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Complete controllability of nonlocal fractional stochastic differential evolution equations with Poisson jumps in Hilbert spaces

Research Article

Kerboua Mourad[a, *] **Debbouche Amar**[b]

[a] *Department of Mathematics, Guelma University, 24000, Guelma, Algeria*

[b] *Department of Mathematics, Guelma University, 24000, Guelma, Algeria*

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Abstract: The objective of this paper is to investigate the complete controllability property of a nonlinear nonlocal fractional stochastic control system with poisson jumps in a separable Hilbert space. By employing a fixed point technique, fractional calculus, stochastic analysis and methods adopted directly from deterministic control problems for the main results. In particular, we discuss the complete controllability of nonlinear nonlocal control system under the assumption that the corresponding linear system is completely controllable. Finally, an example is provided to illustrate the effectiveness of the obtained result.

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1. Introduction

During the past three decades, fractional differential equations and their applications have gained a lot of importance, mainly because this field has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [1–8] .

Stochastic differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the theory and applications of stochastic differential equations, (see [9–14]) and the references therein). To build more realistic models in economics, social sciences, chemistry, finance, physics and other areas, stochastic effects need to be taken into account. Therefore, many real world problems can be modeled by stochastic differential equations. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems. Recently, there is observed an increasing interest in the study of stochastic differential equations with jumps [15, 16] . Luo and Liu [17] established the existence and uniqueness theory of mild solutions to stochastic partial functional differential equations with Markovian switching and Poisson jumps, R. Sakthivel and Y. Ren [18] studied the complete controllability property of a nonlinear stochastic control system with jumps in a separable Hilbert space. It should be noted that most of the literature in this direction was mainly concerned with results on controllability of stochastic equations without jumps.

To the best of our knowledge, the complete controllability for a class of nonlinear nonlocal fractional stochastic dynamical systems with jumps is an untreated topic in the literature and this fact is the motivation of the present work.

The paper is organized as follows: in Section 2, we present some essential facts in fractional calculus, semigroup theory, stochastic analysis and control theory that will be used to obtain our main results. In Section 3, we formulate and prove sufficient conditions for ensuring the complete controllability for nonlinear nonlocal fractional stochastic control systems with jumps. Finally, in Section 4, an example is provided to illustrate the abstract theory.

* Corresponding author.

E-mail address: kerbouamourad@gmail.com (Kerboua Mourad)

2. Formulation of the problem

Let X be a separable Hilbert space with norm $\|\cdot\|_X$. Let Y be another Hilbert space with norm $\|\cdot\|_Y$. We denote by $L(Y,X)$ the set of all linear bounded operators from Y into X which is equipped with the usual operator norm $\|\cdot\|$ .Let (Ω,Γ,P) be a complete probability space equipped with some filtration $\{\Gamma_t\}_t, t\in[0,b]$, satisfying the usual conditions (i.e. it is right continuous and Γ_0 contains all P null-sets). Let $\{w(t): t\geq 0\}$ denote a Y-valued Wiener process defined on the probability space (Ω,Γ,P) with covariance operator Q,that is $E\langle w(t),x\rangle_Y \langle w(s),y\rangle_Y = (t\wedge s)\langle Qx,y\rangle_Y$, for all $x,y\in Y$, where Q is a positive, self-adjoint, trace-class operator on Y. In particular, we denote by $w(t)$ a Y-valued Q-Wiener process with respect to $\{\Gamma_t\}_{t\geq 0}$. Let $\beta_n(t)(n=1,2,3,...)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent on
$(\Omega,\Gamma,\{\Gamma_t\}_{t\geq 0},P)$. Now, $w(t)$ can be defined by [19]

$$w(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n,\quad t\in J,\quad e\in Y,$$

here λ_n are eigenvalues of Q and $e_n, n\in\mathbb{N}$ are the corresponding eigenvectors, i.e. $Qe_n=\lambda_n e_n, n=1,2,...,$

In order to define stochastic integrals with respect to the Q-Wiener process $w(t)$ we introduce the subspace $Y_0=Q^{1/2}(Y)$ of Y which is endowed with the inner product $\langle v_1,v_2\rangle_{Y_0}=\langle Q^{-1/2}v_1,Q^{-1/2}v_2\rangle_Y$. Moreover, it is a Hilbert space.

Let $L_2^0=L_2(Y_0,X)$ denote the space of all Hilbert–Schmidt operators from Y_0 into X. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|^2_{L_2^0}=tr((\mu Q^{1/2})(\mu Q^{1/2})^*)$ for any $\mu\in L_2^0$. Clearly for any bounded operators $\mu\in L(Y,X)$ this norm reduces to $\|\mu\|^2_{L_2^0}=tr(\mu Q\mu^*)$.

Let $\chi:[0,b]\to L_2^0$ be a predictable, Γ_b-adapted process such that $\int_0^t E\|\chi(s)\|^2_{L_2^0}ds<\infty$. Then we can define an X-valued stochastic integral $\int_0^t\chi(s)dw(s)$ which is a continuous square-integrable martingale. Let $q=(q(t)), t\in D_q$, be a stationary Γ_t-Poisson point process with characteristic measure λ. Let $N(dt,d\eta)$ be the Poisson counting measure associated with q, i.e. $N(t,Z)=\sum_{s\in D_q,s\leq t}I_Z(q(s))$ with measurable set $Z\in\overline{B}(Y-\{0\})$, which denotes the Borel σ-field of $Y-\{0\}$. Let $\widetilde{N}(dt,d\eta)=N(dt,d\eta)-dt\lambda(d\eta)$ be the compensated Poisson measure that is independent of $w(t)$. Let $P^2([0,b]\times Z;X)$ be the space of all predictable mappings $g:[0,b]\times Z\times\Omega\to X$ for which $\int_0^b\int_Z E\|g(t,\eta)\|^2_X dt\lambda(d\eta)<\infty$. Then, we can define the X-valued stochastic integral $\int_0^b\int_Z g(t,\eta)\widetilde{N}(dt,d\eta)$, which is a centred square-integrable martingale.

In this paper, we consider a mathematical model given by the following fractional nonlocal stochastic differential equations with poisson jumps and control variable,

$$\begin{aligned} {}^C D_t^q x(t) &= Ax(t)+Bu(t)+f(t,x(t))+\sigma(t,x(t))\frac{dw(t)}{dt} \\ &+ \frac{\int_Z g(t,x(t),\eta)\widetilde{N}(dt,d\eta)}{dt},\quad t\in J=[0,b], \end{aligned} \tag{1}$$

$$x(0)+h(x(t))=x_0, \tag{2}$$

where $0<q<1, {}^C D_t^q$ denotes the Caputo fractional derivative operator of order q. Let X and Y be two Hilbert spaces and the state $x(\cdot)$ takes its values X; A is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators $\{S(t)=e^{At}, t\geq 0\}$ and we suppose that $M_0=\sup_{t\geq 0}\|S(t)\|<\infty$. The control function $u(\cdot)$ is given in $L^2_\Gamma([0,b],U)$ of admissible control functions, U is a Hilbert space. B is a bounded linear operator from U into X; $f: J\times X\to X$, $g: J\times X\times Z\to X$, $\sigma: J\times X\to L_2^0$ and $h: C(J,X)\to X$ are appropriate functions; x_0 is Γ_0 measurable X-valued random variables independent of w.

Definition 2.1.
The fractional integral of order β with the lower limit 0 for a function f is defined as

$$I^\beta f(t)=\frac{1}{\Gamma(\beta)}\int_0^t\frac{f(s)}{(t-s)^{1-\beta}}ds,\quad t>0,\ \beta>0$$

provided the right-handside is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2.
Riemann–Liouville derivative of order β with lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$$^{L}D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\beta+1-n}} ds, \quad t > 0, \; n-1 < \beta < n.$$

Definition 2.3.
The Caputo derivative of order β for a function : $[0,\infty) \to R$ can be written as

$$^{c}D^{\beta} f(t) = ^{L}D^{\beta}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \; n-1 < \beta < n.$$

Remark 2.1. (a) If $f(t) \in C^n[0,\infty)$, then

$$^{c}D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^n(s)}{(t-s)^{\beta+1-n}} ds = I^{n-\beta} f^n(s), \quad t > 0, \; n-1 < \beta < n.$$

(b) The Caputo derivative of a constant is equal to zero.

(c) If f is an abstract function with values in E, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

The following results will be used through out this paper.

Lemma 2.1 ([20]).
Let $G : [0,b] \times \Omega \to L_2^0$ be a strongly measurable mapping such that $\int_0^b E\|G(t)\|_{L_2^0}^p dt < \infty$. Then

$$E\left\| \int_0^t G(s)dw(s) \right\|^p \le L_G \int_0^t E\|G(s)\|_{L_2^0}^p ds$$

for all $0 \le t \le b$ and $p \ge 2$, where L_G is the constant involving p and b

Lemma 2.2.
Consider the following linear fractional stochastic system

$$\begin{aligned} D_t^q x(t) &= Ax(t) + (Bu)(t) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in [0,b], \\ x(0) &= x_0 \end{aligned} \tag{3}$$

Let us now introduce the following operators.

Define the operator $L_0^b : L^2([0,b],U) \to L^2(b,X)$, the controllability operator $\Pi_0^b : L^2(b,X) \to L^2(b,X)$ associated with (3), and the controllability operato Ψ_0^b associated to the linear fractional stochastic system of (3) as

$$L_0^b u = \int_0^b (b-s)^{q-1} \mathscr{S}(b-s)Bu(s)\,ds,$$

$$\Pi_0^b(\cdot) = \int_0^b (b-s)^{2(q-1)} \mathscr{S}(b-s)BB^*\mathscr{S}^*(b-s)E(\cdot \mid \Gamma_s)\,ds,$$

$$\Psi_0^b = \int_0^b (b-s)^{2(q-1)} \mathscr{S}(b-s)BB^*\mathscr{S}^*(b-s)ds,$$

where B^* denotes the adjoint of B and $\mathscr{S}^*(t)$ is the adjoint of $\mathscr{S}(t)$.

Lemma 2.3 ([20]).
If the linear stochastic system (3) is completely controllable, then for some $\eta > 0$,

$$E\left\langle \Pi_0^b z, z \right\rangle \ge \eta E\|z\|^2, \text{ for some } \eta > 0 \text{ and all } z \in L^2(b,X),$$

and consequently

$$E\left\| \left(\Pi_0^b\right)^{-1} \right\|^2 \le \frac{1}{\eta}$$

3. Complete controllability

In this section, we formulate and prove conditions for the existence and the complete controllability results for nonlocal fractional stochastic eqs. (1)-(2) by using a fixed point approach. To prove the required results, we impose some Lipschitz and linear growth conditions on the functions f, σ, h and g.

Further, let $L^2(b, X)$ be the Banach space of all Γ_b-measurable square integrable random variables with values in the Hilbert space X. Let $D(J; L^2(\Gamma, X))$ be the Banach space of the càdlàg (right continuous with left limit) processes from J into $L^2(\Gamma, X)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. Let $H_2(J; X)$ be the closed subspace of $D(J; L^2(\Gamma, X))$ consisting of measurable and Γ_t-adapted X-valued process $x \in D(J; L^2(\Gamma, X))$ endowed with the norm $\|x\|_{H_2} = \left(\sup_{t\in J} E\|x(t)\|_X^2\right)^{1/2}$.

Now, we present the mild solution of the problem (1)-(2).

Definition 3.1 ([21–23]).
A stochastic process $x \in H_2([0, b], X)$ is a mild solution of (1)-(2) if for each $u \in L_\Gamma^2([0, b], U)$, it satisfies the following integral equation,

$$
\begin{aligned}
x(t) =& \mathcal{T}(t)\,(x_0 - h(x)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)[Bu(s) + f(s, x(s))]ds \\
&+ \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)\sigma(s, x(s))dw(s) \\
&+ \int_0^t \int_Z (t-s)^{q-1} g\left(s, x(s), \eta\right) \widetilde{N}\left(ds, d\eta\right),
\end{aligned}
$$

where $\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)S(t^q\theta)d\theta$; $\mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)S(t^q\theta)d\theta$; $\mathcal{S}(t)$ is a C_0-semigroup generated by a linear operator A on X; ξ_q is a probability density function defined on $(0,\infty)$,that is $\xi_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)d\theta = 1$.

Lemma 3.1 (see [24, 25]).
The operators $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are strongly continuous, i.e., for $x \in X$ and $0 \leq t_1 < t_2 \leq b$, we have $||T(t_2)x - T(t_1)x|| \to 0$ and $||\mathcal{S}(t_2)x - S(t_1)x|| \to 0$ as $t_1 \to t_2$.

We impose the following conditions on data of the problem.

(i) For any fixed $t \geq 0, \mathcal{T}(t)$ and $\mathcal{S}(t)$ are bounded linear operators, i.e., for any $x \in X$,

$$||\mathcal{T}(t)x|| \leq M_0\,||x||,\quad ||\mathcal{S}(t)x|| \leq \frac{M_0}{\Gamma(q)}\,||x||.$$

(ii) The functions f, σ and g are Borel measurable functions and satisfy the Lipschitz continuity condition and the linear growth condition for some constant $k > 0$ and arbitrary $x, y \in X$ such that

$$\left|\left|f(t,x) - f(t,y)\right|\right|_X^2 + \left|\left|\sigma(t,x) - \sigma(t,y)\right|\right|_{L_2^0}^2 + \int_Z \left|\left|g(t,x,\eta) - g(t,y,\eta)\right|\right|_X^2 \lambda d\eta \leq k\left|\left|x - y\right|\right|_X^2,$$

$$\left|\left|f(t,x)\right|\right|_X^2 + ||\sigma(t,x)||_{L_2^0}^2 + \int_Z \left|\left|g(t,x,\eta)\right|\right|_X^2 \lambda\left(d\eta\right) \leq k(1 + ||x||_X^2).$$

(iii) There exists a number $\widetilde{N_0} > 0$ and arbitrary $x, y \in X$ such that

$$\left|\left|h(x) - h\left(y\right)\right|\right|_X^2 \leq \widetilde{N_0}\left|\left|x - y\right|\right|_X^2,\quad ||h(x)||_X^2 \leq \widetilde{N_0}\left(1 + ||x||_X^2\right)$$

(iv) The linear stochastic system is completely controllable on J.

(v) There exists a number $\widetilde{L_0} > 0$ such that for arbitrary $x_1, x_2 \in X$,

$$\int_Z \left|\left|g(t,x_1,\eta) - g(t,x_2,\eta)\right|\right|_X^4 \lambda\left(d\eta\right) \leq \widetilde{L_0}\left(||x_1 - x_2||_X^4\right),$$

$$\int_Z \left|\left|g(t,x,\eta)\right|\right|_X^4 \lambda\left(d\eta\right) \leq \widetilde{L_0}\left(1 + ||x||_X^4\right).$$

Definition 3.2.
System (1)-(2) is completely controllable on $[0,b]$ if $\overline{\Re(b)} = L^2(\Gamma_b, X)$, where

$$\Re(b) = \{x(b) = x(b,u) : u \in L^2_\Gamma([0,b],U)\},$$

here $L^2_\Gamma([0,b],U)$, is the closed subspace of $L^2_\Gamma([0,b]\times\Omega;U)$, consisting of all Γ_t adapted, U-valued stochastic processes.

The following lemma is required to define the control function.The reader can refer to [7] for the proof.
Using the assumptions, for an arbitrary process $x(\cdot)$, define the control process

$$\begin{aligned} u(t,x) &= B^*(b-t)^{q-1}\mathscr{S}^*(b-t)\mathbf{E}\{(\Pi_0^b)^{-1}\left(\mathscr{T}(b)(x_0-h(x)-\int_0^t(b-s)^{q-1}\mathscr{S}(b-s)f(s,x(s))ds\right.\\ &- \left.\int_0^t(b-s)^{q-1}\mathscr{S}(b-s)\sigma(s,x(s))dw(s)-\int_0^t\int_Z(b-s)^{q-1}g\left(s,x(s),\eta\right)\widetilde{N}\left(ds,d\eta\right)\right)\mid\Gamma_t\} \end{aligned}$$

Now, let us state and prove the following lemma, which will be used in the proof of the main results.

Lemma 3.2.
There exists positive real constants L_1, L_2 such that for all $x, y \in H_2$ we have

$$E\left\|u(t,x)-u\left(t,y\right)\right\|^2 \le L_1 E\left\|x(t)-y(t)\right\|^2,$$

$$E\|u(t,x)\|^2 \le L_2\left(\frac{1}{b}+E\|x(t)\|^2\right).$$

Proof. Let $x, y \in H_2$. From 2.2 and the conditions on the data, we obtain

$$\begin{aligned} &E\left\|u(t,x)-u\left(t,y\right)\right\|^2 \le 4E\left\|\left\{B^*(b-t)^{q-1}\mathscr{S}^*(b-t)(\Pi_0^b)^{-1}\mathscr{T}(b)(h(x(t))-h\left(y(t)\right)\mid\Gamma_t\right\}\right\|^2\\ &+4E\left\|\left\{B^*(b-t)^{q-1}\mathscr{S}^*(b-t)(\Pi_0^b)^{-1}\int_0^t(b-s)^{q-1}\mathscr{S}(b-s)\left[f(s,x(s))-f(s,y(s))\right]ds\mid\Gamma_t\right\}\right\|^2\\ &+4E\left\|\left\{B^*(b-t)^{q-1}\mathscr{S}^*(b-t)(\Pi_0^b)^{-1}\int_0^t(b-s)^{q-1}\mathscr{S}(b-s)\left[\sigma(s,x(s))-\sigma(s,y(s))\right]dw(s)\mid\Gamma_t\right\}\right\|^2\\ &+4E\left\|\left\{B^*(b-t)^{q-1}\mathscr{S}^*(b-t)(\Pi_0^b)^{-1}\int_0^t\int_Z(b-s)^{q-1}\mathscr{S}(b-s)^{q-1}\right.\right.\\ &\times\left.\left.\left[g\left(s,x(s),\eta\right)-g\left(s,y(s),\eta\right)\right]\widetilde{N}\left(ds,d\eta\right)\mid\Gamma_t\right\}\right\|^2\\ &:= I_1+I_2+I_3+I_4 \end{aligned}$$

where,

$$\begin{aligned} I_1 &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}M_0^2\left(\frac{M_0}{\Gamma(q)}\right)^2\widetilde{N_0}E\left\|x(t)-y(t)\right\|^2\\ I_2 &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}k\int_0^t E\left\|x(s)-y(s)\right\|^2ds\\ I_3 &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}kL_\sigma\int_0^t E\left\|x(s)-y(s)\right\|^2ds\\ I_4 &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\int_0^t\int_Z E\left\|g\left(s,x(s),\eta\right)-g\left(s,y(s),\eta\right)\right\|^2\lambda\left(d\eta\right)ds\\ &+\frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left(\int_0^t\int_Z E\left\|g\left(s,x(s),\eta\right)-g\left(s,y(s),\eta\right)\right\|^4\lambda\left(d\eta\right)ds\right)^{1/2}\\ &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}k\int_0^t E\left\|x(s)-y(s)\right\|^2ds\\ &+\frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\sqrt{\widetilde{L_0}}\left(\int_0^t E\left\|x(s)-y(s)\right\|^4ds\right)^{1/2}\\ &\le \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left(k+\sqrt{\widetilde{L_0}}\right)\int_0^t E\left\|x(s)-y(s)\right\|^2ds \end{aligned}$$

Finally, we get

$$E\left\|u(t,x)-u\left(t,y\right)\right\|^2 \le L_1 E\left\|x(t)-y(t)\right\|^2$$

where $L_1 = \frac{4}{\eta^2}\|B\|^2(b)^{2q-2}\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left(\widetilde{N_0}+2k+kL_\sigma+\sqrt{\widetilde{L_0}}\right)$. The proof of the second inequality is similar to the first one. This completes the proof of the lemma. □

Theorem 3.1.
Assume that the conditions (i)–(v) hold. Further, if the inequality

$$4\widetilde{N_0}M_0^2+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left[\|B\|^2L_2+k+kL_\sigma+\left(k+\sqrt{\widetilde{L_0}}\right)\right]<1$$

is satisfied, then the stochastic control system (1)-(2) *is completely controllable on* $[0,b]$.

Proof. We will show that, using the control, the operator $F:H_2\to H_2$ defined by

$$\begin{aligned}Fx(t) &= \mathscr{T}(t)(x_0-h(x))+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)[Bu(s,x)+f(s,x(s))]ds\\ &+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)\sigma(s,x(s))dw(s)+\int_0^t\int_Z(t-s)^{q-1}g\left(s,x(s),\eta\right)\widetilde{N}\left(ds,d\eta\right),\end{aligned}\tag{4}$$

has a fixed point in H_2. As mentioned in [26] , to prove the complete controllability, it is enough to show that the operator F has a fixed point in H_2. The proof is carried through by the Banach fixed point technique. First, we show that the operator F maps H_2 into itself. Let $x\in H_2$, from (4) we obtain

$$E\|Fx(t)\|_{H_2}^2\le 5\left[\sup_{t\in J}E\|\mathscr{T}(t)(x_0-h(x))\|^2+\sup_{t\in J}\sum_{i=1}^4E\left\|\Pi_i^x(t)\right\|^2\right]\tag{5}$$

Using conditions (i)–(v), 2.2, and with the standard computations, we have

$$\begin{aligned}\sup_{t\in J}\sum_{i=1}^4E\left\|\Pi_i^x(t)\right\|^2 &\le 4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\|B\|^2L_2\left(\frac{1}{b}+\|x\|_{H_2}^2\right)\\ &+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left(k+kL_\sigma+\left(k+\sqrt{\widetilde{L_0}}\right)\right)\left(1+\|x\|_{H_2}^2\right).\end{aligned}\tag{6}$$

Hence $\sup_{t\in J}E\|\mathscr{T}(t)(x_0-h(x))\|^2\le M_0^2\left[\|x_0\|^2+\widetilde{N_0}\left(1+||x||^2\right)\right]$ together with (6) imply that $E\|Fx(t)\|_{H_2}^2<\infty$. Thus, F maps H_2 into itself. Next, we show that F is a contraction in H_2.

For any $x,y\in H_2$, then

$$\begin{aligned}E\left\|(Fx)(t)-(Fy)(t)\right\|^2 &\le 4\sup_{t\in J}\sum_{i=1}^4E\left\|\Pi_i^x(t)-\Pi_i^y(t)\right\|^2\\ &\le 4\widetilde{N_0}M_0^2E\left\|x(t)-y(t)\right\|^2+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\|B\|^2L_2E\left\|x(t)-y(t)\right\|^2\\ &+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}kE\left\|x(t)-y(t)\right\|^2+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}kL_\sigma E\left\|x(t)-y(t)\right\|^2\\ &+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left(k+\sqrt{\widetilde{L_0}}\right)E\left\|x(t)-y(t)\right\|^2\end{aligned}$$

Hence we obtain a positive real number $D=4\widetilde{N_0}M_0^2+4\left(\frac{M_0}{\Gamma(q)}\right)^2\frac{b^{2q-1}}{2q-1}\left[\|B\|^2L_2+k+kL_\sigma+\left(k+\sqrt{\widetilde{L_0}}\right)\right]<1$ such that

$$\sup_{t\in J}E\left\|(Fx)(t)-(Fy)(t)\right\|^2\le D\sup_{t\in J}E\left\|x(t)-y(t)\right\|^2\tag{7}$$

for any $x,y\in H_2$. So, F is a contraction mapping and hence there exists a unique fixed point x in H_2. Thus the fractional stochastic control system (1)-(2) is completely controllable on J. This completes the proof. □

4. Example

Consider the fractional stochastic system with a stochastic process $x(t,z)$ and Poisson jumps in the following form

$$\begin{aligned}{}^cD_t^qx(t,z) &= \frac{\partial^2x(t,z)}{\partial z^2}+\hat{\mu}(t,z)+x(t,z)\\ &+\hat{\sigma}(t,x(t,z))\frac{d\hat{w}(t)}{dt}+\frac{\int_{-1}^{\infty}x(t,z)\eta\widetilde{N}(dt,d\eta)}{dt},\end{aligned}\tag{8}$$

$$\begin{aligned}x(0,z)+\sum_{k=1}^mc_kx(z,t_k) &= x_0(z),\quad z\in[0,\pi],\\ x(t,0) &= x(t,\pi)=0,\quad t\in J,\end{aligned}\tag{9}$$

where $0 < q < 1$, $0 < t_1 < \cdots < t_m = b$ and c_k are positive constants, $k = 1, ..., m$, $\hat{w}(t)$ is a two sided and standard one dimensional Brownian motion defined on the filtered probability space (Ω, Γ, P) and $\widetilde{N}(\cdot,\cdot)$, is a compensated Poisson measure on $[1,\infty]$ with parameter $\lambda(d\eta)dt$ such that $\int_1^\infty \eta(d\eta) < \infty$. . To write the above system into the abstract form of (1)-(2), let $X = U = L^2[0,\pi]$. Define the operator $A: X \to X$ by $Ax = x''$ with domain

$$D(A) = \{x \in X; x, x' \text{ are absolutely continuous}, x'' \in X \text{ and } x(0) = x(\pi) = 0\}.$$

$$Ax = \sum_{n=1}^{\infty} n^2 (x, x_n) x_n, \quad x \in D(A),$$

where $x_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n = 1, 2, ...$ is the orthogonal set of eigen vectors in A. It is well known that A generates a compact, analytic semigroup $\{S(t), t \geq 0\}$ in X and

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, x_n)x_n, \quad ||S(t)|| \leq e^{-t} \text{ for all } t \geq 0.$$

Define $x(t)(z) = x(t,z)$, $\sigma(t, x(t))(z) = \hat{\sigma}(t, x(t,z))$ and $h(x(t))(z) = \sum_{k=1}^{m} c_k x(z, t_k)$. The bounded linear operator $B: U \to X$ by $Bu(t)(z) = \hat{\mu}(t,z)$, $0 \leq z \leq \pi$, $u \in U$. Assume that the operator L_0^b be defined by

$$\left(L_0^b u\right)(z) = \int_0^b (b-s)^{q-1} e^{-n^2(b-s)} \hat{\mu}(s,z)ds,$$

On the other hand, it can be easily seen that the deterministic linear fractiona lcontrol system corresponding to (8)-(9) is completely controllable [27] . Therefore, with the above choices, the system (8)-(9) can bewritten to the abstract form (1)-(2) and all the conditions of Theorem 3.1 are satisfied. Thus by Theorem 3.1, fractional stochastic control system (8)-(9) is completely controllable on $[0, b]$..

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