omega\right) D(w)\,dw \end{array}\right]^{\frac{2}{\lambda+1}}$$

(e) *If* $g_2(\theta) \neq 0$ *and* $\lambda + 1 = 0$, *then system* (3) *has the first integral*

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int^{\arctan\frac{y}{x}} (C(\omega)+D(\omega))\,d\omega\right)$$

Moreover, the curves which are formed by the trajectories of the differential system (3), *written as*

$$\sqrt{x^2+y^2} = h\exp\left(\int^{\arctan\frac{y}{x}} (C(\omega)+D(\omega))\,d\omega\right)$$

(f) *If* $g_2(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then system* (3) *has the first integral* $H(x,y) = \dfrac{y}{x}$,

Moreover, the curves which are formed by the trajectories of the differential system (3), *written as* $y = hx$ *where* $h \in \mathbb{R}$

Proof. In order to prove our results (a), (b) and (c) we write the polynomial differential system (2) in polar coordinates (r,θ), defined by $x = r\cos\theta$, and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} r' = f_1(\theta)\,r^n + f_2(\theta)\,r^{m+1} \\ \theta' = f_3(\theta)\,r^{n-1} \end{cases} \tag{4}$$

where the functions $f_1(\theta)$, $f_2(\theta)$ and $f_3(\theta)$ are given in introduction, and $r' = \dfrac{dr}{dt}$, $\theta' = \dfrac{d\theta}{dt}$.

If $f_3(\theta) \neq 0$, and $\lambda \neq 0$

We take as new independent variable the variable θ, then the differential system (4) becomes the differential equation

$$\frac{dr}{d\theta} = A(\theta)\,r + B(\theta)\,r^{1-\lambda} \tag{5}$$

where the functions $A(\theta)$ and $B(\theta)$ are the ones defined in statement (a) of theorem 1, and $\lambda = n - m - 1$.

We note that the differential equation (5) is a Bernoulli differential equation. By introducing the standard change of variables $\rho = r^\lambda$ the Bernoulli differential equation becomes the linear equation

$$\frac{d\rho}{d\theta} = \lambda\left(A(\theta)\,\rho + B(\theta)\right) \tag{6}$$

The general solution of linear equation (6) is

$$\rho(\theta) = \exp\left(\lambda\int^{\theta} A(\omega)\,d\omega\right)\left[\alpha + \lambda\int^{\theta}\exp\left(-\lambda\int^{w} A(\omega)\,d\omega\right)B(w)\,dw\right]$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{\lambda}{2}}\exp\left(-\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) - \lambda\int^{\arctan\frac{y}{x}}\exp\left(-\lambda\int^{w} A(\omega)\,d\omega\right)B(w)\,dw$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), can be written as

$$x^2+y^2 = \left[\begin{array}{c} h\exp\left(\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ \lambda\exp\left(\lambda\int^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right)\int^{\arctan\frac{y}{x}}\exp\left(-\lambda\int^{w} A(\omega)\,d\omega\right)B(w)\,dw \end{array}\right]^{\frac{2}{\lambda}}$$

Hence statement (a) of Theorem 1.1 is proved.
Suppose now that $f_3(\theta) \neq 0$, and $\lambda = 0$.
Taking as new independent variable the coordinate θ, this differential system (4) writes

$$\frac{dr}{d\theta} = (A(\theta) + B(\theta))\, r \tag{7}$$

The general solution of equation (7) is

$$r(\theta) = \alpha \exp\left(\int^{\theta} (A(\omega) + B(\omega))\, d\omega\right)$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x,y) = \sqrt{x^2+y^2} \exp\left(-\int^{\arctan\frac{y}{x}} (A(\omega) + B(\omega))\, d\omega\right)$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), can be written as

$$\sqrt{x^2+y^2} = h \exp\left(\int^{\arctan\frac{y}{x}} (A(\omega) + B(\omega))\, d\omega\right)$$

Hence statement (b) of Theorem 1.1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then, from (4) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $H(x,y) = \dfrac{y}{x}$ is a first integral of the system, then since all straight lines through the origin are formed by trajectories. Then the curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), can be written as $y = hx$ where $h \in \mathbb{R}$.
This completes the proof of statement (c) of Theorem 1.1.

In order to prove our results (d) , (e) and (f) , we write the polynomial differential system (3) in polar coordinates (r,θ) , defined by $x = r\cos\theta$, and $y = r\sin\theta$, then system (3) becomes

$$\begin{cases} r' = g_1(\theta)\, r^{n+1} + f_2(\theta)\, r^{m+1} \\ \theta' = g_2(\theta)\, r^{n} \end{cases} \tag{8}$$

where $f_2(\theta)$, $g_1(\theta)$, $g_2(\theta)$ the trigonometric functions are given in introduction.

If $g_2(\theta) \neq 0$, and $\lambda + 1 \neq 0$

Taking as new independent variable the coordinate θ, then the differential system (8) becomes the differential equation

$$\frac{dr}{d\theta} = C(\theta)\, r + D(\theta)\, r^{-\lambda} \tag{9}$$

where the functions $C(\theta)$ and $D(\theta)$ are the ones defined in statement (d) of Theorem 1, and $\lambda = n - m - 1$

which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{\lambda+1}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = (\lambda+1)\left(C(\theta)\,\rho + D(\theta)\right) \tag{10}$$

The general solution of linear Eq. (10) is

$$\rho(\theta) = \exp\left((\lambda+1)\int^{\theta} C(\omega)\, d\omega\right)\left[\alpha + (\lambda+1)\int^{\theta} \exp\left(-(\lambda+1)\int^{w} C(\omega)\, d\omega\right) D(w)\, dw\right]$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{\lambda+1}{2}} \exp\left(-(\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\, d\omega\right) - (\lambda+1)\int^{\arctan\frac{y}{x}} \exp\left(-(\lambda+1)\int^{w} C(\omega)\, d\omega\right) D(w)\, dw$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3), can be written as

$$x^2+y^2 = \left[\begin{array}{c} h \exp\left((\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\, d\omega\right) + \\ (\lambda+1)\exp\left((\lambda+1)\int^{\arctan\frac{y}{x}} C(\omega)\, d\omega\right)\int^{\arctan\frac{y}{x}} \exp\left(-(\lambda+1)\int^{w} C(\omega)\, d\omega\right) D(w)\, dw \end{array}\right]^{\frac{2}{\lambda+1}}$$

Hence statement (d) of Theorem 1.1 is proved.

Suppose now that $g_2(\theta) \neq 0$, and $\lambda + 1 = 0$
Taking as new independent variable the coordinate θ, this differential system (8) writes

$$\frac{dr}{d\theta} = (A(\theta) + B(\theta))\, r \tag{11}$$

The general solution of equation (11) is

$$r(\theta) = \alpha \exp\left(\int^{\theta} (C(\omega) + D(\omega))\, d\omega\right)$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$H(x, y) = \sqrt{x^2 + y^2} \exp\left(-\int^{\arctan \frac{y}{x}} (C(\omega) + D(\omega))\, d\omega\right)$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3), can be written as

$$\sqrt{x^2 + y^2} = h \exp\left(\int^{\arctan \frac{y}{x}} (C(\omega) + D(\omega))\, d\omega\right)$$

Hence statement (e) of (1.1) is proved.

Assume now that $g_2(\theta) = 0$ for all $\theta \in \mathbb{R}$, then, from system (8) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (3) are invariant by the flow of this system. Hence, $H(x, y) = \frac{y}{x}$ is a first integral of the system, then since all straight lines through the origin are formed by trajectories. Then the curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (3), can be written as $y = hx$ where $h \in \mathbb{R}$. This completes the proof of statement (f) of Theorem 1.1. □

2. Examples

The following examples are given to illustrate our result.

Example 2.1.
if we take $P_3(x, y) = x^3 - x^2 y + x y^2 - y^3$, $Q_3(x, y) = x^3 + x^2 y + x y^2 + y^3$, $R_0(x, y) = -1$, then system (2) reads

$$\begin{cases} x' = -x + x^3 - x^2 y + x y^2 - y^3 \\ y' = -y + x^3 + x^2 y + x y^2 + y^3 \end{cases}$$

which has the first integral $H(x, y) = (x^2 + y^2 - 1) \exp\left(-2 \arctan \frac{y}{x}\right)$.

The curves which are formed by the trajectories of the differential system written as $x^2 + y^2 - 1 = h \exp\left(2 \arctan \frac{y}{x}\right)$ where $h \in \mathbb{R}$.

Example 2.2.
if we take $P_3(x, y) = 2x^3 + 2x y^2$, $Q_3(x, y) = 4x^3 + y^3 + 4x y^2 + x^2 y$, $R_2(x, y) = 3x^2 + 3y^2$, then system (3) reads

$$\begin{cases} x' = x\left(2x^3 + 2x y^2 + 3x^2 + 3y^2\right) \\ y' = y\left(4x^3 + y^3 + 4x y^2 + x^2 y + 3x^2 + 3y^2\right) \end{cases}$$

which has the first integral $H(x, y) = \frac{(2x+3)(2x+y)}{x(2x+y+3)}$.

The curves which are formed by the trajectories of the differential system written as $\frac{(2x+3)(2x+y)}{x(2x+y+3)} = h$ where $h \in \mathbb{R}$.

Example 2.3.
if we take $P_3(x, y) = x^3 + x y^2$, $Q_3(x, y) = 2x^3 + y^3 + 2x y^2 + x^2 y$, $R_2(x, y) = -x^2 - y^2$, then system (3) reads

$$\begin{cases} x' = x\left(x^3 + x y^2 - x^2 - y^2\right) \\ y' = y\left(2x^3 + y^3 + 2x y^2 + x^2 y - x^2 - y^2\right) \end{cases}$$

which has the first integral $H(x, y) = \frac{-x - y + x y + x^2}{-x + x y + x^2}$.

The curves which are formed by the trajectories of the differential system written as $\frac{-x - y + x y + x^2}{-x + x y + x^2} = h$ where $h \in \mathbb{R}$.

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