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New upper bounds for the energy and signless Laplacian energy of a graph

Research Article

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Abstract: Let $M = (m_{ij})$ be an $n \times n$ real symmetric matrix with eigenvalues $\mu_1(M) \geq \mu_2(M) \geq ... \geq \mu_n(M)$. The energy $Eng(M)$ and spread $Spr(M)$ of M are defined respectively as $\sum_{i=1}^{n} |\mu_i(M) - \frac{Tra(M)}{n}|$ and $\mu_1(M) - \mu_n(M)$, where $Tra(M) := \sum_{i=1}^{n} \mu_i(M)$ is the trace of M. In this note we first present an inequality on the energy and spread of M. Then we obtained new upper bounds for the energy and signless Laplacian energy of a graph by applying that inequality to the adjacency matrix and signless Laplacian matrix of a graph.

MSC: 05C50

Keywords: Upper bound • Energy • Spread

1. Introduction

Let $M = (m_{ij})$ be an $n \times n$ real symmetric matrix with eigenvalues $\mu_1(M) \geq \mu_2(M) \geq ... \geq \mu_n(M)$. We use $Tra(M) := \sum_{i=1}^{n} \mu_i(M)$, $Eng(M) := \sum_{i=1}^{n} |\mu_i(M) - \frac{Tra(M)}{n}|$, and $Spr(M) := \mu_1(M) - \mu_n(M)$ to denote the trace, energy, and spread of M, respectively. We define $S(M) := \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}^2$ and $R(M) := \sum_{i=1}^{n} \left(\mu_i(M) - \frac{Tra(M)}{n}\right)^2$. All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G. The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^{n} |\mu_i(G)|$ (see [2]). The spread, denoted $Spr(G)$, of G is defined as $\mu_1(G) - \mu_n(G)$. The signless Laplacian matrix, denoted $Q(G)$, of G is defined as $D(G) + A(G)$, where $D(G)$ is a diagonal matrix such that its diagonal entries are the degrees of the vertices in G. The eigenvalues $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$ of $Q(G)$ are the signless Laplacian eigenvalues of G. The signless Laplacian energy, denoted $SLEng(G)$, of G is defined as $\sum_{i=1}^{n} |q_i(G) - \frac{2e}{n}|$ (see [3]). The signless Laplacian spread, denoted $SLSpr(G)$, of G is defined as $q_1(G) - q_n(G)$ (see [4]). The independence number, denoted $\alpha(G)$, is defined the size of the largest independent set in G. The 2 – degree, denoted $t(v)$, of a vertex v in G is defined as the sum of degrees of vertices adjacent to v. We use $T(G)$ to denote the maximum 2 – degree of G. Obviously, $T(G) \leq (\Delta(G))^2$, where $\Delta(G)$ is the maximum degree of G. We also use $\chi(G)$ to denote the chromatic number of G. The first Zagreb index, denoted $Z(G)$, of G is defined as $\sum_{v \in V(G)} (d_G(v))^2$. A bipartite graph G is called semiregular if all the vertices in the same vertex part of a bipartition of the vertex set of G have the same degree.

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2. Main results and their proofs

In this section, we first present the following inequality on the energy and spread of a real symmetric matrix.

Theorem 2.1.
Let M be an $n \times n$ real symmetric matrix with eigenvalues $\mu_1(M) \geq \mu_2(M) \geq ... \geq \mu_n(M)$. Suppose c_1 and c_2 are two nonnegative real expressions such that $\mu_1 - \frac{Tra(M)}{n} \leq c_1$ and $\mu_n - \frac{Tra(M)}{n} \geq -c_2$. Then

$$Eng(M) \leq Spr(M) + \sqrt{(n-2)\left(S(M) - \frac{(Tra(M))^2}{n} + 2c_1c_2 - (Spr(M))^2\right)}.$$

Proof. We first notice that

$$R(M) = \sum_{i=1}^{n} (\mu_i(M))^2 - \frac{(Tra(M))^2}{n} = S(M) - \frac{(Tra(M))^2}{n}.$$

From Cauchy – Schwarz inequality and the fact $\nu_1(M) := \mu_1(M) - \frac{Tra(M)}{n} \geq 0 \geq \nu_n(M) := \mu_n - \frac{Tra(M)}{n}$, we have that

$$\begin{aligned}
Eng(M) &= \sum_{i=1}^{n} |\mu_i(M) - \frac{Tra(M)}{n}| \\
&= \mu_1(M) - \mu_n(M) + \sum_{i=2}^{n-1} |\mu_i(M) - \frac{Tra(M)}{n}| \\
&\leq Spr(M) + \sqrt{(n-2)\sum_{i=2}^{n-1}\left(\mu_i(M) - \frac{Tra(M)}{n}\right)^2} \\
&= Spr(M) + \sqrt{(n-2)\left(\sum_{i=1}^{n}\left(\mu_i(M) - \frac{Tra(M)}{n}\right)^2 - (\nu_1(M))^2 - (\nu_n(M))^2\right)} \\
&= Spr(M) + \sqrt{(n-2)\left(R(M) - (\nu_1(M) - \nu_n(M))^2 - 2\nu_1(M)\nu_n(M)\right)} \\
&\leq Spr(M) + \sqrt{(n-2)\left(S(M) - \frac{(Tra(M))^2}{n} + 2c_1c_2 - (Spr(M))^2\right)}.
\end{aligned}$$

Hence, we complete the proof of Theorem 2.1. □

Applying Theorem 2.1 to the adjacency matrix of a graph G, we have the following Theorem 2.2.

Theorem 2.2.
Let G be a graph of order n with e edges. Suppose c_1 and c_2 are two nonnegative real expressions such that $\mu_1 \leq c_1$ and $\mu_n \geq -c_2$. Then

$$Eng(G) \leq Spr(G) + \sqrt{(n-2)(2e + 2c_1c_2 - (Spr(G))^2)}.$$

Applying Theorem 2.1 to the signless Laplacian matrix of a graph G, we have the following Theorem 2.3.

Theorem 2.3.
Let G be a graph of order n with e edges and the first Zagreb index Z. Suppose c_1 and c_2 are two nonnegative real expressions such that $q_1 - \frac{2e}{n} \leq c_1$ and $q_n - \frac{2e}{n} \geq -c_2$. Then

$$SLEng(G) \leq SLSpr(G) + \sqrt{(n-2)\left(Z^2 + 2e - \frac{4e^2}{n} + 2c_1c_2 - (SLSpr(G))^2\right)}.$$

Next, we will use Theorem 2.2 and Theorem 2.3 to obtain new upper bounds for the energy and signless Laplacian energy of a graph, respectively. we need the following existing results as our lemmas to achieve our goals.

The following lemma is Theorem 1 on Page 5 in [5].

Lemma 2.1.
Let G be a connected graph with the maximum $2-$degree T. Then $\mu_1 \leq \sqrt{T}$ with equality if and only if G is either a regular graph or a semiregular bipartite graph.

The following lemma follows from Proposition 2 on Page 174 in [6].

Lemma 2.2.
Let G be a graph. Then $\mu_n \geq -\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ with equality if and only if G is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

The following lemma is Corollary 3.4 on Page 2731 in [7].

Lemma 2.3.
Let G be a graph with independence number α. Then $Spr(G) \geq 2\delta\sqrt{\frac{\alpha}{n-\alpha}}$. If equality holds, then G is a semiregular bipartite graph.

The following lemma is Theorem 1.5 on Page 26 in [8].

Lemma 2.4.
For a graph G of order n with e edges,

$$Spr(G) \leq \mu_1 + \sqrt{2e - \mu_1^2} \leq 2\sqrt{e}.$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $e = 0$ or G is $K_{a,b}$ for some a, b with $e = ab$ and $a + b \leq n$.

The following lemma follows from Theorem 4.6 on Page 163 in [9].

Lemma 2.5.
Let G be a graph. Then $q_1 \leq 2\Delta$.

The following lemma is Proposition 1 on Page 2343 in [10].

Lemma 2.6.
Let G be a graph of order n and $e \geq 1$ edges. Then $SLSpr(G) \geq 2$ with equality if and only if G consists of a union of independent edges and possibly some isolated vertices.

The following lemma is from Theorem 2.1 on Page 507 in [4].

Lemma 2.7.
If G is a connected graph. Then $SLSpr(G) \leq \max\{d(v) + \frac{t(v)}{d(v)} : v \in V(G)\}$ with equality if and only if G is regular bipartite or semiregular bipartite.

The following lemma is from Theorem 3.1 on Page 61 in [11].

Lemma 2.8.
If G is a graph of order n and e edges. Then

$$\max\{d(v) + \frac{t(v)}{d(v)} : v \in V(G)\} \leq \left(\frac{2e}{n-1} + n - 2\right).$$

The following lemma follows from Theorem 1 on Page 245 in [12] and Theorem 3.6 on Page 40 in [13].

Lemma 2.9.
If G is a graph of order $n \geq 2$ and $e \geq 1$ edges. Then

$$Z(G) \leq e\left(\frac{2e}{n-1} + n - 2\right)$$

with equality if and only if G is $K_{1,n-1}$ or K_n or the union of K_{n-1} and one isolated vertex.

From Lemma 2.1, we can choose $c_1 = \sqrt{T}$, where T is the maximum 2 – degree of a graph, in Theorem 2.2. From Lemma 2.2, we can choose $c_2 = \sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ in Theorem 2.2. Then, by Theorem 2.2, Lemma 2.1 - Lemma 2.4, we can obtain the following new upper bounds for the energy of a graph.

Theorem 2.4.
Let G be a connected graph of order n with e edges. Then

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \sqrt{T\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2 \alpha}{n-\alpha)}\right)} \\ &\leq 2\sqrt{e} + \sqrt{2(n-2)\left(e + \Delta\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2 \alpha}{n-\alpha}\right)}, \end{aligned}$$

where T and α are respectively the maximum 2 – degree and independence number of G.
Moreover, the first inequality becomes an equality if and only if G is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$; the second inequality becomes an inequality if and only if n is even and G is $K_{\frac{n}{2}, \frac{n}{2}}$.

From Lemma 2.5, we can choose $c_1 = 2\Delta$ in Theorem 2.3. Since Q is positive semidefinite, $q_n - \frac{2e}{n} \geq -\frac{2e}{n}$. Thus we can choose $c_2 = \frac{2e}{n}$ in Theorem 2.3. Then, by Theorem 2.3, Lemma 2.5 - Lemma 2.9, we can obtain the following new upper bound for the signless Laplacian energy of a graph.

Theorem 2.5.
Let G be a connected graph with $n \geq 2$ vertices and $e \geq 1$ edges. Then

$$SLEng(G) \leq \frac{2e}{n-1} + n - 2 + \sqrt{(n-2)\left(\frac{2e^2}{n-1} + \frac{8e\Delta - 4e^2}{n} + en - 4\right)}$$

with equality if and only if G is K_2.

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