Int. J. Adv. Appl. Math. and Mech. 3(1) (2015) 57 – 61 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Analytical approximate solution for inhomogeneous wave equation on cantor sets by local fractional variational iteration method

Research Article

Hassan Kamil Jassim *

Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq

Received 09 May 2015; accepted (in revised version) 19 August 2015

Abstract: This paper applies the local fractional variational iteration method (LFVIM) to find the analytical approximate solution of linear and nonlinear wave equation on Cantor sets within local fractional operators. The LFVIM has been found to be particularly valuable as a tool for the solution of differential equations in engineering, science, and applied mathematics. To illustrate the simplicity and reliability of the method, two examples are provided. The results obtained reveal that the method is capable and easy to apply.

MSC: 26A33 • 34A12 • 35R11

Keywords: Local fractional variational iteration method • Wave equation • Local fractional operator

1. Introduction

The local fractional calculus [1–3], as a new branch of fractional calculus, was successfully applied to describe the fractal problems from science and engineering. For example, the local fractional diffusion equations defined on Cantor sets [4, 5], the local fractional Fokker Planck equation defined on Cantor sets [6, 7], the local fractional SchrÃűdinger equation [8], local fractional Navier-Stokes equations on cantor sets [9], the local fractional Laplace equation [10, 11], the local fractional heat-conduction equation [12–16] were discussed.
There are a variety of problems in physics, chemistry and biology have their mathematical setting as linear and nonlinear ordinary or partial differential equations. Many of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations. Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are formulated by partial differential equations [17].
In this paper, we consider the linear and nonlinear wave equation defined on Cantor sets with local fractional derivative given by

$$L_{tt}^{(2\alpha)} u(x,t) - L_{xx}^{(2\alpha)} u(x,t) + f(u) = g(x,t), 0 < \alpha \le 1, \tag{1}$$

$$L_{tt}^{(2\alpha)} u(x,t) - L_{xx}^{(2\alpha)} u(x,t) + F(u) = g(x,t), 0 < \alpha \le 1, 0 < x \le l, t > 0, \tag{2}$$

with the initial-boundary conditions

$$\begin{aligned} &u(x,0) = \varphi(x), \\ &\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = \psi(x), \\ &u(0,t) = \kappa(t), u(l,t) = \gamma(t) \end{aligned} \tag{3}$$

where $f(u)$ and $F(u)$are linear and nonlinear functions respectively, $g(x,t)$ is source term of nondifferentiable function, and $\varphi(x), \psi(x), \kappa(t), \gamma(t)$ are continuous functions. The main aim of this paper is to discuss the linear and nonlinear wave equation defined on Cantor sets by the local fractional variational iteration method. The paper is organized as follows. In Section 2, we give analysis of the methods used. In Section 3, we present the applications for the wave equation defined on Cantor sets. Finally, in Section 4, we present our conclusions.

* *E-mail address:* hassan.kamil28@yahoo.com

2. Local Fractional Variational Iteration Method

We consider a general nonlinear local fractional partial differential equation:

$$L_\alpha u(x,t) + R_\alpha u(x,t) + N_\alpha u(x,t) = g(x,t), \tag{4}$$

where L_α denotes linear local fractional derivative operator of order 2α , R_α denotes a lower order local fractional derivative operator, N_α denotes nonlinear local fractional operator, and $g(x,t)$ is the nondifferentiable source term.
According to the rule of local fractional variational iteration method, the correction local fractional functional for (4) is constructed as [12, 15]:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left\{L_\alpha u_n(x,\xi) + R_\alpha u_n(x,\xi) + N_\alpha u_n(x,\xi) - g(x,\xi)\right\}(d\xi)^\alpha, \tag{5}$$

where $\dfrac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$ is a fractal Lagrange multiplier.
Making the local fractional variation, we get

$$\delta^\alpha u_{n+1}(x,t) = \delta^\alpha u_n(x,t) + \delta^\alpha \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left\{L_\alpha u_n(x,\xi) + R_\alpha u_n(x,\xi) + N_\alpha u_n(x,\xi) - g(x,\xi)\right\}(d\xi)^\alpha. \tag{6}$$

The extremum condition of $u_{n+1}(x,t)$ is given by

$$\delta^\alpha u_{n+1}(x,t) = 0. \tag{7}$$

In view of (7), we have the following stationary conditions:

$$1 - \left[\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha}\right]^{(\alpha)}|_{\xi=t} = 0, \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha}|_{\xi=t} = 0, \left[\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha}\right]^{(2\alpha)}|_{\xi=t} = 0. \tag{8}$$

So, from(**??**), we get

$$\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha} = \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \tag{9}$$

The initial value $u_0(x,t)$ is given by

$$u_0(x,t) = u(x,0) + \frac{t^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(x,0). \tag{10}$$

In view of (9), we have

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{L_\alpha u_n(x,\xi) + R_\alpha u_n(x,\xi) + N_\alpha u_n(x,\xi) - g(x,\xi)\right\}(d\xi)^\alpha, \tag{11}$$

Finally, from(11), we obtain the solution of (4) as follows:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \tag{12}$$

3. Illustrative Examples

In this section, we given some illustrative examples for solving the linear and nonlinear wave equation on Cantor sets within local fractional operator by using local fractional variational iteration method.

Example 3.1.
Let us consider the following wave equation on Cantor sets with local fractional operator

$$L_{tt}^{(2\alpha)}u(x,t) - L_{xx}^{(2\alpha)}u(x,t) = \sin_\alpha(x^\alpha), 0 < x \le \pi, t > 0, \tag{13}$$

with the initial-boundary conditions

$$\begin{aligned} u(x,0) &= \sin_\alpha(x^\alpha),\\ \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} &= \sin_\alpha(x^\alpha),\\ u(0,t) &= u(\pi,t) = 0 \end{aligned} \tag{14}$$

In view of (11), we have the correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{\partial^{2\alpha} u_n(x,\xi)}{\partial \xi^{2\alpha}} - \frac{\partial^{2\alpha} u_n(x,\xi)}{\partial x^{2\alpha}} - \sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha. \tag{15}$$

Considering the given initial values, we can select $u_0(x,t) = \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha)$ .Using this selection into (15), we obtain the following successive approximations

$$u_0(x,t) = \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha), \tag{16}$$

$$\begin{aligned} u_1(x,t) &= u_0(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{\partial^{2\alpha} u_0(x,\xi)}{\partial \xi^{2\alpha}} - \frac{\partial^{2\alpha} u_0(x,\xi)}{\partial x^{2\alpha}} - \sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{\xi^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha) - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \sin_\alpha(x^\alpha), \end{aligned} \tag{17}$$

$$\begin{aligned} u_2(x,t) &= u_1(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{\partial^{2\alpha} u_1(x,\xi)}{\partial \xi^{2\alpha}} - \frac{\partial^{2\alpha} u_1(x,\xi)}{\partial x^{2\alpha}} - \sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ -\frac{\xi^{3\alpha}}{\Gamma(1+3\alpha)} \sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} \sin_\alpha(x^\alpha) - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \sin_\alpha(x^\alpha) + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} \sin_\alpha(x^\alpha), \end{aligned} \tag{18}$$

$$\vdots$$

$$u_n(x,t) = \sin_\alpha(x^\alpha) + \sin_\alpha(x^\alpha) \left[\sum_{k=0}^{n} (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)} \right]. \tag{19}$$

Finally, the solution is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= \sin_\alpha(x^\alpha) + \sin_\alpha(x^\alpha) \left[\sum_{k=0}^{\infty} (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)} \right] \\ &= \sin_\alpha(x^\alpha) + \sin_\alpha(x^\alpha) \sin_\alpha(t^\alpha). \end{aligned} \tag{20}$$

Example 3.2.
Consider the following wave equation on Cantor sets with local fractional operator

$$L_{tt}^{(2\alpha)} u(x,t) - L_{xx}^{(2\alpha)} u(x,t) + u(x,t) = 2\sin_\alpha(x^\alpha), 0 < x \le \pi, t > 0, \tag{21}$$

with the initial-boundary conditions

$$\begin{aligned} & u(x,0) = \sin_\alpha(x^\alpha), \\ & \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = 1, \\ & u(0,t) = \sin_\alpha(t^\alpha), \\ & u(\pi,t) = \sin_\alpha(t^\alpha). \end{aligned} \tag{22}$$

In view of (11), we have the correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{\partial^{2\alpha} u_n(x,\xi)}{\partial \xi^{2\alpha}} - \frac{\partial^{2\alpha} u_n(x,\xi)}{\partial x^{2\alpha}} + u_n(x,t) - 2\sin_\alpha(x^\alpha) \right\} (d\xi)^\alpha. \tag{23}$$

Considering the given initial values, we can select $u_0(x,t) = \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)}$ .Using this selection into (23), we obtain the following successive approximations

$$u_0(x,t) = \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{24}$$

$$\begin{aligned} u_1(x,t) &= u_0(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{\frac{\partial^{2\alpha}u_0(x,\xi)}{\partial\xi^{2\alpha}} - \frac{\partial^{2\alpha}u_0(x,\xi)}{\partial x^{2\alpha}} + u_0(x,t) - 2\sin_\alpha(x^\alpha)\right\}(d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right\}(d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \end{aligned} \tag{25}$$

$$\begin{aligned} u_2(x,t) &= u_1(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{\frac{\partial^{2\alpha}u_1(x,\xi)}{\partial\xi^{2\alpha}} - \frac{\partial^{2\alpha}u_1(x,\xi)}{\partial x^{2\alpha}} + u_1(x,t) - 2\sin_\alpha(x^\alpha)\right\}(d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{-\frac{\xi^{3\alpha}}{\Gamma(1+3\alpha)}\right\}(d\xi)^\alpha \\ &= \sin_\alpha(x^\alpha) + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}, \end{aligned} \tag{26}$$

$$\vdots$$

$$u_n(x,t) = \sin_\alpha(x^\alpha) + \sum_{k=0}^{n}(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}. \tag{27}$$

Hence, we obtain the solution of (21) as

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t) \\ &= \sin_\alpha(x^\alpha) + \sum_{k=0}^{\infty}(-1)^k\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)} \\ &= \sin_\alpha(x^\alpha) + \sin_\alpha(t^\alpha). \end{aligned} \tag{28}$$

Example 3.3.

Consider the following nonlinear wave equation on Cantor sets with local fractional operator

$$L_{tt}^{(2\alpha)}u(x,t) - L_{xx}^{(2\alpha)}u(x,t) - u(x,t) - u^2(x,t) = -\frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{x^{2\alpha}}{\Gamma^2(1+\alpha)}\frac{t^{2\alpha}}{\Gamma^2(1+\alpha)}, 0 < x \le \pi, t > 0, \tag{29}$$

subject to the initial-boundary conditions

$$\begin{aligned} &u(x,0) = 0, \\ &\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ &u(0,t) = 0, \\ &u(\pi,t) = \frac{\pi^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{30}$$

In view of (11) the local fractional iteration algorithm can be written as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left\{\begin{array}{l}\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}} - \frac{\partial^{2\alpha}u_n(x,\xi)}{\partial x^{2\alpha}} - u_n(x,\xi) - u_n^2(x,\xi) + \\ \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{\xi^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma^2(1+\alpha)}\frac{\xi^{2\alpha}}{\Gamma^2(1+\alpha)}\end{array}\right\}(d\xi)^\alpha.$$

Considering the given initial values, we can select $u_0(x,t) = \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}$ .Using this selection into above formula, we obtain the following successive approximations

$$\begin{aligned} u_0(x,t) &= \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ u_1(x,t) &= \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ u_2(x,t) &= \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ &\vdots \\ u_n(x,t) &= \frac{x^\alpha}{\Gamma(1+\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{31}$$

Hence, we obtain the solution of (29) as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \quad (32)$$

4. Conclusions

In this work, the local fractional variational iteration method has been successfully applied to obtain the analytical solution of linear and nonlinear wave equation within local fractional operator. It is clearly seen that the used method is straightforward, powerful and efficient.

Acknowledgements

The author acknowledges Ministry of Higher Education and Scientific Research in Iraq for its support this work.

References

[1] X. J. Yang, Advanced Local Fractional Calculus and Its Applications, World Science Publisher, New York, 2012.
[2] A. Babakhani and V. D. Gejji, âĂIJOn calculus of local fractional derivatives, Journal of Mathematical Analysis and Applications 270(1) (2002) 66–79.
[3] X. J. Yang, Local Fractional Functional Analysis and Its Applications, Asian Academic, Hong Kong, 2011.
[4] A. Carpinteri, A. Sapora, âĂIJDiffusion problems in fractal media defined on Cantor sets, ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik 90(3) (2010) 203–210.
[5] X. Yang, D. Baleanu, W. P. Zhong, Approximate solutions for diffusion equations on Cantor space-time, Proceedings of the Romanian Academy, Series A 14(2) (2013) 127–133.
[6] X. J. Yang, D. Baleanu, Local fractional variational iteration method for Fokker Planck equation on a Cantor set, Acta Universitaria 23(2) (2013) 3–8.
[7] S. H. Yan, X. H. Chen, G. N. Xie, C. Cattani, X. J. Yang, Solving Fokker Planck equations on Cantor sets using local fractional decomposition method Abstract and Applied Analysis 2014, Article ID 396469 (2014)1–6.
[8] H. Jafari, H. K. Jassim, Local Fractional Series Expansion Method for Solving Laplace and Schrodinger Equations on Cantor Sets within Local Fractional Operators, International Journal of Mathematics and Computer Research 2(11) (2014) 736–744.
[9] X. Yang, D. Baleanu, J. A. Tenreiro Machado, Systems of navier-stokes equations on cantor sets, Mathematical Problems in Engineering 2013, Article ID 769724 (2013) 1–8.
[10] S. Yan, H. Jafari, H. K. Jassim, Local fractional Adomain decomposition and function decomposition methods for Laplace equation within local fractional operators, Advances in Mathematical Physics 2014, Article ID 161580 (2014) 1–7.
[11] S. Wang, Y. Yang, H. K. Jassim, Local fractional function decomposition method for solving inhomogeneous wave equations with local fractional derivative, Abstract and Applied Analysis 2014, Article ID 176395 (2014) 1–7.
[12] A. M. Yang, Y. Z. Zhang, X. L. Zhang, The nondifferentiable solution for local fractional Tricomi equation arising in fractal transonic flow by local fractional variational iteration method, Advances in Mathematical Physics 2014, ID 161760 (2014) 1–6.
[13] A. Yang, C. Zhang, H. Jafari, C. Cattani, Y. Jiao, Picard successive approximation method for solving differential equations arising in fractal heat transfer with local fractional derivative, Abstract and Applied Analysis 2014, Article ID 395710, (2014) 1–5.
[14] H. Jafari, H. K. Jassim, Local Fractional Adomian Decomposition Method for Solving Two Dimensional Heat conduction Equations within Local Fractional Operators, Journal of Advance in Mathematics 9(4) (2014) 2574–2582.
[15] J. H. He, F. J. Liu, Local fractional variational iteration method for fractal heat transfer in silk cocoon hierarchy, Nonlinear Science Letters A 4(1) (2013) 15–20.
[16] C. F. Liu, S. S. Kong, S. J. Yuan, Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem, Thermal Science 7(3) (2013) 715–721.
[17] A.M. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications. Beijing and Springer-Verlag Berlin Heidelberg, 2011.