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Asymptotic perturbation analysis for nonlinear oscillations in viscoelastic systems with hardening exponent

Research Article

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Abstract: The response of many living and engineering structural systems to dynamic loads is often translated in differential equations including some parameters which may change with time. This change may affect in a dramatic way the qualitative behavior of the dynamic response of these systems. It is then suitable to investigate the asymptotic behavior of these systems when some dynamic parameters tend to their critical value. The problem, given a structural model which depends on a strain hardening exponent, is to verify if a small perturbation in this parameter produces a small qualitative change in the dynamic response of the system. To this end, asymptotic perturbation and numerical analyses are performed. The study showed that a small change in the strain hardening exponent does not produce a significant change in the qualitative behavior of the dynamic response of the structural model. The current research work has permitted from a theoretical point of view to note the accuracy of the theory of averaging and the stability of the system. Thus, from a practical point of view, the current model may serve as an alternative to other models for an easy numerical simulation of the dynamics of some mechanical systems experiencing a weak viscoelastic response in view of prediction and operation performance.

MSC: 34C15 • 34C29

Keywords: Viscoelastic mechanical systems • Nonlinear oscillations • Averaging perturbation analysis

1. Introduction

In modern engineering design, the dynamics of many structural systems are often represented in terms of differential equations containing some parameters. These parameters can be subject of variation under working situation of structural systems. This variation of parameter values in structural systems may be attributed in general to defaults of manufacturing and uncertainties resulting from measures. So, the typical behavior of solutions of the model equation may substantially change as the relative value of parameters changes. The model equation may then transit from an operating regime to another. Other phenomena, like instability and bifurcation, may also occur due to change in dynamic parameters. This situation may affect the safety and the performance of a structural system. In this regard, it appears reasonable for preventing eventual catastrophes to theoretically investigate the effects due to change in dynamic parameters. In particular, an interesting physical problem is the study of the effect of a small perturbation in the value of parameter which governs the transition from a nonlinear regime to linear regime of a model equation. Numerous researchers have studied this problem for some living and engineering systems by application of asymptotic perturbations theory [1, 2]. This theory includes several techniques based on the idea of small parameter meaning a weak nonlinearity [3], like harmonic balance, Lindstedt-Poincaré, multiple-scale and averaging methods [1, 4, 5]. These methods are widely used for solving the problem of computing approximate solutions to mathematical models and also for analyzing the influence of characteristic parameters on the qualitative behavior and evolutions of dynamic systems in the fields of nonlinear mechanics and theoretical physics, since phenomena in these areas of science are in general described by means of nonlinear differential equations [6] for which exact solutions are not always

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available. Particularly, averaging is well suited for studying damped nonlinear oscillations in mechanical systems, after Nayfeh and Mook [1] and Jordan and Smith [5]. So, the effects of changes in damping coefficient on the dynamic behavior of mechanical systems have been a subject of intensive mathematical studies, as the damping mechanism in nonlinear mechanics is not yet well understood and captured with accuracy. As known in nonlinear mechanics, large deformations may lead to hardening or softening effects which may affect the damping nonlinearity [1, 7, 8]. But, a system parameter that has received less attention in the research field applying the theory of perturbation to nonlinear mechanical systems seems to be the strain hardening exponent. In such a situation, it is logical to develop an asymptotic perturbation analysis for nonlinear oscillations in viscoelastic systems with hardening exponent. Let us now consider the model of a nonlinear viscoelastic oscillator recently developed by Monsia and Kpomahou [8] for which the equation of motion under free vibrations is

$$\ddot{u} + (l-1)\frac{\dot{u}^2}{u} + \lambda\dot{u} + \omega_0^2 u = 0 \tag{1}$$

where $u(t)$ represents the displacement of the system and the dot over a symbol designates the derivative with respect to time. In Eq. (1), ω_0 is the natural frequency, λ is the linear viscous damping coefficient and l denotes the strain hardening exponent. The problem is, then, to know how does a small change in the parameter l affect the dynamic response of the system. More precisely, the problem that merits to be studied is to understand how does the solution of Eq. (1) behave as $l \to 1$, that is to say, behave in the limit $\alpha \to 0$, with $\alpha = 1 - l$, a positive parameter. In this situation, an interesting question to be elucidated becomes: does a small perturbation in the exponent l produce a small change in the qualitative behavior of the dynamic response of the system under consideration? The investigation of this question is motivated by several facts. First of all, in mechanical systems the damping coefficient is often considered as the varying parameter [9]. Here, the coefficient l governs the nonlinear dissipative properties of the dynamical system under study. Moreover, l is an experimental coefficient. In other words, it is known from experiments, so it is then subjected to uncertainties. Finally, it is worth to note that the exponent l secures the transition of the dynamics of the system from nonlinear to linear regime when it reaches its critical value $l = 1$, that is, when α attains zero. The preceding situation could lead to changes in the value of exponent l, which may affect in large fashion the expected dynamic behavior, the safety and the performance of the viscoelastic system under question in working conditions. Hence, it is reasonable to explore the effects of change in the parameter l on the dynamic response of the studied system. It is more again reasonable to investigate the asymptotic behavior of the system, that is to say, the dynamic behavior of the model as the exponent l tends to its critical value, unity, or as $\alpha \to 0$. In this perspective, it is postulated that the dynamic behavior of the solution to equation (1) does not change significantly in the limit $\alpha \to 0$. In other words, it is predicted that a small change in l doesn' t produce a large change in the qualitative behavior of the dynamic response of the viscoelastic system under consideration. This will allow the use of the current structural model for easy numerical simulations for predicting the long time dynamics of some viscoelastic mechanical systems in engineering design. This will serve also for obtaining useful information in view of appropriate solutions to performance problems arising from existing viscoelastic mechanical systems. Mathematically, as previously mentioned, such a prediction may be achieved through the application of the asymptotic perturbation theory and the limit analysis of the exact analytical solution as $\alpha \to 0$ (section 2). To check the obtained results, it is needed to compare graphically the asymptotic perturbation solution with the solution obtained by numerical integration of the approximate equation on the one hand, and with the exact analytical solution on the other hand (section 3). The validity of the accuracy of the prediction previously formulated will be discussed (section 4) and general conclusions and future works based on obtained results will be addressed (section 5).

2. Methods

In this section, the approximate analytical solution to equation (1) will be determined by application of the averaging method, since this analytical technique has been mathematically proved to be consistent to provide reliable asymptotic solution for damped nonlinear oscillatory systems [1, 5]. As the exact analytical solution to equation (1) is available, it is also suitable to develop the limiting model resulting from this exact analytical solution in the limit $l \to 1$, that is, $\alpha \to 0$. This in order to verify if the asymptotic perturbation solution is close to the exact analytical solution as $\alpha \to 0$. This analysis should allow us to test analytically the accuracy of the averaging perturbation theory.

2.1. Averaging perturbation analysis

In order to perform an asymptotic perturbation solution by averaging, it is convenient to reduce the equation (1) to the non-dimensional form.

2.1.1. Reduction of the equation to the dimensionless form

By introducing the following dimensionless variables

$$\varepsilon(t) = \frac{u(t)}{L_0}$$

where L_0 is the initial length of the system, to say, the initial length of the viscoelastic oscillator, and

$$\tau = \omega_0 t$$

a dimensionless time, the equation (1) transforms after a few mathematical computation into

$$\varepsilon'' + \left[(l-1)\frac{\varepsilon'}{\varepsilon} + 2\mu\right]\varepsilon' + \frac{1}{l}\varepsilon = 0 \tag{2}$$

where $2\mu = \dfrac{\lambda}{\omega_0}$, and the prime denotes differentiation with respect to the dimensionless independent variable τ. The dimensionless dependent variable $\varepsilon(\tau)$ is the normalized displacement, a strain, that is a deformation measure experienced by the viscoelastic system. Following the preceding part, $\alpha = 1 - l$, consists of a relevant physical coefficient that may play the role of small, positive dimensionless parameter for the asymptotic perturbation analysis of equation (2). Regarding then the coefficient $\alpha = 1 - l$, as a small parameter,it is possible to develop an approximate analytical solution to Eq.(2) by application of the averaging theory. In this perspective, it is necessary to formulate in explicit fashion the mathematical problem resulting from Eq.(2).

2.1.2. Mathematical problem

In this subsection the problem is formulated as an approximate differential equation in replacement of Eq. (2) in the sense of weak nonlinearity. To do so, consider the quantity [1, 5]

$$\mu = \alpha\beta$$

where $\alpha = 1 - l$, is a small, positive dimensionless parameter, to say, $0 < \alpha \ll 1$. This statement leads to the Cauchy initial value problem

$$\varepsilon'' + \varepsilon = \alpha\left[\frac{\varepsilon'^2}{\varepsilon} - 2\beta\varepsilon' - \varepsilon\right] \tag{3}$$

that should satisfy the initial conditions

$\varepsilon(\tau = 0) = \varepsilon_0$ and $\varepsilon'(\tau = 0) = v_0$

For $\alpha = 0$, Eq.(3) becomes the well known equation of the classical linear harmonic oscillator, that is the classical prototype for any dynamical system exhibiting periodic motion [10]. The solution in this case is

$$\varepsilon(\tau) = a\cos(\tau + \theta)$$

where a and θ are constants. In this situation, Eq.(3) may be regarded as a small nonlinear perturbation of the linear harmonic oscillator equation for small parameter α. It is easy to see that the equation (3) is written in the standard form

$$\varepsilon'' + \varepsilon = \alpha f(\varepsilon, \varepsilon', \beta)$$

where

$$f(\varepsilon, \varepsilon', \beta) = \frac{\varepsilon'^2}{\varepsilon} - 2\beta\varepsilon' - \varepsilon$$

So, the classical theory of averaging may be applied in order to carry out an approximate analytical solution. The mathematical problem is, instead solving of the equation (2), to solve the approximate equation (3). In other words, the original equation (2) will be replaced by its approximate equation (3) for which an analytical solution is accessible by application of the asymptotic perturbation analysis. In this moment, the problem consists to observe the original system as a small perturbation of a system with a well-known dynamic behavior [1, 5]. Hence, some assumptions are required.

2.1.3. Hypothesis on approximate solution

In averaging theory [1, 5] the desired solution to Eq.(3) is assumed to be, for $\alpha \neq 0$, of the form

$$\varepsilon(\tau) = a(\tau)\cos[\tau + \theta(\tau)] \tag{4}$$

where the amplitude $a(\tau)$ and the phase $\theta(\tau)$ are functions of dimensionless time τ. At the present time, it is important to note that two new time-dependent variables are introduced in the solution (4). Therefore, it is needed to impose upon them an additional restriction.

2.1.4. Restriction on approximate asymptotic solution

As supplementary condition it is suitable to assume that $a(\tau)$ and $\theta(\tau)$ are slow functions of τ. So, the differentiation of Eq.(4) with respect to τ gives

$$\varepsilon'(\tau) = -a(\tau)\sin[\tau + \theta(\tau)] \tag{5}$$

by setting

$$a'(\tau)\cos[\tau+\theta(\tau)] - a\theta'(\tau)\cos[\tau+\theta(\tau)] = 0 \tag{6}$$

Now, it is possible to transform Eq.(3) into a set of two first-order differential equations.

2.1.5. Differential system

The substitution of the equations (4), (5) and (6) into the equation (3) leads to the following equivalent differential system

$$\begin{cases} \frac{da}{d\tau} = -\alpha f(a\cos\psi, -a\sin\psi)\sin\psi \\ \frac{d\psi}{d\tau} = 1 - \frac{\alpha}{a} f(a\cos\psi, -a\sin\psi)\cos\psi \end{cases} \tag{7}$$

where $f(a\cos\psi, -a\sin\psi) = a\dfrac{\sin^2\psi}{\cos\psi} + 2a\beta\sin\psi - a\cos\psi$ and $\psi = \tau + \theta(\tau)$

In Eq.(7) the unknowns are $a(\tau)$ and $\psi(\tau)$. The system of equations (7) is yet equivalent to the equation (3). But, the objective is to replace the differential system (7) by it average value.

2.1.6. Approximate differential system

To establish the approximate differential system, it is assumed that, as $a(\tau)$ and $\theta(\tau)$ are being slowly varying functions of τ, the values of $a'(\tau)$ and $\theta'(\tau)$ should not change significantly during one cycle of motion of period 2π [11]

In this perspective the equation (7) may be averaged over one period with respect to $\theta(\tau)$ to give

$$\begin{cases} \frac{da}{d\tau} = -\alpha\beta a \\ \frac{d\theta}{d\tau} = 0 \end{cases} \tag{8}$$

The system of equations (8) is very easy to be integrated rather than the original system (7) since the equations in the differential system (8) are separable equations [11].

2.1.7. Integration of the differential system

The integration of the differential system (8) is immediate since the first equation is linear in a and the second is null. Hence, $a(\tau)$ and $\psi(\tau)$ may be expressed as

$$a(\tau) = a_0\exp(-\alpha\beta\tau) \tag{9}$$

and

$$\psi(\tau) = \tau + \theta_0 \tag{10}$$

where a_0 and θ_0 are the constants of integration which should be determined from initial conditions

2.1.8. Approximate solution

Noting at $\tau = 0$
$\varepsilon = \varepsilon_0$
and
$\varepsilon' = v_0$
it is possible to find
$a_0 = \sqrt{\varepsilon_0^2 + v_0^2}$
and
$\theta_0 = -\arctan\left(\dfrac{v_0}{\varepsilon_0}\right)$
to recover the definitive approximate solution

$$\varepsilon(\tau,\alpha) = \sqrt{\varepsilon_0^2 + v_0^2}\exp(-\alpha\beta\tau)\cos\left[\tau - \arctan\left(\frac{v_0}{\varepsilon_0}\right)\right] \tag{11}$$

with $\varepsilon_0 \neq 0$

The equation (11) represents the desired asymptotic perturbation solution directly obtained by application of the averaging theory. However, it is required to check the adequacy and accuracy of this procedure. To do so, in a first time the approximate result obtained should be compared with the exact analytical solution to the equation (2) in the limit $\alpha \to 0$, and in the second time, with the solution which results from numerical integration of Eq. (3). It is then, before to progress in this study, suitable to determine the asymptotic behavior of the exact analytical solution to the equation (2) as $\alpha \to 0$.

2.2. Limiting solution

It is needed to recall that following Monsia and Kpomahou [8] the equation (2) may exhibit three damped solutions following the value of the parameter μ. So, the damped oscillatory solution to Eq. (2) obtained for $\mu < 1$, that is to say, the under-damped solution to Eq. (2), is only concerned in this research contribution due to the averaging procedure [5] used in the preceding section and the fact that numerous mechanical systems are weakly damped. In this regard the under-damped solution to Eq. (2) may be written in the form [8]

$$\varepsilon(\tau, l) = C^{\frac{1}{l}} \exp\left(-\frac{\mu\tau}{l}\right) \left[\cos\left(\omega_d \tau - \psi_0\right)\right]^{\frac{1}{l}} \tag{12}$$

where

$$\omega_d = \sqrt{1-\mu^2}$$

$$C = \varepsilon_0^l \sqrt{\frac{1 + \frac{2\mu l v_0}{\varepsilon_0} + \left(\frac{l v_0}{\varepsilon_0}\right)^2}{1-\mu^2}}$$

and

$$\psi_0 = \arctan\left(\frac{\mu + \frac{l v_0}{\varepsilon_0}}{\sqrt{1-\mu^2}}\right)$$

with $\varepsilon_0 \neq 0$

The objective, here, is to seek the limit of the solution (12) as $\alpha \to 0$. To this end, each term of solution (12) should be evaluated as $\alpha \to 0$.

Knowing $l = 1-\alpha$ and $\mu = \alpha\beta$, the quantity C as a function of α becomes

$$C = \varepsilon_0^{1-\alpha} \sqrt{\frac{1 + \frac{2\alpha\beta(1-\alpha) v_0}{\varepsilon_0} + \left[\frac{(1-\alpha) v_0}{\varepsilon_0}\right]^2}{1-\alpha^2\beta^2}}$$

or

$$C = \varepsilon_0^{1-\alpha} \sqrt{\frac{1 + \frac{v_0^2}{\varepsilon_0^2} + 2\alpha\left(\beta - \frac{v_0}{\varepsilon_0}\right)\frac{v_0}{\varepsilon_0} + \alpha^2\left(\frac{v_0}{\varepsilon_0} - 2\beta\right)\frac{v_0}{\varepsilon_0}}{1-\alpha^2\beta^2}}$$

Therefore, as $\alpha \to 0$,

$$C^{\frac{1}{l}} = C^{\frac{1}{1-\alpha}} \sim \varepsilon_0 \sqrt{1 + \frac{v_0^2}{\varepsilon_0^2}}$$

or

$$C^{\frac{1}{1-\alpha}} \sim \sqrt{\varepsilon_0^2 + v_0^2}$$

In the same way, $\exp\left(-\frac{\mu\tau}{l}\right)$ may be written as

$$\exp\left(-\frac{\mu\tau}{l}\right) = \exp\left(-\frac{\mu\tau}{1-\alpha}\right)$$

or

$$\exp\left(-\frac{\mu\tau}{l}\right) = \exp\left(-\frac{\alpha\beta\tau}{1-\alpha}\right)$$

to give

$\exp\left(-\frac{\mu\tau}{l}\right) \sim \exp(-\alpha\beta\tau)$ as $\alpha \to 0$. On the other hand

$$\arctan\left(\frac{\mu + \frac{l v_0}{\varepsilon_0}}{\sqrt{1-\mu^2}}\right) = \arctan\left[\frac{\frac{v_0}{\varepsilon_0} + \alpha\left(\beta - \frac{v_0}{\varepsilon_0}\right)}{\sqrt{1-\alpha^2\beta^2}}\right]$$

that is to say,

$$\arctan\left(\frac{\mu + \frac{l v_0}{\varepsilon_0}}{\sqrt{1-\mu^2}}\right) \sim \arctan\left(\frac{v_0}{\varepsilon_0}\right), \text{ as } \alpha \to 0,$$

to lead also to

$$\left[\cos\left(\sqrt{1-\mu^2}\,\tau - \arctan\left(\frac{\mu + \frac{l v_0}{\varepsilon_0}}{\sqrt{1-\mu^2}}\right)\right)\right]^{\frac{1}{l}} \sim \cos\left(\tau - \arctan\left(\frac{v_0}{\varepsilon_0}\right)\right)$$

in the limit $\alpha \to 0$

Finally, the exact analytical under-damped solution to equation (2) behaves as

$$\varepsilon(\tau, \alpha) = \sqrt{\varepsilon_0^2 + v_0^2} \exp(-\alpha\beta\tau) \cos\left(\tau - \arctan\left(\frac{v_0}{\varepsilon_0}\right)\right) \tag{13}$$

in the limit $\alpha \to 0$. It is worth noting that the solution (13) is identical to the solution (11) obtained by averaging theory. Now, it is also interesting to note that nonlinear problems translated in nonlinear ordinary differential equations, may also be solved by numerical integration with the help of computer. This allows us to check the obtained analytical result and to enhance the understanding of the dynamic response of the viscoelastic system under small perturbations in the strain hardening exponent. Hence, the problem to be investigated is the graphical comparison of the asymptotic perturbation solution (11) with the result obtained by numerical integration of equation (3) on the one hand and with the exact analytical under-damped solution (12), by numerical simulations on the other hand.

3. Numerical applications

In this section the aim is to evaluate numerically the reliability and the accuracy of the developed asymptotic perturbation solution. It is then suitable to compare this approximate analytical result with the solution to the approximate equation (3) obtained by numerical integration. It is also suitable, given the exact analytical solution (12), to perform a comparison of this solution with the averaging solution (11). This may lead to gain more understanding on the agreement between their dynamics.

3.1. Comparison of numerical result with approximate analytical solution

Here, a graphical comparison of result based on numerical integration of Eq.(3) using the Matlab's routine ode 45 with the averaging solution to Eq.(3) is carried out in order to check the accuracy of the averaging procedure that is used. Therefore Fig. 1 shows the comparison of result obtained from numerical evaluation of Eq.(3) with the approximate analytical solution obtained on the basis of averaging theory. The numerical solution is represented in solid line and the amplitude as given by (11) is plotted in dashed line from $\tau = 0$ to $\tau = 80$ under the arbitrary initial conditions $\varepsilon_0 = 1$ and $\upsilon_0 = 0.1$, with the value $\beta = 0.39$. The agreement is found to be consistent for $\alpha = 0.013$ with a mean squared error $mse = 3.3376e-005$. At the present time, a comparison of the asymptotic perturbation solution with the exact analytical solution (12) is needed for a complete numerical evaluation of the tested prediction.

3.2. Comparison of exact result with approximate analytical solution

This subsection aims to compare graphically the exact result with the approximate analytical solution obtained from asymptotic perturbation analysis, on the basis of numerical simulations that are run by using a Matlab's code. In this regard Fig. 2 shows the comparison of the approximate evolution equation (11) with the exact solution (12) from $\tau = 0$ to $\tau = 20$, with the values $\mu = 0.3571$, $\varepsilon_0 = 1.01$, and $\upsilon_0 = 0.94$. The agreement is satisfactory for $l = \frac{1}{3}$. The solid line represents the exact analytical solution (12) while the averaging perturbation solution (11) is plotted in circles. Having presented these numerical and analytical evaluations of the investigated question, it is now possible to discuss the validity of the accuracy of the formulated prediction.

4. Discussion

The objective of this part is to proceed to the analysis and discussion of the results obtained from analytical and numerical evaluations in order to confirm the adequacy and exactness of the hypothesis under question. It is, in effect, assumed that a small perturbation in the strain hardening exponent that governs the effect of nonlinear damping does not produce a significant change in the qualitative nature of the dynamic behavior of the model of viscoelastic nonlinear oscillator under study. In this perspective an approximate analytical solution obtained from the averaging perturbation theory based on the idea of small parameter is performed. For this, it is assumed that the strain hardening exponent may assure the role of small parameter. The obtained asymptotic perturbation result clearly shows that in this case the viscoelastic nonlinear oscillator model may be considered as a small nonlinear perturbation of the well- known classical linear harmonic oscillator. It suffices to note that for $\alpha = 0$, the solution (11) converges to the periodic solution of a linear harmonic oscillator equation, that is, to the non-perturbed harmonic oscillation. Therefore, the perturbation here consists of a modification of harmonic oscillations amplitude that is multiplied by $\exp(-\alpha\beta\tau)$, which is a function of the small parameter α. At sufficiently long time, this factor vanishes, that is to say, $\exp(-\alpha\beta\tau) \to 0$ as $\tau \to \infty$, for $\beta > 0$. So, the system asymptotically approaches its equilibrium value, zero, as shown in Fig. 2. Moreover, Fig. 2 shows a deviation between exact and approximate analytical solutions at short time but, at sufficiently long time, the approximate asymptotic perturbation solution is close to the exact analytical solution, as expected, since the basic logic of the averaging technique is to replace the original system with another system which is more simple than the original model for the large time behavior [12]. Fig. 2 also reveals that the exact solution (12) converges fast to the asymptotic equilibrium more than the approximate analytical solution obtained from averaging theory, which requires more time to approach the asymptotic equilibrium value, as expected, since the averaging procedure corresponds to the slow dynamics of the structural model [1, 4, 12]. It is not difficult to immediately see that the solution (12) may present some power law singularity, so it may lead in numerical simulations in view of practical purpose to some problems of stability and convergence. This highlights, in order to avoid these complex difficulties, the usefulness to design an appropriate approximate model for the long time behavior of the system under study in replacement of the power law solution (12). Fig. 2 shows again, as a noteworthy result, that a small perturbation in the strain hardening exponent doesn' t alter the stability behavior of the dynamic system, as suggested by analytical results. Fig. 1 shows a satisfactory agreement between the solution obtained from numerical integration of the approximate equation (3) and the asymptotic perturbation solution (11) that is close to the exact analytical solution as $\alpha \to 0$. As can be seen in Fig. 1, the slowly varying amplitude approximation of averaging theory holds. In other words, the oscillations amplitude changes with time, but very slowly. Thus, the above demonstrates by analytical and numerical evaluations the validity of the accuracy of the approximate analytical result obtained by application of the asymptotic perturbation method. In this regard, the coefficient α effectively assures the role of small parameter in the determination of approximate model for the replacement of the original model through an averaging perturbation analysis. In such a situation, it is reasonable to affirm that, a small perturbation in α, that is, in the strain hardening exponent l, doesn' t produce a large change in the qualitative nature of the dynamic response of the viscoelastic system under consideration. So, it may be said that the pursued objective is reached.

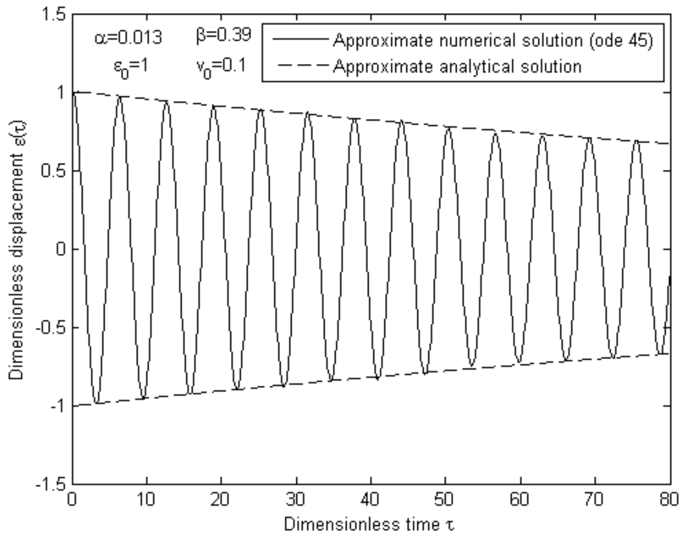

Fig. 1. Validation of the approximate analytical results against the numerical solution ($\alpha = 0.013, \beta = 0.39, \varepsilon_0 = 1$, $v_0 = 0.1$). Solid line: numerical solution; dashed line: averaging solution.

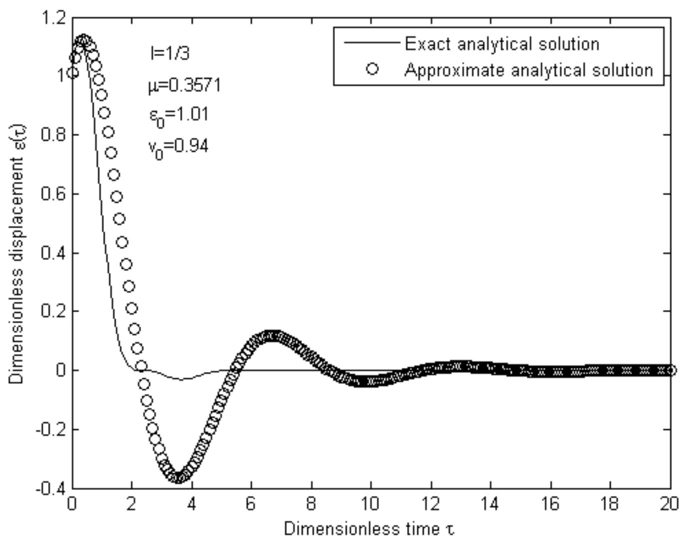

Fig. 2. Graphical representation showing the agreement between the long time dynamics of the exact solution and the averaging perturbation result($l = \frac{1}{3}, \mu = 0.3571, \varepsilon_0 = 1.01, v_0 = 0.94$). Solid line: exact solution; circle line: averaging theory solution.

5. Conclusions and future research

A model of a single degree of freedom nonlinear viscoelastic oscillator including a strain hardening exponent as parameter which governs the nonlinear damping property of the system is considered in this research contribution. In such a situation a necessary question for nonlinear mechanics and engineering applications is to understand the effect of small change in this parameter on the behavior of the dynamic system response. The investigation of this question has needed to perform an averaging perturbation analysis. An approximate analytical solution is then obtained. The accuracy of this solution is verified by its comparison with the solution obtained by numerical integration of the corresponding equation on the one hand, and with the exact analytical result in the limit that the strain hardening exponent reaches its critical value, on the other hand. In addition, the asymptotic perturbation solution is graphically compared with the exact analytical result by numerical simulations. The analysis has shown that the strain hardening exponent secures the role of small parameter. In other words, a small perturbation in the value of the strain hardening exponent does not affect in a dramatic fashion the qualitative nature of the dynamic behavior of the nonlinear viscoelastic oscillator under question. The comparison of the asymptotic perturbation result with the exact analytical solution by numerical simulations, confirms also the fact that a small change in the strain hardening exponent does not alter the stability behavior of the system, as expected from analytical results. In this regard, the current work allowed us to test numerically and analytically the accuracy of the averaging theory. It may also be observed that the proposed structural model is a small nonlinear perturbation, here, a nonlinear damping, of the linear harmonic oscillator. Thus, the approximate asymptotic model which is developed in this paper may be useful, in engineering applications, for representing with a simple explicit expression the dynamics of a variety of mechanical systems undergoing a weak viscoelastic response. It then appears interesting to investigate, as future works, the effects of variation in initial conditions and other parameters on the long time behavior of the dynamic response of the current structural model. An interesting question which should be also investigated in a future work is to know what happens if the equation model is non-homogeneous. These studies seem to be important for nonlinear mechanics and engineering applications, since it is well known that, under forced vibrations, second-order mechanical systems may exhibit a rich variety of physical phenomena, such as loss of stability, that is to say, instability and bifurcation, sub or super-harmonic resonances and chaotic motion.

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