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A parameter-uniform non-standard finite difference method for a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term

Research Article

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Abstract: In this paper, a non-standard finite difference scheme is presented to solve a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term on uniform mesh. The leading term of each equation is multiplied by a small positive parameter with different magnitudes. Boundary and weak interior layers appear in the solution of the problem. The method is proved to be uniformly convergent with respect to the singular perturbation parameters. Numerical results are provided to illustrate the theoretical results and compares well with the existing standard finite difference method on Shishkin mesh.

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Keywords: Singular perturbation problem • Weakly coupled system • Convection-diffusion • Discontinuous source term • Non-standard finite difference scheme • Parameter-uniform

1. Introduction

Singular perturbation problems (SPPs) arise in diverse areas of science and engineering including fluid mechanics, fluid dynamics, chemical reactor theory, aero dynamics, combustion, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, control theory, oceanography, nuclear engineering. Modeling of steady and unsteady viscous flow problems with high Reynolds number, convective heat transport problems with large péclet numbers, magneto-hydrodynamics duct problems at high Hartman number, drift diffusion equation of semiconductor device modeling, boundary layer problems, Wentzel, Kramers and Brillouin (WKB) problems are some of the applications of SPPs. The system of SPPs have applications in electro analytical chemistry, predator prey population dynamics, modeling of optimal control situations and resistance-capacitor electrical circuits. These problems are characterized by the presence of small positive parameters multiplying the highest order derivatives of the differential equations. Due to these small parameters, it is very difficult to obtain satisfactory numerical solutions. It is a well-known fact that the solution of SPPs exhibit a multi-scale character i.e., the solution varies very rapidly in a narrow region where as it varies slowly and uniformly out side these regions. This leads to boundary and/or interior layers in the solution of the problems. Classical numerical methods fail to produce good approximations for these problems. Therefore, it is important to develop numerical methods for these problems, whose accuracy does not depend upon the perturbation parameter(s) which are called as parameter-uniform numerical methods. There are two important approaches widely used in the literature for the development of uniformly convergent numerical methods, namely, Fitted operator method (FOM) and fitted mesh method (FMM). In FOM, because of the uniform mesh, the layers will be resolved automatically without having to decompose the solution. But FMMs use standard classical finite difference schemes on specially designed piece-wise uniform mesh. A priori knowledge about the location and width of the boundary layer(s) are required for

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the construction of the mesh. For a variety of possible exponentially fitting techniques, one can refer the book by Doolan et al. [1]. Mickens [2, 3] gave a novel approach of non-standard finite difference method (NSFDM) and the basic idea is to modify the denominator of the discrete second order derivatives with suitable functions in the governing differential equations, which comes under the category of FOM. The information on the use of NSFDMs for various types of problems arising in many different fields can be found in the survey article [4].

There are several articles available in the literature describing various numerical methods to solve the SPPs, but they mostly dealt with singularly perturbed problems containing single Eq. [5, 6]. A few authors have developed robust numerical methods for a coupled system of singularly perturbed convection-diffusion [7, 8] and reaction-diffusion [9, 10] problems. Linss and Stynes [11] provided a survey article on the current research about the numerical solution of systems of singularly perturbed differential equations which deals with finite difference methods. Very few works can be seen in the literature for system of SPPs with non-smooth data, see for example [12, 13] and the references there in. Most of these works are based on a piecewise uniform Shishkin mesh using standard finite difference scheme. Using mesh equidistribution technique, Das and Natesan [14] considered a system of singularly perturbed weakly coupled convection-diffusion equations having different parameters of different magnitudes and obtained first order uniform convergence with respect to the diffusion parameters. Munyakazi [15] proposed a uniformly convergent numerical method based on non-standard finite difference scheme for the coupled system of convection-diffusion equations for smooth case and obtained first order convergence. Tamilselvan et al.[16] developed a numerical method for singularly perturbed system of second order ordinary differential equations of convection-diffusion type with a discontinuous source term having equal diffusion parameters using fitted mesh method on piecewise uniform Shishkin mesh and the method was proved to be almost first order parameter-uniform convergence. Motivated by the works [1, 2, 8, 15, 16], in this paper, we have developed a parameter uniform NSFDM on uniform mesh for a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term having different diffusion parameters.

Consider the problem of finding $y_1, y_2 \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^1(\Omega) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$ such that

$$P_1\overline{y}(x) \equiv -\varepsilon y_1''(x) - a_1(x)y_1'(x) + b_{11}(x)y_1(x) + b_{12}(x)y_2(x) = f_1(x),\ \forall\, x \in \Omega^- \cup \Omega^+ \tag{1}$$

$$P_2\overline{y}(x) \equiv -\mu y_2''(x) - a_2(x)y_2'(x) + b_{21}(x)y_1(x) + b_{22}(x)y_2(x) = f_2(x),\ \forall\, x \in \Omega^- \cup \Omega^+ \tag{2}$$

with the boundary conditions

$$y_1(0) = p, \quad y_1(1) = q, \quad y_2(0) = r, \quad y_2(1) = s, \tag{3}$$

where ε and μ are small parameters such that $0 < \varepsilon \le \mu < 1$, and assume that

$$a_1(x) \ge \alpha_1 > 0, \quad a_2(x) \ge \alpha_2 > 0, \tag{4}$$

$$b_{12}(x) \le 0, \quad b_{21}(x) \le 0, \tag{5}$$

$$b_{11}(x) + b_{12}(x) \ge \beta_1 > 0, \quad b_{21}(x) + b_{22}(x) \ge \beta_2 > 0, \quad \forall x \in \overline{\Omega}. \tag{6}$$

$$|[f_1](d)| \le C, \ |[f_2](d)| \le C. \tag{7}$$

Here $\Omega = (0,1), \Omega^- = (0,d), \Omega^+ = (d,1), d \in \Omega$ and $\overline{y} = (y_1, y_2)^T$. It is also assumed that the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x)$ are sufficiently smooth on $\overline{\Omega}$, and the source terms $f_1(x), f_2(x)$ are sufficiently smooth on $\overline{\Omega} \setminus \{d\}$. Further, $f_1(x), f_2(x)$ and their derivatives are assumed to have right and left limits at the point $x = d$. In general, this discontinuity gives rise to weak interior layers in the solution of the problem.

The above coupled system of Eqs. (1)-(3) can be written in vector form as

$$\mathbf{P}_{\varepsilon,\mu}\overline{y}(x) \equiv \begin{pmatrix} -\varepsilon \dfrac{d^2}{dx^2} & 0 \\ 0 & -\mu \dfrac{d^2}{dx^2} \end{pmatrix} \overline{y}(x) - A(x)\overline{y'}(x) + B(x)\overline{y}(x) = \overline{f}(x), \quad \forall\, x \in \Omega^- \cup \Omega^+$$

with the boundary conditions

$$\overline{y}(0) = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \overline{y}(1) = \begin{pmatrix} q \\ s \end{pmatrix},$$

where

$$\overline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix}, B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}, \quad \overline{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

The norm which is used for studying the convergence of numerical solution to the exact solution of the singular perturbation problem is the maximum norm defined by $\|\phi_k(x)\|_\Omega = \sup_{x\in\Omega} |\phi_k(x)|, k = 1,2$ and $\|\overline{\phi}\|_\Omega = \max\{\|\phi_1\|, \|\phi_2\|\}$ for a function ϕ defined on a domain Ω. The jump at d is denoted in any function w with $[w](d) = w(d^+) - w(d^-)$. Throughout the paper, C denotes a generic positive constant which is independent of the singular perturbation parameters ε, μ and of the mesh parameter h.

The rest of the paper is organized as follows: In Section 2, some theoretical results like maximum principle, stability and bounds on the solution derivatives for the continuous problem are discussed. The discrete version of the problem using non-standard finite difference scheme on uniform mesh is presented in Section 3. Error analysis is carried out in Section 4. Numerical results obtained from the test problem and comparison with the existing results are provided in Section 5. Finally, Section 6 gives the conclusion of this paper.

2. Theoretical results

In this section, the maximum principle, stability result and bounds on the solution and its derivatives are established for the boundary value problem (BVP) (1)-(3).

Lemma 2.1 (Maximum principle).
Suppose that $\overline{y}(x) = (y_1(x), y_2(x))^T, y_1, y_2 \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$ *satisfies* $\overline{y}(0) \geq \overline{0}, \overline{y}(1) \geq \overline{0}, \boldsymbol{P}_{\varepsilon,\mu}\overline{y}(x) \geq \overline{0}$,*and* $[\overline{y}'](d) \leq \overline{0}, \forall\, x \in \Omega^- \cup \Omega^+$. *Then* $\overline{y}(x) \geq \overline{0}, \forall\, x \in \overline{\Omega}$.

Proof. Let $y_1(m) = \min_{x\in\overline{\Omega}}\{y_1(x)\}$ and $y_2(n) = \min_{x\in\overline{\Omega}}\{y_2(x)\}$. Without loss of generality, assume that $y_1(m) \leq y_2(n)$. If $y_1(m) \geq 0$, then there is nothing to prove. So, let $y_1(m) < 0$, then it will be shown that this leads to a contradiction.

Note that $m \neq \{0,1\}$, and $y_1'(m) = 0, y_1''(m) \geq 0$.
Therefore, either $m \in \Omega^- \cup \Omega^+$ or $m = d$.

Case-(i): Let $m \in \Omega^- \cup \Omega^+$. Then, we have

$$\begin{aligned} P_1\overline{y}(m) &= -\varepsilon y_1''(m) - a_1(m)y_1'(m) + b_{11}(m)y_1(m) + b_{12}(m)y_2(m) \\ &= -\varepsilon y_1''(m) - a_1(m)y_1'(m) + (b_{11}(m) + b_{12}(m))y_1(m) + (y_2(m) - y_1(m))b_{12}(m) \\ &< 0, \end{aligned}$$

which contradicts the hypothesis of the Lemma.

Case-(ii): Let $m = d$.

Assume that there exists a neighborhood $N_h = (d-h, d)$ such that $y_1(x) < 0$ and $y_1(x) < y_2(x), \forall\, x \in N_h$.

Let $x_1 \neq d, x_1 \in N_h$ be a point such that $y_1(x_1) > y_1(d)$.

It follows from the mean value theorem that, for some $x_2 \in N_h, y_1'(x_2) < 0$, and for some $x_3 \in N_h, y_1''(x_3) > 0$.
Also note that $y_1(x_3) < 0, y_1'(x_3) = 0$, since $x_3 \in N_h$.
Thus,

$$\begin{aligned} P_1\overline{y}(x_3) &= -\varepsilon y_1''(x_3) - a_1(x_3)y_1'(x_3) + b_{11}(x_3)y_1(x_3) + b_{12}(x_3)y_2(x_3) \\ &= -\varepsilon y_1''(x_3) - a_1(x_3)y_1'(x_3) + (b_{11}(x_3) + b_{12}(x_3))y_1(x_3) + (y_2(x_3) - y_1(x_3))b_{12}(x_3) \\ &< 0, \end{aligned}$$

which is a contradiction.
Similarly, $P_2\overline{y}(x)$ can be dealt.
Hence, $\overline{y}(x) \geq \overline{0}, \forall\, x \in \overline{\Omega}$. □

Lemma 2.2 (Stability).
If $y_1, y_2 \in \mathscr{C}^0(\overline{\Omega}) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$, *then* $\forall\, x \in \overline{\Omega}$,

$$|y_j(x)| \leq \max\{\|\overline{y}(0)\|, \|\overline{y}(1)\|\} + \frac{1}{\beta}\|\overline{f}\|_{\Omega^-\cup\Omega^+}, \quad j = 1, 2,$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Define two barrier functions $\overline{\omega}^{\pm}(x) = (\omega_1^{\pm}(x), \omega_2^{\pm}(x))^T$ as

$$\overline{\omega}^{\pm}(x) = M\overline{e} \pm \overline{y}(x),$$

where $\overline{e} = \begin{pmatrix}1\\1\end{pmatrix}$ is the unit column vector and

$$M\overline{e} = \max\{\|\overline{y}(0)\|, \|\overline{y}(1)\|\} + \frac{1}{\beta}\|\overline{f}\|_{\Omega^-\cup\Omega^+}.$$

Clearly, $\overline{\omega}^{\pm}(0) \geq \overline{0}, \quad \overline{\omega}^{\pm}(1) \geq \overline{0}$.
Also,

$$\begin{aligned} P_1\omega_1^{\pm}(x) &= M(b_{11}(x) + b_{12}(x)) \pm P_1\overline{y}(x) \\ &\geq \beta_1 M \pm P_1\overline{y}(x) \\ &\geq \beta \max\{\|y_1(0)\|, \|y_1(1)\|\} + \|f_1\| \pm f_1(x) \\ &\geq 0. \end{aligned}$$

Similarly, it can be proved that $P_2\overline{\omega}^{\pm}(x) \geq 0$.
Hence, $\mathbf{P}_{\varepsilon,\mu}\overline{\omega}^{\pm}(x) \geq \overline{0}, \forall\, x \in \Omega^- \cup \Omega^+$. Further, $[(\overline{\omega}^{\pm})'](d) = \pm[\overline{y}'](d) = \overline{0}$.
Therefore, by maximum principle, we have $\overline{\omega}^{\pm}(x) \geq \overline{0}$, which implies the desired result. □

Lemma 2.3 (Bounds on the solution derivatives).
Let $\overline{y}(x) = (y_1(x), y_2(x))^T$ be the solution of the BVP (1)-(3). Then $\forall\ x \in \Omega^- \cup \Omega^+$, we have

$$|y_1^{(k)}(x)| \le C\varepsilon^{-k}, \quad |y_2^{(k)}(x)| \le C\mu^{-k}, \quad k = 1,2,$$
$$|y_1'''(x)| \le C\varepsilon^{-1}(\varepsilon^{-2} + \mu^{-1}), \quad |y_2'''(x)| \le C\mu^{-1}(\mu^{-2} + \varepsilon^{-1}).$$

Proof. By the mean value theorem, there exists a point $z \in (0, \varepsilon)$ such that

$$y_1'(z) = \frac{y_1(\varepsilon) - y_1(0)}{\varepsilon}.$$

Therefore, we get

$$|\varepsilon y_1'(z)| \le 2\|y_1\| \tag{8}$$

Integrating the first equation of the system

$$-\varepsilon y_1''(x) - a_1(x)y_1'(x) + b_{11}(x)y_1(x) + b_{12}(x)y_2(x) = f_1(x)$$

from 0 to z, we get

$$\left|\varepsilon(y_1{}'(z) - y_1{}'(0))\right| = \left|\int_0^z b_{11}(t)y_1(t)dt + \int_0^z b_{12}(t)y_2(t)dt - [a_1(t)y_1(t)]_0^z + \int_0^z a_1'(t)y_1(t)dt - \int_0^z f_1(t)dt\right| \tag{9}$$

Using (8), it follows that

$$|\varepsilon y_1'(0)| \le \|f_1(z)\| + C(\|y_1(z)\| + \|y_2(z)\|)$$

Using (9) with $z = x$, we get

$$|\varepsilon y_1'(x)| \le C(\|y_1(x)\| + \|y_2(x)\| + \|f_1(x)\|)$$

Thus, we have

$$|y_1'(x)| \le C\varepsilon^{-1}(\|y_1(x)\| + \|y_2(x)\| + \|f_1(x)\|), \forall\ x \in \Omega^- \cup \Omega^+$$

Since $y_1(x), y_2(x)$ and $f_1(x)$ are bounded, $\forall\ x \in \Omega^- \cup \Omega^+$ we have

$$|y_1'(x)| \le C\varepsilon^{-1} \tag{10}$$

Similarly, we can get

$$|y_2'(x)| \le C\mu^{-1} \tag{11}$$

From the first equation of the system, we have

$$\varepsilon y_1''(x) = -a_1(x)y_1'(x) + b_{11}(x)y_1(x) + b_{12}(x)y_2(x) - f_1(x) \tag{12}$$

Using (10) in (12), we get

$$|y_1''(x)| \le C\varepsilon^{-2} \tag{13}$$

Similarly, we can get

$$|y_2''(x)| \le C\mu^{-2} \tag{14}$$

To bound the third derivative, differentiate both sides of the first equation of the system, we have

$$-\varepsilon y_1'''(x) = a_1(x)y_1''(x) + y_1'(x)a_1'(x) - b_{11}(x)y_1'(x) - b_{11}'(x)y_1(x) - b_{12}(x)y_2'(x) - b_{12}'(x)y_2(x) + f_1'(x) \tag{15}$$

Applying (10) and (13) and using the assumption that the functions $a_1(x), b_{11}(x), b_{12}(x)$ and $f_1(x)$ and their derivatives are bounded, we have

$$|y_1'''(x)| \le C\varepsilon^{-1}(\varepsilon^{-2} + \mu^{-1}), \quad \forall\ x \in \Omega^- \cup \Omega^+ \tag{16}$$

Similarly, we can prove that

$$|y_2'''(x)| \le C\mu^{-1}(\mu^{-2} + \varepsilon^{-1}), \quad \forall\ x \in \Omega^- \cup \Omega^+ \tag{17}$$

□

3. Discrete problem

Consider the following uniform mesh on the interval $\overline{\Omega} = [0,1]$ with N mesh points, where N is a positive integer.

$$x_0 = 0, \quad x_i = x_0 + ih, \quad i = 0(1)N, \quad h = N^{-1}, \quad x_N = 1.$$

The interior points of the mesh are denoted by

$$\Omega^N_{\varepsilon,\mu} = \{x_i : 1 \le i \le \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \le i \le N-1\}.$$

Clearly $x_{\frac{N}{2}} = d$ and $\overline{\Omega}^N_{\varepsilon,\mu} = \{x_i\}_0^N \cup \{d\}$. Let $\overline{Y} = (Y_1, Y_2)^T$ be the approximate solution of $\overline{y}(x) = (y_1, y_2)^T$.
Let the discrete operator of $\mathbf{P}_{\varepsilon,\mu}$ be denoted by $\mathbf{P}^N_{\varepsilon,\mu}\overline{Y} = \begin{pmatrix} P^N_{1,\varepsilon}\overline{Y} \\ P^N_{2,\mu}\overline{Y} \end{pmatrix}$.
A non-standard finite difference scheme is constructed on the uniform mesh $\Omega^N_{\varepsilon,\mu}$ as follows:
For $x_i \ne d$,

$$P^N_{1,\varepsilon}\overline{Y}(x_i) \equiv -\varepsilon\Delta^2 Y_1(x_i) - a_1(x_i)D^+Y_1(x_i) + b_{11}(x_i)Y_1(x_i) + b_{12}(x_i)Y_2(x_i) = f_1(x_i), \tag{18}$$
$$P^N_{2,\mu}\overline{Y}(x_i) \equiv -\mu\Delta^2 Y_2(x_i) - a_2(x_i)D^+Y_2(x_i) + b_{21}(x_i)Y_1(x_i) + b_{22}(x_i)Y_2(x_i) = f_2(x_i), \tag{19}$$

and for $x_i = d$,

$$\mathbf{P}^N_{\varepsilon,\mu}\overline{Y}(d) = \tilde{f}(d) \tag{20}$$

with the boundary conditions

$$Y_{10} = y_1(0), \quad Y_{1N} = y_1(1), \quad Y_{20} = y_2(0), \quad Y_{2N} = y_2(1), \tag{21}$$

where $\Delta^2 Y_j(x_i) = \dfrac{Y_j(x_{i-1}) - 2Y_j(x_i) + Y_j(x_{i+1})}{(s_j(x_i))^2}$, $D^+Y_j(x_i) = \dfrac{Y_j(x_{i+1}) - Y_j(x_i)}{h}$, $j = 1,2$, $\tilde{f}(d) = \begin{pmatrix} f_1(d) \\ f_2(d) \end{pmatrix}$,
and $(s_{1,i})^2, (s_{2,i})^2, f_j(d)$ are given by

$$(s_{1,i})^2 = (s_1(x(i)))^2 = (s_{1,i}(h,\varepsilon))^2 = \frac{h\varepsilon}{a_{1,i}}\left(\exp\left(\frac{a_{1,i}h}{\varepsilon}\right) - 1\right), \tag{22}$$
$$(s_{2,i})^2 = (s_2(x(i)))^2 = (s_{2,i}(h,\mu))^2 = \frac{h\mu}{a_{2,i}}\left(\exp\left(\frac{a_{2,i}h}{\mu}\right) - 1\right), \tag{23}$$
$$f_j(d) = \frac{f_j(d-h) + f_j(d+h)}{2}, \; j = 1,2. \tag{24}$$

It is easy to verify that $(s_{1,i}(h,\varepsilon))^2 = h^2 + O\left(\frac{h^3}{\varepsilon}\right)$ and $(s_{2,i}(h,\varepsilon))^2 = h^2 + O\left(\frac{h^3}{\mu}\right)$.
Eqs. (18)-(21) lead to the following $(2N-2)\times(2N-2)$ system of linear equations:

$$\begin{pmatrix} T_1 & D_1 \\ D_2 & T_2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{25}$$

where the tri-diagonal matrices T_1, T_2, the diagonal matrices D_1, D_2 and the vectors F_1, F_2 are defined as follows:

$$(T_1)_{i,i} = \frac{2\varepsilon}{(s_{1,i})^2} + \frac{a_{1,i}}{h} + b_{11,i}, \quad i = 1(1)(N-1),$$
$$(T_1)_{i,i+1} = -\frac{\varepsilon}{(s_{1,i})^2} - \frac{a_{1,i}}{h}, \quad i = 1(1)(N-2),$$
$$(T_1)_{i,i-1} = -\frac{\varepsilon}{(s_{1,i})^2}, \quad i = 2(1)(N-1),$$
$$F_{1,1} = f_{1,1} + \left(\frac{\varepsilon}{(s_{1,i})^2}\right)Y_{10},$$
$$F_{1,i} = f_{1,i}, \quad i = 2(1)(\frac{N}{2} - 1) \;\&\; (\frac{N}{2} + 1)(1)(N-2),$$
$$F_{1,\frac{N}{2}} = \frac{f_1(d-h) + f_1(d+h)}{2},$$
$$F_{1,N-1} = f_{1,N-1} + \left(\frac{\varepsilon}{(s_{1,N-1})^2} + \frac{a_{1,N-1}}{h}\right)Y_{1N},$$
$$D_{1\,i,i} = b_{12}(x_i), \quad i = 1(1)(N-1).$$

$$(T_2)_{i,i} = \frac{2\mu}{(s_{2,i})^2} + \frac{a_{2,i}}{h} + b_{22,i}, \quad i = 1(1)(N-1),$$
$$(T_2)_{i,i+1} = -\frac{\mu}{(s_{2,i})^2} - \frac{a_{2,i}}{h}, \quad i = 1(1)(N-2),$$
$$(T_2)_{i,i-1} = -\frac{\mu}{(s_{2,i})^2}, \quad i = 2(1)(N-1),$$
$$F_{2,1} = f_{2,1} + \left(\frac{\mu}{(s_{2,i})^2}\right) Y_{20},$$
$$F_{2,i} = f_{2,i}, \quad i = 2(1)(\frac{N}{2}-1) \;\&\; (\frac{N}{2}+1)(1)(N-2),$$
$$F_{2,\frac{N}{2}} = \frac{f_2(d-h) + f_2(d+h)}{2},$$
$$F_{2,N-1} = f_{2,N-1} + \left(\frac{\mu}{(s_{2,N-1})^2} + \frac{a_{2,N-1}}{h}\right) Y_{2N},$$
$$D_{2\,i,i} = b_{21}(x_i), \quad i = 1(1)(N-1).$$

Note that the matrices T_1 and T_2 in Eq. (25) are both M-matrices.

The solution of the matrix system (25) is solved iteratively using the following simultaneous iterative scheme

$$T_1 Y_1^{(k)} = F_1 - D_1 Y_2^{(k)}, \tag{26}$$
$$T_2 Y_2^{(k+1)} = F_2 - D_2 Y_1^{(k)}. \tag{27}$$

An arbitrary initial guess for Y_2 was taken to be $(0.1, 0.1, ..., 0.1)^T$ and the stopping criteria were set to be

$$\| Y_2^{(k+1)} - Y_2^{(k)} \| < 10^{-15} \quad \text{and} \quad \| Y_1^{(k+1)} - Y_1^{(k)} \| < 10^{-15}.$$

It is easy to see that the iterative scheme (26),(27) converges to the solution of the matrix system (25) by adopting the procedure given in [17].
Analogous to the continuous results stated in Lemma 2.1 and Lemma 2.2, the following results can be proved.

Lemma 3.1 (Discrete Maximum Principle).
For any mesh function $\overline{\Psi}(x_i)$, assume that $\overline{\Psi}_0 \geq \overline{0}, \overline{\Psi}_N \geq \overline{0}, \boldsymbol{P}^N_{\varepsilon,\mu}\overline{\Psi} \geq \overline{0}\ \forall\ x_i \in \Omega^N$ and $D^+\overline{\Psi}_{\frac{N}{2}} - D^-\overline{\Psi}_{\frac{N}{2}} \leq \overline{0}$. Then $\overline{\Psi}(x_i) \geq \overline{0}\ \forall\ x_i \in \overline{\Omega}^N$. □

Lemma 3.2 (Discrete Stability).
If $\overline{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$ is any mesh function such that $\overline{Z}_0 \geq \overline{0}$, $\overline{Z}_N \geq \overline{0}$, and $|P^N_{\varepsilon,\mu}\overline{Z}(x_i)| \leq \overline{g}$, then

$$|Z_j(x_i)| \leq \max\{\|\overline{Z}(x_0)\|, \|\overline{Z}(x_N)\|\} + \frac{1}{\beta}\|\overline{g}\|_{\Omega^{-N}\cup\Omega^{+N}},\ \forall\ x_i \in \overline{\Omega}^N, \quad j = 1, 2,$$

where $\overline{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $\|\overline{g}\| = \max\{\|g_1\|, \|g_2\|\}$ *and* $\beta = \min\{\beta_1, \beta_2\}$. □

4. Error analysis

Lemma 4.1.
At each mesh point $x_i \neq d$, the error satisfies the following estimate:

$$|\boldsymbol{P}^N_{\varepsilon,\mu}(\overline{Y} - \overline{y})(x_i)| \leq Ch.$$

Proof. At each grid point x_i, $i = 1, ..., (\frac{N}{2}-1)\ \&\ (\frac{N}{2}+1), ..., N-1$, consider the mesh function $(\overline{Y} - \overline{y})(x_i)$, where $\overline{Y}$ and $\overline{y}$ are the solutions of discrete problem and continuous problem respectively.

The local truncation error of the scheme in the first component of the solution is given by

$$P^N_{1,\varepsilon}(\overline{Y} - \overline{y})(x_i) = \varepsilon y''_{1,i} + a_{1,i} y'_{1,i} - \varepsilon\left(\frac{y_{1,i-1} - 2y_{1,i} + y_{1,i+1}}{(s_{1,i})^2}\right) - \frac{a_{1,i}}{2h}(y_{1,i+1} - y_{1,i}). \tag{28}$$

Using $(s_{1,i})^2$ from Eq. (22) and the Taylor series expansions of $y_{1,i-1}, y_{1,i+1}$, we get

$$P^N_{1,\varepsilon}(\overline{Y}-\overline{y})(x_i) = \varepsilon y''_{1,i} - \frac{\varepsilon}{h^2}\left(1 - \frac{a_{1,i}h}{\varepsilon} + \frac{a^2_{1,i}h^2}{\varepsilon^2} - \frac{a^3_{1,i}h^3}{\varepsilon^3}\right) h^2 y''_{1,i} - \frac{a_{1,i}h}{2} y''_{1,i} - \frac{a_{1,i}h^2}{6} y''_1(\xi), \quad \xi \in (x_i, x_{i+1})$$

$$= \frac{a_{1,i}h}{2} y''_{1,i} - \left[a^2_{1,i}\frac{y''_{1,i}}{\varepsilon} - a^3_{1,i}h\frac{y''_{1,i}}{\varepsilon^2} + \frac{a_{1,i}}{6} y'''_1(\xi)\right] h^2, \quad \xi \in (x_i, x_{i+1}).$$

Thus

$$|P^N_{1,\varepsilon}(\overline{Y}-\overline{y})(x_i)| \leq \frac{a_{1,i}h}{2}|y''_{1,i}| + \left[a^2_{1,i}|\frac{y''_{1,i}}{\varepsilon}| + a^3_{1,i}h|\frac{y''_{1,i}}{\varepsilon^2}| + \frac{a_{1,i}}{6}|y'''_1(\xi)|\right] h^2. \tag{29}$$

Using the Lemma-7 given in [18] for continuous case of reaction-diffusion problem, the following results hold good for convection-diffusion problem:

$$\lim_{\varepsilon\to 0} \max_{1\leq i\leq N-1} \frac{\exp\left(-\frac{Cx_i}{\varepsilon}\right)}{\varepsilon^k} = 0, \tag{30}$$

$$\lim_{\varepsilon\to 0} \max_{1\leq i\leq N-1} \frac{\exp\left(-\frac{C(1-x_i)}{\varepsilon}\right)}{\varepsilon^k} = 0, \tag{31}$$

for a fixed mesh and for all integers k.
Now, using the Lemma 2.3 and Eqs. (30)-(31) for $x_i \in \Omega^N_{\varepsilon,\mu}$, the absolute values of the quantities in the inequality (29) are bounded by a constant , then we have

$$|P^N_{1,\varepsilon}(\overline{Y}-\overline{y})(x_i)| \leq Ch. \tag{32}$$

Similarly, the bound on the truncation error in the second component of the solution can be obtained as follows:

$$|P^N_{2,\mu}(\overline{Y}-\overline{y})(x_i)| \leq Ch. \tag{33}$$

Combining the results (32) and (33), we get

$$|\mathbf{P}^N_{\varepsilon,\mu}(\overline{Y}-\overline{y})(x_i)| \leq Ch. \tag{34}$$

□

Lemma 4.2.
At the point of discontinuity $x_i = d$, the error satisfies the following estimate:

$$|\boldsymbol{P}^N_{\varepsilon,\mu}(\overline{Y}-\overline{y})(d)| \leq Ch.$$

Proof. Similar to the procedure adopted in [19, 20], the proof is given as follows:
At the point $x_i = d$, we have

$$\begin{aligned}
P^N_{1,\varepsilon}(\overline{Y}-\overline{y})(d) &= P^N_{1,\varepsilon}\overline{Y}(d) - \overline{y}(d)\\
&= f_1(d) - P^N_{1,\varepsilon}\overline{y}(d)\\
&= f_1(d) + \frac{\varepsilon}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t} y''_1(s)\,ds\,dt - \frac{\varepsilon}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t} y''_1(s)\,ds\,dt\\
&+ \frac{a_1(d)}{h}\int_{t=d}^{d+h} y'_1(t)\,dt - b_{11}(d)y_1(d) - b_{12}(d)y_2(d)\\
&= -\frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}(f_1 + a_1y'_1 - b_{11}y_1 - b_{12}y_2)(s)\,ds\,dt\\
&+ \frac{a_1(d)}{h}\int_{t=d}^{d+h} y'_1(t)\,dt + \frac{f_1(d-h)}{2} + \frac{f_1(d+h)}{2} - b_{11}(d)y_1(d) - b_{12}(d)y_2(d)\\
&= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{p=s}^{d+h} + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{p=d-h}^{s}(f_1 + a_1y'_1 - b_{11}y_1 - b_{12}y_2)'(p)\,dp\,ds\,dt\\
&- \frac{h^2}{2}((f_1 + a_1y'_1 - b_{11}y_1 - b_{12}y_2)(d-h) + (f_1 + a_1y'_1 - b_{11}y_1 - b_{12}y_2)(d+h))\\
&+ \frac{a_1(d)}{h}\int_{t=d}^{d+h} y'_1(t)\,dt + \frac{1}{2}\int_{t=d}^{d-h}(b_{11}y_1 + b_{12}y_2)'(t)\,dt + \frac{1}{2}\int_{t=d}^{d+h}(b_{11}y_1 + b_{12}y_2)'(t)\,dt
\end{aligned}$$

which implies that

$$\left|P^N_{1,\varepsilon}(\overline{Y}-\overline{y})(d)\right| \le Ch. \tag{35}$$

Similarly, we obtain

$$\left|P^N_{2,\mu}(\overline{Y}-\overline{y})(d)\right| \le Ch. \tag{36}$$

Combining the results (35) and (36), we get

$$\left|\mathbf{P}^N_{\varepsilon,\mu}(\overline{Y}-\overline{y})(d)\right| \le Ch. \tag{37}$$

□

Theorem 4.1.
If $\overline{y}$ is the solution of the problem (1)-(3) and $\overline{Y}$ is the approximation of $\overline{y}$ obtained using the scheme (18)-(21), then there exists a constant C, independent of ε, μ and h, such that

$$\sup_{0<\varepsilon\le\mu<1}\max_{0\le i\le N}|(\overline{Y}-\overline{y})(x_i)| \le Ch, \quad \forall x_i \in \overline{\Omega}_{\varepsilon,\mu} \tag{38}$$

Proof. From Lemma 4.1, using the maximum principle and proper choice of the constant C, it is easy to prove that

$$|(\overline{Y}-\overline{y})(x_i)| \le Ch. \tag{39}$$

Similarly, from Lemma 4.2, it can be seen that

$$|(\overline{Y}-\overline{y})(d)| \le Ch. \tag{40}$$

Combining the results (39) and (40), the desired result follows. □

5. Numerical results

To show the applicability and efficiency of the proposed method, the following test problem is considered:

Example 5.1.

$$-\varepsilon y_1''(x) - 0.8y_1'(x) + 3y_1(x) - y_2(x) = f_1(x) = \begin{cases} 2, & 0 \le x < 0.5, \\ -1, & 0.5 \le x \le 1. \end{cases},$$
$$-\mu y_2''(x) - y_2'(x) - y_1(x) + 3y_2(x) = f_2(x) = \begin{cases} 1.8, & 0 \le x < 0.5, \\ -0.8, & 0.5 \le x \le 1. \end{cases},$$
$$y_1(0) = 0, \quad y_1(1) = 2, \quad y_2(0) = 0, \quad y_2(1) = 2.$$

Example 5.1 was also used in [16] for $\varepsilon = \mu$ case.
Due to the fact that the exact solution of the test problem is not available, the maximum point-wise errors at all mesh points using two mesh differences are computed by

$$E^N_{j_{\varepsilon,\mu}} = \max_{0\le i\le N}\left|Y^N_{j,i} - \tilde{Y}^{2N}_{j,i}\right|, \quad j = 1,2,$$

which is the difference between the values of the i^{th} component of the solution of a mesh of N mesh points and the interpolated value of the solution, at the same point, on a mesh of $2N$ points. The range of parameters is taken as $\varepsilon = \mu = \{10^{-1}, 10^{-2}, ..., 10^{-15}\}$ satisfying the condition $0 < \varepsilon \le \mu < 1$.
The parameter-uniform maximum errors E^N_j are obtained by

$$E^N_j = \max_{\varepsilon}\max_{\mu} E^N_{j_{\varepsilon,\mu}}, \quad j = 1,2.$$

Further, the rates of uniform convergence p^N_j are calculated by

$$p^N_j = \log_2\left(\frac{E^N_j}{E^{2N}_j}\right), \quad j = 1,2.$$

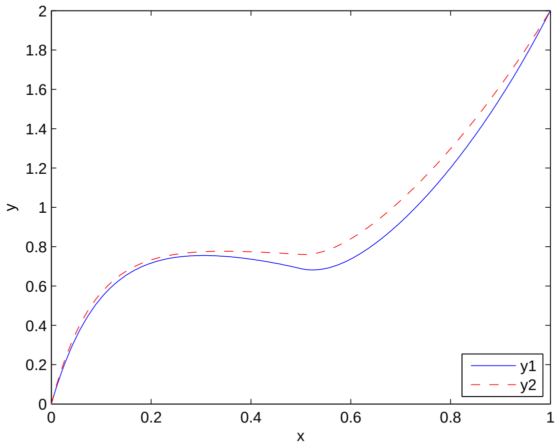

Fig. 1. Numerical solution of Example 5.1 for $\varepsilon = 10^{-1}, \mu = 10^{-1}$ and $N = 256$

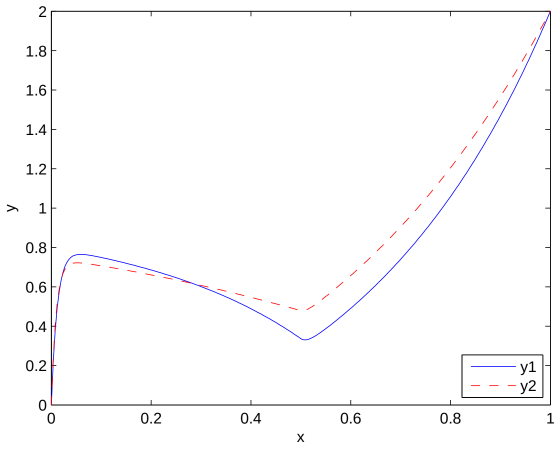

Fig. 2. Numerical solution of Example 5.1 for $\varepsilon = 10^{-2}, \mu = 10^{-1}$ and $N = 256$

Table 1. Maximum point-wise errors $E_{1_\varepsilon}^N$, parameter-uniform errors E_1^N and the uniform rates of convergence p_1^N for the solution y_1 of Example 5.1

μ	Number of Mesh Points N						
	64	128	256	512	1024	2048	4096
10^{-1}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5552E-04
10^{-2}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5552E-04
10^{-3}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5552E-04
10^{-4}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5552E-04
10^{-5}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5552E-04
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
10^{-15}	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5556E-04
E_1^N	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03	7.1056E-04	3.5556E-04
p_1^N	0.97	0.98	0.99	1.00	1.00	1.00	-

Table 2. Maximum point-wise errors $E_{2_\varepsilon}^N$, parameter-uniform errors E_2^N and the uniform rates of convergence p_2^N for the solution y_2 of Example 5.1

μ	Number of Mesh Points N						
	64	128	256	512	1024	2048	4096
10^{-1}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7405E-04
10^{-2}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7405E-04
10^{-3}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7405E-04
10^{-4}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7405E-04
10^{-5}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7405E-04
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
10^{-15}	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7406E-04
E_2^N	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03	5.4786E-04	2.7406E-04
p_2^N	0.98	0.99	0.99	1.00	1.00	1.00	-

Table 3. Comparison of maximum errors(M.E.) and rates of convergence(R.C.) obtained by our method and those in [16] for the solution component y_1 of Example 5.1

$\varepsilon = \mu = 2^{-30}$	Number of Mesh Points N				
	64	128	256	512	1024
[16]	1.7890E-02	1.1151E-02	6.4060E-03	3.6164E-03	2.0081E-03
	0.6820	0.7997	0.8249	0.8487	-
Ours	2.1692E-02	1.1109E-02	5.6231E-03	2.8290E-03	1.4189E-03
	0.97	0.98	0.99	1.00	-

Table 4. Comparison of maximum errors(M.E.) and rates of convergence(R.C.) obtained by our method and those in [16] for the solution component y_2 of Example 5.1

$\varepsilon = \mu = 2^{-30}$	Number of Mesh Points N				
	64	128	256	512	1024
[16]	1.5904E-02	9.6979E-03	5.9419E-03	3.4324E-03	1.9536E-03
	0.7136	0.7067	0.7917	0.8131	-
Ours	1.7025E-02	8.6425E-03	4.3540E-03	2.1852E-03	1.0947E-03
	0.98	0.99	0.99	1.00	-

6. Conclusions

In this paper, a non-standard finite difference method (NSFDM) on a uniform mesh is developed for a weakly coupled system of singularly perturbed convection-diffusion equations with discontinuous source term. Error estimates are provided and the scheme is proved to be uniformly convergent of order one with respect to the singular perturbation parameters. A test example is presented which support the theoretical results. From the Table 1 and Table 2, it can be seen that the error is robust w.r.t. the parameters ε and μ, and is converging to zero as N is increased. The results obtained by the present method proves better than the results of existing standard finite difference method on Shishkin mesh as shown in Table 3 and Table 4. Left boundary layer and weak interior layer can be seen from the Fig. 1 and Fig. 2 as per the prediction.

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