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Analytical solutions for system of fractional partial differential equations by homotopy perturbation transform method

Research Article

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Abstract: In this letter, the homotopy perturbation transform method is used to obtain analytical approximate solutions to the systems of nonlinear fractional partial differential equations. The proposed method was derived by combining Laplace transform and homotopy perturbation method. It yields solutions in convergent series forms with easily computable terms. The fractional derivative is described in the Caputo sense. Illustrative examples demonstrate the efficiency of new method.

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Keywords: Fractional partial differential equations • Homotopy perturbation method • Laplace transform • Caputo fractional derivative

1. Introduction

Fractional partial differential equations started to be used in describing of real world phenomena [1]. The analytic results on uniqueness and existence of solutions of fractional differential equation were favorite topics of many researches [2].

Recently, there are many methods used to solve fractional partial differential equations such as, Adomian decomposition method [3–5], variational iteration method [6–8], homotopy perturbation method [8, 9], differential transform method [10, 11], iterative method [12], homotopy analysis method [13] and another methods. There are several definitions of a fractional derivative and integral of order μ , the Caputo fractional derivative of $f(x,y,t)$ is defined as [2]:

$$D_t^{\mu} f(x,y,t) = \frac{\partial^{\mu} f(x,y,t)}{\partial t^{\mu}} = \begin{cases} J^{m-\mu}\left[\frac{\partial^m f(x,y,t)}{\partial t^m}\right], & m-1<\mu\leq m, \\ \frac{\partial^m f(x,y,t)}{\partial t^m}, & \mu=m, m\in N. \end{cases} \tag{1}$$

where J^{μ} denotes the Riemann-Liouville fractional integral $f(x,y,t)$ defined by:

$$J_t^{\mu} f(x,y,t) = \frac{1}{\Gamma(\mu)}\int_0^t \frac{f(x,y,\tau)d\tau}{(t-\tau)^{1-\mu}}, t>0, \tag{2}$$

$$J_t^0 f(x,y,t) = f(x,y,t). \tag{3}$$

The Caputo fractional derivative is considered here because it allows traditional initial conditions to be included in the formulation of the problem [2]. In this paper, our aims are to present the coupling method of Laplace transform and homotopy perturbation method, which is called as the fractional homotopy perturbation transform method, and to use it to solve the system of nonlinear fractional differential equations.

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2. Fractional Homotopy Perturbation Transform Method

We illustrate the basic idea of this method, by considering a system of fractional partial differential equations with the initial conditions of general form:

$$\frac{\partial^{\alpha_i} u_i(x,y,t)}{\partial t^{\alpha_i}} + N_i(U) = g_i(x,y,t),\, i = 1,2,...,n \tag{4}$$

with the initial conditions

$$u_i(x,y,0) = f_i(x,y), \tag{5}$$

where $U = \left[u_1(x,y,t), u_2(x,y,t), \ldots, u_n(x,y,t)\right]$, N_i are nonlinear local fractional differential operator, and $g_i(x,y,t)$ are the source term .
The method consists of first applying the Laplace transform to both sides of (4) and then by using initial conditions (5), we have

$$L\left[\frac{\partial^{\alpha_i} u_i(x,y,t)}{\partial t^{\alpha_i}}\right] + L[N_i(U)] = L\left[g_i(x,y,t)\right]. \tag{6}$$

Using the property of the Laplace transform, we have

$$L\left[u_i(x,y,t)\right] = \frac{f_i(x,y)}{s^{\alpha_i}} + \frac{1}{s^{\alpha_i}} L\left[g_i(x,y,t)\right] - \frac{1}{s^{\alpha_i}} L[N_i(U)]. \tag{7}$$

Applying inverse Laplace transform on both sides of (7), we get

$$u_i(x,y,t) = K_i(x,y,t) - L^{-1}\left(\frac{1}{s^{\alpha_i}} L[N_i(U)]\right), \tag{8}$$

where $K_i(x,y,t) = L^{-1}\left(\frac{f_i(x,y)}{s^{\alpha_i}} + \frac{1}{s^{\alpha_i}} L\left[g_i(x,y,t)\right]\right)$.The second step in homotopy perturbation transform method is that we represent solution as an infinite series given below

$$u_i(x,y,t) = \sum_{n=0}^{\infty} p^n u_{in}(x,y,t), \tag{9}$$

and the nonlinear term can be decomposed as

$$N_i(U) = \sum_{n=0}^{\infty} p^n H_{in}(U), \tag{10}$$

for some He's polynomials $H_{in}(U)$ [14], that are given

$$H_{in}(U) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N_i\left(\sum_{m=0}^{\infty} p^m u_{im}(x,y,t)\right)\right]_{p=0}, n = 0,1,\ldots \tag{11}$$

Substituting (9) and (10) in (8), we get

$$\sum_{n=0}^{\infty} p^n u_{in}(x,y,t) = K_i(x,y,t) - p\left[L^{-1}\left(\frac{1}{s^{\alpha_i}} L\left[\sum_{n=0}^{\infty} p^n H_{in}(U)\right]\right)\right], \tag{12}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Equating the terms with identical powers in p in (12), we obtain the following approximations:

$$\begin{aligned}
&p^0 : u_{i0}(x,y,t) = K_i(x,y,t)\\
&p^1 : u_{i1}(x,y,t) = L^{-1}\left(\frac{1}{s^{\alpha_i}} L[H_{i0}(U)]\right)\\
&p^2 : u_{i2}(x,y,t) = L^{-1}\left(\frac{1}{s^{\alpha_i}} L[H_{i1}(U)]\right)\\
&\vdots\\
&p^{n+1} : u_{i(n+1)}(x,y,t) = L^{-1}\left(\frac{1}{s^{\alpha_i}} L[H_{in}(U)]\right).
\end{aligned} \tag{13}$$

The best approximation for the solution are

$$u_i(x,y,t) = \lim_{p\to 1} u_{in}(x,y,t) = u_{i0}(x,y,t) + u_{i1}(x,y,t) + u_{i2}(x,y,t) + \cdots \tag{14}$$

3. Applications

In this section, we shall apply fractional homotopy perturbation transform method for solving system of nonlinear fractional partial differential equations.

Example 3.1.
Consider the system of nonlinear fractional partial differential equations:

$$\begin{aligned} &D_t^{\alpha} u - v_x w_y = 1, \\ &D_t^{\beta} v - w_x u_y = 5, \\ &D_t^{\sigma} w - u_x v_y = 5, 0 < \alpha, \beta, \sigma \le 1. \end{aligned} \tag{15}$$

with the initial condition

$$\begin{aligned} &u(x, y, 0) = x + 2y, \\ &v(x, y, 0) = x - 2y, \\ &u(x, y, 0) = -x + 2y, \end{aligned} \tag{16}$$

Taking Laplace transform both of sides (15), subject to the initial conditions, we have

$$\begin{aligned} u(x, y, s) &= \frac{1}{s^{2\alpha}} + \frac{1}{s^{\alpha}}(x + 2y) + \frac{1}{s^{\alpha}} L\left[v_x w_y\right], \\ v(x, y, s) &= \frac{5}{s^{2\beta}} + \frac{1}{s^{\beta}}(x - 2y) + \frac{1}{s^{\beta}} L\left[w_x u_y\right], \\ w(x, y, s) &= \frac{5}{s^{2\sigma}} + \frac{1}{s^{\sigma}}(-x + 2y) + \frac{1}{s^{\sigma}} L\left[u_x v_y\right]. \end{aligned} \tag{17}$$

Applying inverse Laplace transform, we get

$$\begin{aligned} u(x, y, t) &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + x + 2y + L^{-1}\left(\frac{1}{s^{\alpha}} L\left[v_x w_y\right]\right), \\ v(x, y, t) &= \frac{5t^{\beta}}{\Gamma(1+\beta)} + x - 2y + L^{-1}\left(\frac{1}{s^{\beta}} L\left[w_x u_y\right]\right), \\ w(x, y, t) &= \frac{5t^{\sigma}}{\Gamma(1+\sigma)} - x + 2y + L^{-1}\left(\frac{1}{s^{\sigma}} L\left[u_x v_y\right]\right). \end{aligned} \tag{18}$$

In view of the algorithm given in (13), we obtained the components as follows

$$\begin{aligned} &u_0(x, y, t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + x + 2y, \\ p^0 : &v_0(x, y, t) = \frac{5t^{\beta}}{\Gamma(1+\beta)} + x - 2y, \\ &w_0(x, y, t) = \frac{5t^{\sigma}}{\Gamma(1+\sigma)} - x + 2y, \end{aligned} \tag{19}$$

$$\begin{aligned} &u_1(x, y, t) = L^{-1}\left(\frac{1}{s^{\alpha}} L\left[v_{0x} w_{0y}\right]\right) = \frac{2t^{\alpha}}{\Gamma(1+\alpha)}, \\ p^1 : &v_1(x, y, t) = L^{-1}\left(\frac{1}{s^{\beta}} L\left[w_{0x} u_{0y}\right]\right) = -\frac{2t^{\beta}}{\Gamma(1+\beta)}, \\ &w_1(x, y, t) = L^{-1}\left(\frac{1}{s^{\sigma}} L\left[u_{0x} v_{0y}\right]\right) = \frac{2t^{\sigma}}{\Gamma(1+\sigma)}, \end{aligned} \tag{20}$$

$$\begin{aligned} &u_2(x, y, t) = L^{-1}\left(\frac{1}{s^{\alpha}} L\left[v_{1x} w_{0y} + v_{0x} w_{1y}\right]\right) = 0, \\ p^2 : &v_2(x, y, t) = L^{-1}\left(\frac{1}{s^{\beta}} L\left[w_{1x} u_{0y} + w_{0x} u_{1y}\right]\right) = 0, \\ &w_2(x, y, t) = L^{-1}\left(\frac{1}{s^{\sigma}} L\left[u_{1x} v_{0y} + u_{0x} v_{1y}\right]\right) = 0. \end{aligned} \tag{21}$$

Therefore, the solution of system (15) is given below

$$\begin{aligned} u(x,y,t) &= \frac{3t^{\alpha}}{\Gamma(1+\alpha)} + x + 2y, \\ v(x,y,t) &= \frac{3t^{\beta}}{\Gamma(1+\beta)} + x - 2y, \\ w(x,y,t) &= \frac{3t^{\sigma}}{\Gamma(1+\sigma)} - x + 2y. \end{aligned} \tag{22}$$

If $\alpha = \beta = \sigma = 1$ in (22) we have the exact solution of (15), which is

$$\begin{aligned} u(x,y,t) &= x + 2y + 3t, \\ v(x,y,t) &= x - 2y + 3t, \\ w(x,y,t) &= -x + 2y + 3t. \end{aligned} \tag{23}$$

Example 3.2.
Let us consider the system of nonlinear fractional partial differential equations:

$$\begin{aligned} &D_t^{\alpha} u - v_x w_y - v_y w_x = -u, \\ &D_t^{\beta} v - u_x w_y + u_y w_x = v, \\ &D_t^{\sigma} w - u_x v_y + u_y v_x = w, 0 < \alpha, \beta, \sigma \le 1. \end{aligned} \tag{24}$$

with the initial conditions

$$\begin{aligned} u(x,y,0) &= e^{x+y}, \\ v(x,y,0) &= e^{x-y}, \\ w(x,y,0) &= e^{-x+y}. \end{aligned} \tag{25}$$

Applying the same procedure as applied in previous example we construct the following:

$$\begin{aligned} u(x,y,t) &= e^{x+y} + L^{-1}\left(\frac{1}{s^{\alpha}} L\left[v_y w_x - v_x w_y - u\right]\right), \\ v(x,y,t) &= e^{x-y} + L^{-1}\left(\frac{1}{s^{\beta}} L\left[v - u_y w_x - u_x w_y - u\right]\right), \\ w(x,y,t) &= e^{-x+y} + L^{-1}\left(\frac{1}{s^{\sigma}} L\left[w - u_x v_y - u_y v_x - u\right]\right). \end{aligned} \tag{26}$$

In view of the algorithm given in (13), we obtained the components as follows

$$\begin{aligned} &u_0(x,y,t) = e^{x+y}, \\ p^0 : \; &v_0(x,y,t) = e^{x+y}, \\ &w_0(x,y,t) = e^{x+y}. \end{aligned} \tag{27}$$

$$\begin{aligned} &u_1(x,y,t) = L^{-1}\left(\frac{1}{s^{\alpha}} L\left[v_{0y} w_{0x} - v_{0x} w_{0y} - u_0\right]\right) = -\frac{t^{\alpha}}{\Gamma(1+\alpha)} e^{x+y}, \\ p^1 : \; &v_1(x,y,t) = L^{-1}\left(\frac{1}{s^{\beta}} L\left[v_0 - u_{0y} w_{0x} - u_{0x} w_{0y}\right]\right) = \frac{t^{\beta}}{\Gamma(1+\beta)} e^{x-y}, \\ &w_1(x,y,t) = L^{-1}\left(\frac{1}{s^{\sigma}} L\left[w_0 - u_{0x} v_{0y} - u_{0y} v_{0x}\right]\right) = \frac{t^{\sigma}}{\Gamma(1+\sigma)} e^{-x+y}, \end{aligned} \tag{28}$$

$$\begin{aligned} &u_2(x,y,t) = L^{-1}\left(\frac{1}{s^{\alpha}} L\left[(v_{1y} w_{0x} - v_{0y} w_{1x}) - (v_{1x} w_{0y} + v_{0x} w_{1y}) - u_1\right]\right) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} e^{x+y}, \\ p^2 : \; &v_2(x,y,t) = L^{-1}\left(\frac{1}{s^{\beta}} L\left[v_1 - (u_{0y} w_{1x} + u_{1y} w_{0x}) - (u_{0x} w_{1y} + u_{1x} w_{0y})\right]\right) = \frac{t^{2\beta}}{\Gamma(1+2\beta)} e^{x-y}, \\ &w_2(x,y,t) = L^{-1}\left(\frac{1}{s^{\sigma}} L\left[w_1 - (u_{1x} v_{0y} + u_{0x} v_{1y}) - (u_{1y} v_{0x} + u_{0y} v_{1x}\right]\right) = \frac{t^{2\sigma}}{\Gamma(1+2\sigma)} e^{-x+y}, \end{aligned} \tag{29}$$

and so on, in this manner the rest components of the decomposition series were obtained. Substituting the values of (27)-(29) into (9) gives the solution in a series form and a closed form by

$$\begin{aligned} u(x,y,t) &= E_{\alpha}(-t^{\alpha})e^{x+y}, \\ v(x,y,t) &= E_{\beta}(t^{\beta})e^{x-y}, \\ w(x,y,t) &= E_{\sigma}(t^{\sigma})e^{-x+y}. \end{aligned} \tag{30}$$

If $\alpha = \beta = \sigma = 1$ in (30) we have the exact solution of (24), which is

$$\begin{aligned} u(x,y,t) &= e^{x+y-t}, \\ v(x,y,t) &= e^{x-y+t}, \\ w(x,y,t) &= e^{-x+y+t}. \end{aligned} \tag{31}$$

The exact solution of the given problems in this paper is the same results as that obtained by the variational iteration method [7].

4. Conclusions

In this work, we employed the homotopy perturbation transform method for solving nonlinear system of fractional partial differential equations. The proposed method is successfully implemented by using the initial conditions only. It may be consulted that the homotopy perturbation transform method is very powerful and efficient technique in finding exact solutions for wide classes of problems.

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