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Hardening nonlinearity effects on forced vibration of viscoelastic dynamical systems to external step perturbation field

Research Article

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Abstract: Many engineering structural elements are subjected during manufacture process or service to step excitation field. Vibration analysis of structural systems to step perturbation field has then become a vital requirement for modern structural engineering. The objective in this paper is to demonstrate that the hardening or softening nonlinearity affects some properties of viscoelastic dynamical systems under step loading field. The accuracy and reliability of the predicted performance is verified by comparison with numerical results. Phase diagrams are simulated to also verify the linear stability analysis according to the Lyapunov theory. Analytical and numerical results are found to be in satisfactory agreement. It is again found that the steady state forced response follows a power function of excitation rather than the linear law predicted by the viscously damped linear harmonic oscillator. So, it is shown that the stiffening or softening nonlinearity affects not only the transient response and the system stiffness but also its natural frequency and equilibrium position. Therefore the present model can be used in reduction and control of forced vibrations, and in control of long time dynamics of mechanical systems to step excitation field. It finally, may also be used for identification of viscoelastic material properties from stationary output.

MSC: 74H45 • 35B35

Keywords: Hardening nonlinearity • Steady state solution • Local stability

1. Introduction

The study of forced vibration to external dynamical perturbation field of viscoelastic nonlinear systems is today subject to a particular attention in several pure and applied fields of science, and also in many fields of engineering. This interest seems to be motivated by the fact that viscoelastic components are widely used in engineering and industrial applications due to their high vibration damping and isolation properties [1]. For example, viscoelastic materials are used in vibration absorption in automobile and machine system suspensions. In this context, the classical viscously damped linear harmonic oscillator is widely used in engineering design for characterizing the forced vibration response to dynamical perturbation field of mechanical systems undergoing viscoelastic deformation. But, it is experimentally observed and confirmed by analytical works that real mechanical systems exhibit many new phenomena such as frequency dependent amplitude, bifurcation of equilibriums, sub- and super-harmonic resonances, and chaos, related to softening or hardening property, in response to external applied dynamical perturbation field in comparison with the predictions of the damped linear harmonic oscillator [2–4]. The effects of nonlinearity on the vibration response of mechanical systems have been the subject of many works due to their almost omnipresence in real-world phenomena [2, 5, 6]. These works are often performed in the context of linear viscous damping and nonlinear stiffness captured in terms of cubic (Duffing-type) or polynomial nonlinearity, which are separately introduced in equations governing the dynamical response of the studied mechanical system [2, 7, 8]. In such a situation, these model equations could not be used to investigate adequately and satisfactorily the softening or hardening nonlinearity effects on the dynamics of real mechanical systems since these exhibit viscoelastic response in which the

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damping is influenced by the hardening or softening, that is to say in which the damping and stiffness nonlinearities are closely interconnected [8]. Modeling the dynamics of viscoelastic structures requires the use of adequate and satisfactory constitutive law of state which continues to be a challenge for structural engineering. Indeed, there is no in the range of large deformations, a universal theory of viscoelasticity that works for each material in all practical situations of loading [9]. This possibly may be attributed to the rich diversity of dynamical phenomena exhibited by material systems in geometrically nonlinear operating regime. As well known, viscoelastic materials like polymers, rubbers, concrete, soils and soft living tissues, even under moderate external loading levels, may show large deformations. As previously underlined, the hardening or softening effects have often been investigated in terms of cubic nonlinearity. In such a situation, there exist almost no analytical studies in the literature on the topic on the effects of strain hardening exponent on the forced vibration response to external applied dynamical perturbation of mechanical structural systems in the range of nonlinear viscoelastic deformations. It is then justified to perform an analysis on the effects of changes in the hardening exponent on the behavior of the forced vibration of viscoelastic nonlinear dynamical systems to external applied dynamical loading and in particular, to step excitation field, as one of the most important viscoelastic behavior experienced by a mechanical structure is the creep response. Creep is defined as a time-dependent irreversible deformation process. Creep expresses the gradual increase of deformation with time of a material system subject to a dynamical perturbation field undergoing a sudden change in its magnitude at time $t = 0$, from zero to F_0 which is sustained at a constant value. Such a dynamical excitation is widely used in structural design to experimentally characterize the dynamical behavior of viscoelastic structures [10–12]. It is also interesting to note that creep, due to deformation increasing and irreversible change in the structural shape with time, reduces the performance of many engineering structures. It is also a general cause of structural instabilities leading to failure of mechanical structures [11]. In this context, it is convenient to design generalized nonlinear oscillator that can incorporate the classical viscoelastic linear harmonic oscillator as a special case [13] in order to explore the effect of hardening exponent on the forced vibration of mechanical systems under step excitation field. That being so, given now the structural model recently developed within the framework of Bauer's dynamical theory of viscoelasticity, which closely relates stiffness, damping and inertial nonlinearities, by [8] for representing the nonlinear dynamics of viscoelastic systems with single degree of freedom

$$u^{l-1}\ddot{u} + (l-1)u^{l-2}\dot{u}^2 + \lambda u^{l-1}\dot{u} + \frac{\omega_0^2}{l}u^l = \frac{1}{lm}F(t,u,\dot{u},\ddot{u}) \tag{1}$$

where $F(t,u,\dot{u},\ddot{u}) = L_0^{1+l}\sigma(t)$ is the product of an external applied force with the quantity L_0^{l-1} given that $\sigma(t)$ is the scalar stress function and L_0 the original length of the system. In equation (1) the variable $u(t)$ designates the time displacement, ω_0 an angular frequency, λ the viscous damping coefficient, m the mass of the system, and l is the strain hardening exponent. The dot over a symbol means the time derivative. The model (1) reduces, setting $l = 1$, to that of classical equation of forced vibration of a linear viscously damped structure with single degree of freedom [14, 15] so that the geometric nonlinearity control parameter assures the transition of the structural system (1) from linear to nonlinear dynamics. For the linear regime of operation, that is for $l = 1$, the response of the structural system (1) to an external step perturbation field consists of a superposition of a transient motion with a steady state response. The transient response is the solution of the homogeneous equation, which vanishes after a sufficient time ($t \to \infty$). So, the steady state response describes the long time motion of the structural system. The transient vibrations depend on the initial conditions of the structural motion. By contrast, the stationary or steady state forced response is independent of the initial conditions, but depends only on the amplitude of excitation F_0 and the linear stiffness k [15]. A non-dimensional form of this classical viscously damped linear harmonic oscillator eliminates the constant k from the steady state value of the response showing that the dimensionless equilibrium state deformation is equal to amplitude of dimensionless excitation E_0. As a result, the most important response in forced motion to a step forcing function of the classical viscously damped linear harmonic oscillator, that is, the steady state response, varies linearly with excitation. So with that, the structural dynamics problem of interest in comparison with the classical linear prediction is, more precisely, to examine how does the softening or hardening nonlinearity, that is the strain hardening exponent l, affect the steady state response of the structural system (1) under external applied step excitation. In other words, it is desired to investigate how does the equilibrium deformation vary with excitation E_0 in presence of hardening or softening nonlinearity captured by the coefficient l. So the question that can be envisaged here becomes: Does the steady state deformation of the viscoelastic nonlinear dynamical system (1) to an external step excitation field depends on the strain hardening exponent l? The response to this question will allow us to verify the experimental observations following which a variation in stiffness due to stiffening or softening produces a change in the equilibrium deformation of mechanical systems [3, 4]. This question will allow also to establish if the current structural model incorporates the classical viscously damped linear oscillator as a special case. From a practical point of view, the preceding question will enable finally to use the hardening or softening nonlinearity parameter l in control of steady state deformation response of mechanical systems subject to external applied step excitation field on one hand, and to use forced vibration experiment under external applied step force for material system parameter identification from steady state data, on the other hand. Control of steady state response is of great importance, since it represents the essential dynamics, to say the long time dynamics of mechanical systems under forced vibration. Control of equilibrium state is also of great importance, since it is often needed for practical purpose in structural design to perform a linearization of nonlinear dynamical systems around the equilibrium position. In this perspective the present research contribution predicts that the steady state deformation of the current structural model to step loading depends on the hardening exponent $l \neq 0$. That being said, to demonstrate the predicted performance, exact analytical solutions depending on the viscous damping value λ to equation (1) and its stability analysis is first of all achieved (section 2), and secondly, the accuracy and reliability of analytical results is verified by comparison with the numerical evaluations (section 3). Finally, the predicted results are examined and discussed (section 4) in order to conclude on the theoretical and practical implications of the research work (section 5).

2. Methods

This part is devoted to carry out exact analytical solutions to equation (1) following the viscous damping value λ (Paragraph 2.1) and linear stability analysis of the structural model (Paragraph 2.2).

2.1. Exact analysis

To proceed to an exact analysis of the structural model it is convenient to first reduce it to the non-dimensional form (Sub-paragraph 2.1.1) and to clearly define the mathematical problem to be solved (Sub-paragraph 2.1.2). The nonlinear problem is then reduced to a classical linear equation (Sub-paragraph 2.1.3) for which solution may be obtained by application of elementary standard mathematical methods of integration of differential equations in terms of superposition of a steady state response (Sub-paragraph 2.1.4) with a transient response (Sub-paragraph 2.1.5).

2.1.1. Reduction of the equation to the non-dimensional form

Interest for non-dimensional form of differential equations resides in the possibility to reduce the number of model parameters and to make easy their comparison. In doing so, the non-dimensionalization process facilitates the mathematical problem treatment and it numerical simulation. Therefore, introducing the dimensionless independent variable

$$\tau = \omega_0 t$$

and dependent variable

$$\varepsilon(\tau) = \frac{u(t)}{L_0}$$

where L_0 designates the original length of the system, into Eq.(1), leads to the dimensionless non-homogeneous equation

$$\varepsilon^{l-1}\varepsilon'' + (l-1)\varepsilon^{l-2}\varepsilon'^2 + 2\mu\varepsilon^{l-1}\varepsilon' + \frac{1}{l}\varepsilon^l = \frac{E(\tau)}{l} \tag{2}$$

where $2\mu = \dfrac{\lambda}{\omega_0}$, $E(\tau) = \dfrac{\sigma_\tau}{a}$ and $a = \omega_0^2 c$.

The prime over a symbol means differentiation with respect to dimensionless variable τ and $c \neq 0$, denotes the inertia coefficient [8], $E(\tau)$ denotes the dimensionless external applied perturbation in order to study the model response in forced vibration experiments. Thus the normalized displacement $\varepsilon(\tau)$ has the nature of strain within the structural system. There is then a need to clarify the mathematical problem to be answered as regards Eq.(2)

2.1.2. Mathematical problem

The objective, here is to clearly set the mathematical problem in solving process. Assuming that the dynamics of the dimensionless dependent variable $\varepsilon(\tau)$ should obey the conditions at $\tau = 0$, $\varepsilon(\tau) = \varepsilon_0$, and $\varepsilon'(\tau) = v_0$, and also that the dynamical perturbation field $E(\tau)$ verifies

$$E(\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ E_0 & \text{for } \tau \geq 0 \end{cases}$$

that is

$$E(\tau) = E_0 H(\tau)$$

where $H(\tau)$ is the so called Heaviside unit function, the mathematical problem under consideration reduces to the Cauchy initial value problem for the second-order non-homogeneous nonlinear differential equation

$$\varepsilon'' + (l-1)\frac{\varepsilon'^2}{\varepsilon} + 2\mu\varepsilon' + \frac{1}{l}\varepsilon = \frac{E_0}{l}\varepsilon^{1-l} \tag{3}$$

with initial conditions being mentioned above and $\mu \geq 0$. Equation (3) shows that the factor $\dfrac{1}{l}$ may be interpreted as the dimensionless natural frequency and $(l-1)$ as the dimensionless non-viscous damping term due to the terms ε and $\dfrac{\varepsilon'^2}{\varepsilon}$,respectively. So, in Eq.(3) the hardening exponent l closely relates the damping and restoring nonlinear forces. The equation (3) is not directly integrable by quadrature. Thus, exact analytical solving of Eq.(3) needs a new change of variable which may lead to classical mathematical problem solving in closed form analytical solution.

2.1.3. Reduction of the nonlinear equation of motion to classical linear equation

In this section, a differential equation transformation will be applied to carry out exact analytical solutions to equation (3). This process provides the advantage to apply then the well known and established standard mathematical integration methods for classical linear ordinary differential equations with constant coefficient. So, there will be sufficient to make the following suitable substitution

$$h(\tau) = \varepsilon^{l}(\tau) \tag{4}$$

into equation (3), to obtain after a few mathematical manipulation, the classical second-order inhomogeneous linear differential equation with constant coefficients

$$h^{''}(\tau) + 2\mu h^{'}(\tau) + h(\tau) = E_0 \tag{5}$$

From this analysis, the following useful result has been obtained.

Theorem 2.1.
Let us consider $h(\tau) = \varepsilon^{l}(\tau)$. If $\varepsilon(\tau)$ satisfies (3), then $h(\tau)$ satisfies (5).

The equation (5) is well known in physics and mechanical engineering literatures, since it is widely used for modeling forced vibrations to step loading of viscously damped structures with single degree of freedom under the assumption of small displacement or low excitation level [14, 15]. In other words, Eq.(5) is the so called differential equation of Kelvin-Voigt viscoelastic linear harmonic oscillator under forced vibration. So with that, its exact analytical solution is well known to be the sum of the general solution of the associated homogeneous differential equation with a particular solution of the non-homogeneous differential equation.

2.1.4. Particular solution to inhomogeneous linear equation

The particular solution to Eq.(5) is the constant solution, that is the steady state solution of the equation (5) corresponding to the expression $h(\tau)$ in the limit that $\tau \to +\infty$. So, it becomes immediate to observe that the stationary solution $h_s(\tau)$ may be written as

$$h_s(\tau) = E_0$$

from which the desired exact analytical solution to nonlinear differential equation (3) may be easily expressed.

2.1.5. Exact analytical solution to nonlinear equation of motion

Depending on the above, forced solution to nonlinear differential equation (3) may be computed according to Eq. (4) as

$$\varepsilon(\tau) = [h_t(\tau) + h_s(\tau)]^{1/l}$$

or

$$\varepsilon(\tau) = [h_t(\tau) + E_0]^{1/l}$$

where $h_t(\tau)$ denotes the homogeneous solution to Eq.(5), that is the transient solution. In this context the time history of the dimensionless displacement $\varepsilon(\tau)$ depends on the strength of the viscous damping factor μ. Let us now find the solutions to Eq.(3) following the different viscous damping cases.

2.1.5.1. Over-damped nonlinear forced solution

This case corresponds to a strong viscous damping that is, $\mu > 1$. According to [8] the over-damped nonlinear forced solution may be written in the form

$$\varepsilon(\tau) = \varepsilon_0\left[\sqrt{\frac{\left(\frac{l v_0}{\varepsilon_0}\right)^2 + \frac{2l\mu v_0}{\varepsilon_0}\left(1-\frac{E_0}{\varepsilon_0^l}\right) + \left(1-\frac{E_0}{\varepsilon_0^l}\right)^2}{\mu^2-1}}\exp(-\mu\tau)\sinh\left(\sqrt{\mu^2-1}\,\tau - \operatorname{arctanh}\left(-\frac{\left(1-\frac{E_0}{\varepsilon_0^l}\right)\sqrt{\mu^2-1}}{\frac{l v_0}{\varepsilon_0}+\mu\left(1-\frac{E_0}{\varepsilon_0^l}\right)}\right)\right) + \frac{E_0}{\varepsilon_0^l}\right]^{1/l} \tag{6}$$

The solution (6) is the sum of an exponentially decaying term and a constant term so that it asymptotically approaches in the limit $\tau \to +\infty$ without oscillations the equilibrium position

$$\varepsilon_s(\tau) = E_0^{1/l} \tag{7}$$

which defines the steady state response of the viscoelastic nonlinear dynamical system under creep conditions, as the factor $\exp(-\mu\tau) \to 0$. It can be with clearness seen that the steady state response of the viscously over-damped structural system follows $\frac{1}{l}$ th power law of excitation E_0. Now, it is convenient to analytically check the validity of this law (7) for the case of viscously critically damped forced response to step loading of the viscoelastic nonlinear dynamical system under question.

2.1.5.2. Critically damped nonlinear forced solution

This case of damping is associated with $\mu = 1$. Following [8] the forced solution to Eq.(3) may be given by

$$\varepsilon_0 \left[\left(1 - \frac{E_0}{\varepsilon_0^l} + \left(1 - \frac{E_0}{\varepsilon_0^l} + \frac{l v_0}{\varepsilon_0} \right) \tau \right) \exp(-\tau) + \frac{E_0}{\varepsilon_0^l} \right]^{1/l} \tag{8}$$

As can be seen in the solution (8) the first term in the brackets vanishes as the dimensionless time $\tau \to +\infty$ in accordance with the factor $\exp(-\tau)$, and the dimensionless displacement asymptotically converges without oscillations to the steady response

$$\varepsilon_s(\tau) = E_0^{1/l} \tag{9}$$

Therefore, the steady state law previously observed holds also for the forced response to step excitation of the viscoelastic nonlinear dynamical system under study. It remains now to verify the validity of this law for the viscously under-damped forced response of the structural system.

2.1.5.3. Under-damped nonlinear forced solution

The under-damped nonlinear forced response signifies that the structural system is weakly viscously damped. This case will occur if $\mu < 1$. Since mechanical systems are in general weakly damped, the case $\mu < 1$ is then of primary importance in vibration analysis in engineering design. According to [8], the solution to Eq.(3) in this case may be of the form

$$\varepsilon(\tau) = \varepsilon_0 \left[\sqrt{\frac{\left(\frac{l v_0}{\varepsilon_0}\right)^2 + \frac{2 l \mu v_0}{\varepsilon_0}\left(1 - \frac{E_0}{\varepsilon_0^l}\right) + \left(1 - \frac{E_0}{\varepsilon_0^l}\right)^2}{1 - \mu^2}} \exp(-\mu\tau) \cos\left(\sqrt{1-\mu^2}\,\tau - \arctan\left(\frac{\frac{l v_0}{\varepsilon_0} + \mu\left(1 - \frac{E_0}{\varepsilon_0^l}\right)}{\left(1 - \frac{E_0}{\varepsilon_0^l}\right)\sqrt{1-\mu^2}} \right) \right) + \frac{E_0}{\varepsilon_0^l} \right]^{1/l} \tag{10}$$

Contrarily to Eqs.(6) and (8), as can be clearly seen, the solution (10) consists of a sum of an exponentially damped sinusoidal function which will cause the oscillatory behavior of the forced response of the structural system, and a constant term which is independent of the dimensionless time. So, the system asymptotically converges with oscillations to the stationary position

$$\varepsilon_s(\tau) = E_0^{1/l} \tag{11}$$

in the limit that $\tau \to +\infty$.

The equation (11) follows the same law as equations (7) and (9).

A special case of the viscously under-damped forced response may be found by setting in Eq.(10), $\mu = 0$. Hence, the solution (10) becomes

$$\varepsilon(\tau) = \varepsilon_0 \left[\sqrt{\left(\frac{l v_0}{\varepsilon_0}\right)^2 + \left(1 - \frac{E_0}{\varepsilon_0^l}\right)^2} \cos\left(\tau - \arctan\left(\frac{\frac{l v_0}{\varepsilon_0}}{1 - \frac{E_0}{\varepsilon_0^l}} \right) \right) + \frac{E_0}{\varepsilon_0^l} \right]^{1/l} \tag{12}$$

The solution (12) is the non-viscously damped forced response of the structural system to step perturbation field. The forced response, here, consists of a purely sinusoidal function and a constant term which is independent of the dimensionless time τ. Due to the form of solution (12) the existence of periodic vibrations for the structural system can be predicted. The criteria for existence of periodic solutions to a generalized form of equation (3) with $\mu = 0$, and $l < 1$, have been studied by [16]. However, it should be mentioned that periodic solutions to equation (12) can also be found for $l > 1$ as numerically shown below. In this context, the following theorem has been shown.

2.1.6. Statement of theorem

From the above, it is shown that the solution to the structural model (3) depends on the value of the viscous damping coefficient μ. But, the equilibrium state is the same for all solution such that $\mu > 0$. These main results are then stated in terms of the following theorem.

Theorem 2.2.
Let us consider the differential equation (3). *If the viscous damping factor* μ *is strictly positive, then all solutions to* (3) *are either non-oscillatory or oscillatory and asymptotically approach the unique equilibrium state* $\varepsilon_s(\tau)$ *such that* $\varepsilon_s(\tau) = E_0^{1/l}$, *as* $\tau \to +\infty$.

According to the above, it is seen that the equilibrium state is given by $\varepsilon_s(\tau) = E_0^{1/l}$. Taken the natural logarithm of this equation, leads to $\ln\varepsilon_s = \frac{1}{l}\ln E_0$. So, the stiffening control parameter l is the inverse of the slope of the equilibrium deformation versus excitation curve on log-log plot. It can be estimated from the measured logarithmic equilibrium deformation versus logarithmic excitation data. In this situation l indicates the rate of change of equilibrium deformation with excitation, that is how quickly the steady state deformation changes as the excitation E_0 varies. Alternatively, if the material system constant l is known it is then possible to estimate the magnitude of the excitation E_0 in the system from the stationary deformation $\varepsilon_s(\tau)$. As l increases, the equilibrium deformation decreases for a given value of excitation E_0, and the system becomes more stiffened. Conversely, as l decreases, the equilibrium deformation increases and the system becomes more softened for given excitation E_0.

The preceding sections have been devoted to perform exact analytical solution to the structural model. However, finding analytical solutions to a design problem is not sufficient in view of practical purpose, since a solution which is significantly perturbed for small changes in system parameter [17] and initial conditions is of very little usefulness or interest for vibration analysis in the perspective of predictive maintenance and existing problem diagnostic. Thus, stability analysis of structural models has become a key part of vibration analysis in modern engineering design. This stability study is in general achieved within the framework of dynamical system theory. This theory allows a geometric description of the system dynamics. In other words it translates the time evolution of the system under question in terms of trajectories described by a geometric point associated to the system state in a space called state space, with time. For a second order dynamical system, the state space reduces to a phase plane of position x and velocity $y = x^{'}(\tau)$. Let us, now, qualitatively perform dynamical analysis of properties of solutions to the nonlinear structural model.

2.2. Dynamical analysis

The section is designed with the objective of showing the local stability behavior of the structural model restricted to the only case where the viscous damping factor $\mu > 0$. In the context of initial small perturbations of the system from equilibrium, some useful informations may be obtained from the linearized system associated to the real nonlinear structural model. This approach is known as local stability analysis according to Lyapunov or the indirect method of Lyapunov. This is also effective only in the case of hyperbolic equilibrium. In other words, the local stability of the initial nonlinear system is equivalent to that of the associated linear model in the condition that the stationary state is a hyperbolic equilibrium. Otherwise, the stability of the original nonlinear system could not be predicted from that of the linear system. So, first of all, it will be needed to set the state space formulation of the nonlinear structural model (Sub-paragraph 2.2.1) and deduce, secondly, the equilibrium solution (Sub-paragraph 2.2.2). Finally, from the evaluation of the Jacobi matrix at the equilibrium (Sub-paragraph 2.2.3) the stability of the associated characteristic polynomial is demonstrated, from which the local stability behavior of the initial nonlinear structural model is proved (Sub-paragraph 2.2.4)

2.2.1. State-space model

The state-space representation will allow the reduction of the single differential equations (3) to a set of first order differential equations suitable for dynamical system analysis. Therefore, defining the state variables

$$x(\tau) = \varepsilon(\tau)$$

and

$$y(\tau) = x^{'}(\tau)$$

the model (3) reduces to the dynamical system of two first-order differential equations

$$\begin{aligned}\begin{pmatrix} x^{'}(\tau) \\ y^{'}(\tau) \end{pmatrix} &= \begin{pmatrix} y(\tau) \\ -(l-1)\dfrac{y^2}{x} - 2\mu y - \dfrac{1}{l}x + \dfrac{E_0}{l}x^{1-l} \end{pmatrix} \\ &= F(x, y) \end{aligned} \tag{13}$$

with $x \neq 0$, due to the Newtonian singularity in the equations. The form (13) of the structural model (3) is not only convenient for dynamical system theory, but it also is well appropriate for many numerical integration schemes with the help of computers [8]. The system of equations (13) consists of the trajectory equations of the dynamical system in the (x, y) phase plane for a specified set of initial conditions. This system may be then used to define the equilibrium position, since it corresponds to its particular stationary solution.

2.2.2. Equilibrium position

The equilibrium state $P(x^*, y^*)$ of the dynamical system is defined as the stationary solution found under the conditions that the dimensionless time derivatives

$$\frac{dx}{d\tau}|_{x=x^*} = \frac{dy}{d\tau}|_{y=y^*} = 0.$$

So, setting $x^{'}(\tau) = y^{'}(\tau) = 0$ into the system of equations (13), should give the stationary solution

$$\begin{cases} x^* = E_0^{1/l} \\ y^* = 0 \end{cases} \tag{14}$$

Since the set $x = 0$, must be avoided, the solution (14) is the unique equilibrium point of the system (13). In other words, equilibrium occurs only when the deformation history attains the value $E_0^{1/l}$ in accordance with the steady state solution obtained by the previous exact analytical solutions to equation (3). Having obtained the equilibrium position, it becomes possible to evaluate the Jacobian matrix of the first-order system (13) at this equilibrium in order to perform a stability analysis of the initial nonlinear model by a linearization technique.

2.2.3. Jacobian matrix

The Jacobi matrix will result from the linearization of the system (13) under the assumption of initial small perturbations according to Lyapunov theory. So, for a general point (x, y) the set of perturbations $\delta x = x - x^*$, $\delta y = y - y^*$ into the system (13), leads after expanding $F(x, y)$ in a Taylor series and neglecting the terms of order greater than the first, to the following system of first-order linear differential equations which represents the small perturbations behavior from the equilibrium (x^*, y^*)

$$\begin{pmatrix} \delta x^{'} \\ \delta y^{'} \end{pmatrix} = A \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ (l-1)\frac{y^2}{x^2} - \frac{1}{l} + \frac{E_0}{l}(1-l)x^{-l} & -2\mu \end{pmatrix}$$

is the Jacobi matrix of the vector field at a general point (x, y). So, the evaluation of A at the equilibrium position (x^*, y^*) gives

$$M = \begin{pmatrix} 0 & 1 \\ -1 & -2\mu \end{pmatrix} \tag{15}$$

The knowledge of the Jacobian matrix M provides the ability, through its associated characteristic polynomial, to investigate the stability behavior of the initial nonlinear dynamical system by analyzing the properties of the equilibrium state.

2.2.4. Local stability analysis

The local or linear stability analysis of the original nonlinear dynamical system may be achieved by solving the stability problem of the characteristic polynomial associated to the Jacobi matrix. In this context, the characteristic equation, first of all, will be determined (Part 2.2.4.1) and secondly, the stability behavior of the equilibrium will be investigated (Part 2.2.4.2). Finally, the main results will be formulated in terms of theorem (Part 2.2.4.3).

2.2.4.1. Characteristic polynomial

From the matrix M the characteristic polynomial takes the form

$$P(s) = det(M - sI) = s^2 + 2\mu s + 1 \tag{16}$$

where I denotes the identity matrix, s the eigenvalue and det means determinant. The damping factor $\mu > 0$ secures the positiveness of all coefficients of the characteristic polynomial, and then the hyperbolic nature of the equilibrium position. In other words, the necessary condition for stability is assured. By doing so, it now suffices that the roots of the characteristic polynomial have a strictly negative real part [18] to conclude to the local stability of the initial nonlinear system according to Routh-Hurwitz theorem. The problem then is to find the roots s_1 and s_2 of $P(s)$ to verify the sufficient condition of stability and typical nature of the equilibrium.

2.2.4.2. Properties of equilibrium position

To investigate the behavior of the equilibrium, it is before all thing, needed to determine the roots of the characteristic equation (16). To do so, it suffices to solve the following equation $P(s) = 0$, that is

$$s^2 + 2\mu s + 1 = 0 \tag{17}$$

Depending on the value of viscous damping coefficient μ, this equation yields three types of eigenvalue with $\mu > 1$, $\mu = 1$ and $\mu < 1$.

Case with $\mu > 1$

This case corresponds to a strong viscous damping defining the over-damped nonlinear forced response of the dynamical system. The roots of the characteristic equation (17) are real, distinct and negative, and have the form $s_1 = -\mu - \sqrt{\mu^2 - 1}$ and $s_2 = -\mu + \sqrt{\mu^2 - 1}$. So, the equilibrium state is asymptotically stable and then is said to be a stable node or a sink. The phase trajectories of the viscously over-damped system are predicted to be a family of parabola that converges to the equilibrium point as $\tau \to +\infty$.

Case with $\mu = 1$

The nonlinear dynamical system is then viscously critically damped. The eigenvalues are repeated negative real having the expression $s_1 = s_2 = -1$. So, the phase diagrams would consist of trajectories that approach asymptotically the equilibrium as $\tau \to +\infty$. Therefore the equilibrium position is asymptotically stable and is called stable degenerate node or sink [19].

Case with $\mu < 1$

The eigenvalues are two complex conjugate numbers of the form $s_1 = -\mu - i\sqrt{1-\mu^2}$ and $s_2 = -\mu + i\sqrt{1-\mu^2}$. The trajectories of the viscously under-damped system in the (x, y) phase plane are predicted to be a family of spirals that asymptotically converge to the equilibrium position (x^*, y^*) as $\tau \to +\infty$. So, the viscously under-damped system is asymptotically stable and equilibrium state is termed a stable focus or spiral sink. The above has shown then the following theorem.

2.2.4.3. Statement of theorem

Theorem 2.3.
Assume that μ is a positive constant. Then the unique equilibrium $\left(E_0^{1/l}, 0\right)$ to the system (13) is locally asymptotically stable and all solutions to system (13) converge to the equilibrium $\left(E_0^{1/l}, 0\right)$ as $\tau \to +\infty$.

In this situation, the local asymptotic stability of the initial nonlinear structural system has been proved under the condition that the damping factor μ has to be strictly positive. On the other hand, under some conditions, μ may be restricted to be zero [8]. This is allowed, since mechanical systems are in general weakly damped [14, 15]. For $\mu = 0$, the equilibrium point $\left(E_0^{1/l}, 0\right)$ becomes a non hyperbolic equilibrium, that is, the linearization approach predicts a centre, so that the preceding linear stability analysis according to Lyapunov does not hold exactly. In this case an alternative technique should be used. It nevertheless remains that the phase portraits of solution (12) for $\mu = 0$, may be represented, from which the global qualitative behavior may be drawn. The solution (12) clearly predicts for some values of l the existence of periodic solutions which should exhibit closed trajectories surrounding the center, that is the non hyperbolic equilibrium $\left(E_0^{1/l}, 0\right)$ in the (x, y) phase plane. In such a situation the viscously dissipative nonlinear dynamical system becomes, suddenly, at $\mu = 0$, a non-viscously conservative system leading to a great qualitative behavior change of phase diagrams, so that μ may be regarded as a bifurcation parameter. Let us, now, verify these results by comparing with numerical applications.

3. Numerical applications

Finding analytical results in structural dynamics is of course of great importance but, there is also a great interest to assure the reliability and accuracy of results and analytical approaches that have been performed. In such a situation, analytical results can be tested against numerical experiments to establish their validity. By doing so, the accuracy of exact analytical solutions is verified against results obtained by Runge-Kutta numerical integration using a computer code programmed in Matlab. On the other hand, the reliability of phase paths predicted by the linear stability analysis is verified by graphical plots of exact analytical and numerical solutions in the (x, y) phase plane using also a computer code programmed in Matlab. Given that the response of a dynamical system depends on its properties specified by model parameters, comparing analytical predictions with numerical results reduces to the problem of parameter estimation in mathematical model or differential equations. In other words, the problem to be solved consists of finding the most advantageous set of model parameters able to achieve the performance objective assigned to the nonlinear dynamical system under question. The problem to be answered is then the parameter optimization problem known as inverse problem. In this regard simulated data points for some time range employing Matlab's ODE solvers are generated from reasonable initial conditions and trial set of parameters. So, the model performance is compared with this data set considered as the observed data, using a nonlinear least squares algorithm implemented in Matlab. Therefore, the Matlab's Curve Fitting Toolbox which exploits the trust-region-reflective algorithm is used for improving the value of model parameters. In this context, the comparison of analytical and numerical results will

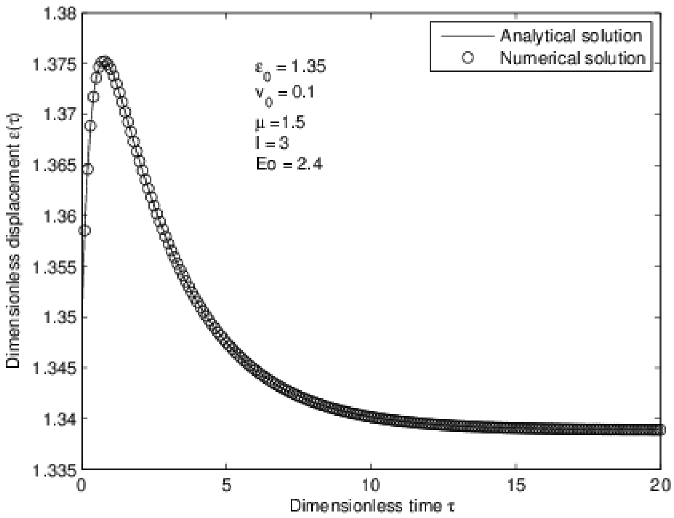

Fig. 1. Comparison between analytical (solid line) and numerical (circles) solutions for viscously over-damped nonlinear forced response

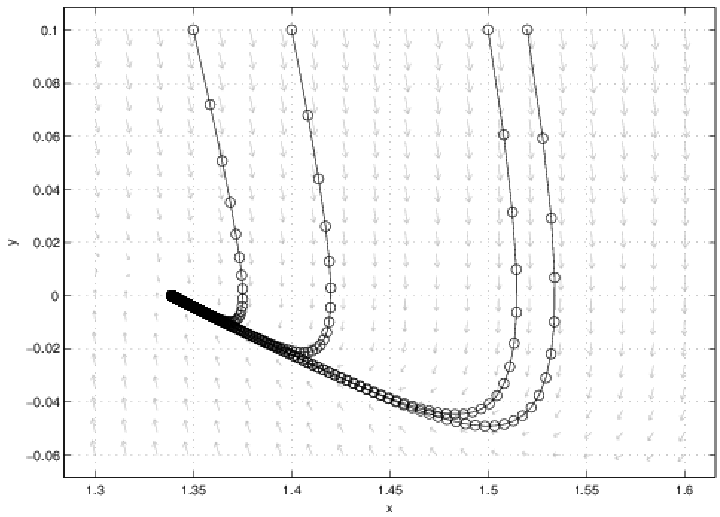

Fig. 2. Phase orbits of the analytical (solid line) and numerical (circles) solutions for viscously over-damped nonlinear forced response

be carried out under the basic estimation criterion such that $\mu > 1$ for the viscously over-damped nonlinear dynamical system (Paragraph 3.1), $\mu = 1$, for the viscously critically damped nonlinear dynamical system (Paragraph 3.2), $\mu < 1$, for the viscously under-damped nonlinear dynamical system (Paragraph 3.3) and $\mu = 0$, for the non-viscously damped nonlinear dynamical system (Paragraph 3.4). Finally, sensitivity analysis is performed to numerically verify the predicted effect of viscous damping (Paragraph 3.5) and stiffening (Paragraph 3.6) on the steady state deformation of the nonlinear dynamical system.

3.1. Comparison of theory with numerical experiment for viscously over-damped nonlinear forced response

In this section both time variation of deformation (Sub-paragraph 3.1.1) and phase curves (Sub-paragraph 3.1.2) are performed to verify the accuracy of the exact solution (6) against result obtained from direct numerical integration of equation (3) under the condition that the viscous damping factor $\mu > 1$.

3.1.1. Time variation of deformation

The exact analytical solution (6) is graphically compared with the numerical integration of equation (3) over the time range from $\tau = 0$ to $\tau = 20$, under arbitrary initial conditions $\varepsilon(\tau = 0) = \varepsilon_0 = 1.35$, and $\upsilon_0 = 0.1$. Fig. 1 shows the comparison result for reasonable system parameters such that $\mu = 1.5$, $l = 3$ and $E_0 = 2.4$. The solid line represents

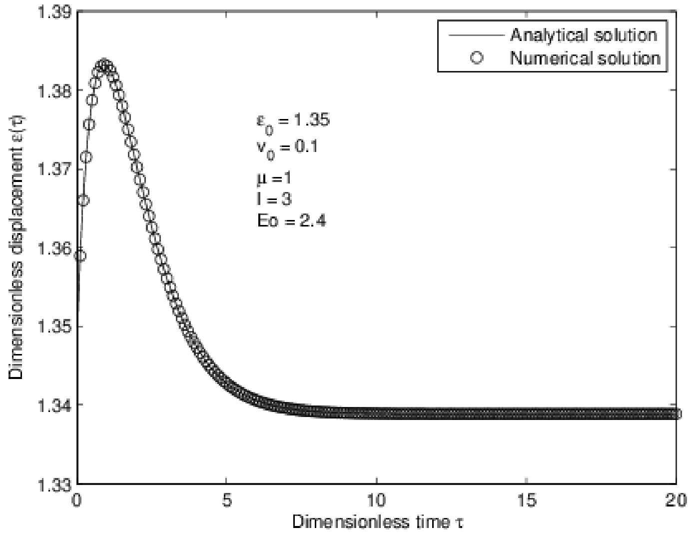

Fig. 3. Comparison between analytical (solid line) and numerical (circles) solutions for viscously critically damped nonlinear forced response

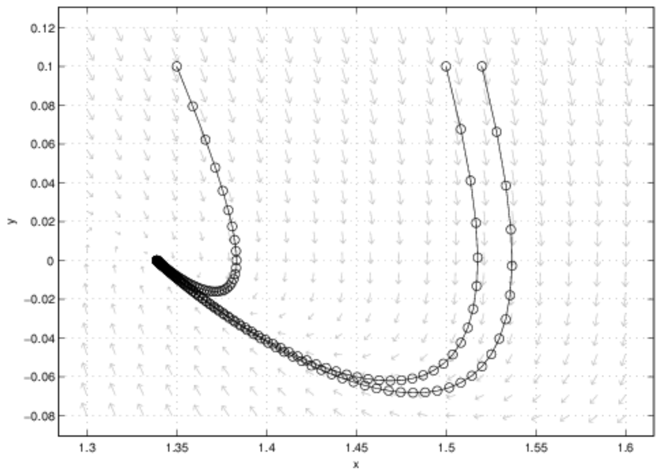

Fig. 4. Phase orbits of the analytical (solid line) and numerical (circles) solutions for viscously critically damped nonlinear forced response

the exact analytical solution and the circles express the numerical solution. It is found that the Matlab's mean squared error is $mse = 1.324e - 12$, showing a consistent agreement between theory and numerical experiment.

3.1.2. Phase portraits

The phase orbits of the exact solution (6) and numerical solution to equation (3) are shown in Fig. 2. These trajectories are simulated under the conditions that $\mu = 1.5$, $l = 3$, $E_0 = 2.4$ and $v_0 = 0.1$, for several initial values of the deformation $\varepsilon_0 = 1.35, 1.4, 1.5, 1.52$. Numerical solution phase diagrams are plotted in circles while exact analytical solutions phase paths are represented in solid line. The phase plane evaluation shows also a satisfactory agreement.

3.2. Comparison of analytical and numerical results for viscously critically damped nonlinear forced response

The structural model is numerically tested in this part to show its performance to simulate and predict the typical critically damped nonlinear forced response of mechanical systems subject to step excitation. The typical time variation of deformation (Sub-paragraph 3.2.1) and its phase portraits (Sub-paragraph 3.2.2) are graphically then performed.

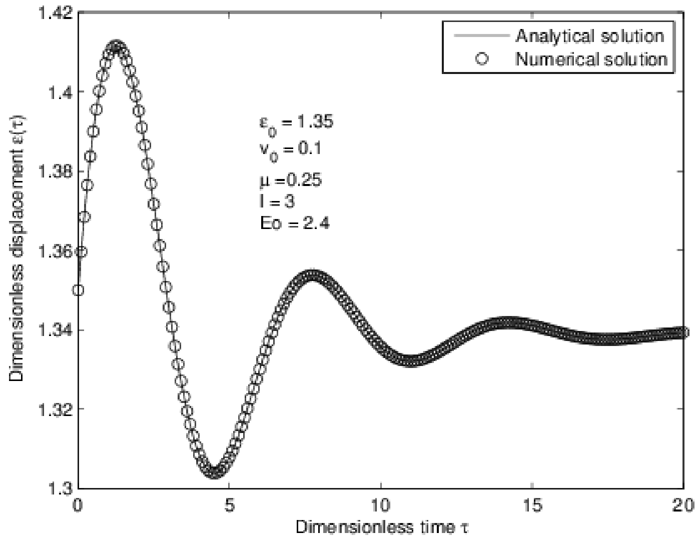

Fig. 5. Comparison between analytical (solid line) and numerical (circles) solutions for viscously under- damped nonlinear forced response

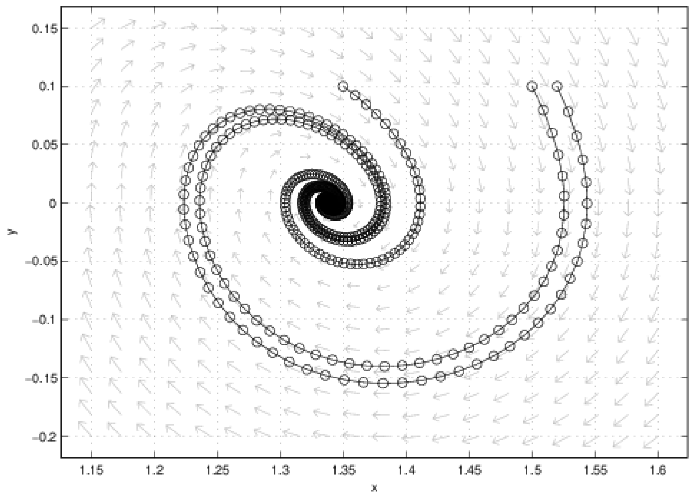

Fig. 6. Phase orbits of the analytical (solid line) and numerical (circles) solutions for viscously under-damped nonlinear forced response

3.2.1. Time variation of deformation

Fig. 3 shows the typical deformation response as a function of dimensionless time τ to step perturbation field of the structural model for the solution (8), which is compared with the result obtained by numerical integration of equation (3) under the conditions that $\mu = 1$, $l = 3$ and $E_0 = 2.4$, over a time interval $\tau = 0$ to $\tau = 20$. Initial conditions are found to be $\varepsilon_0 = 1.35$, and $v_0 = 0.1$. The solid line represents the exact analytical solution (8) and circle line the result of numerical integration of equation (3). The simulation has revealed a good agreement between exact analytical theory and numerical result, as indicated by the Matlab's mean squared error $mse = 5.1044e - 12$.

3.2.2. Phase portraits

The phase portraits obtained from the solution (8) and equation (3) in the case $\mu = 1$, are illustrated in Fig. 4. The phase trajectories are simulated with the values $l = 3$, $E_0 = 2.4$, and $v_0 = 0.1$, for several initial conditions $\varepsilon_0 = 1.35, 1.5, 1.52$.

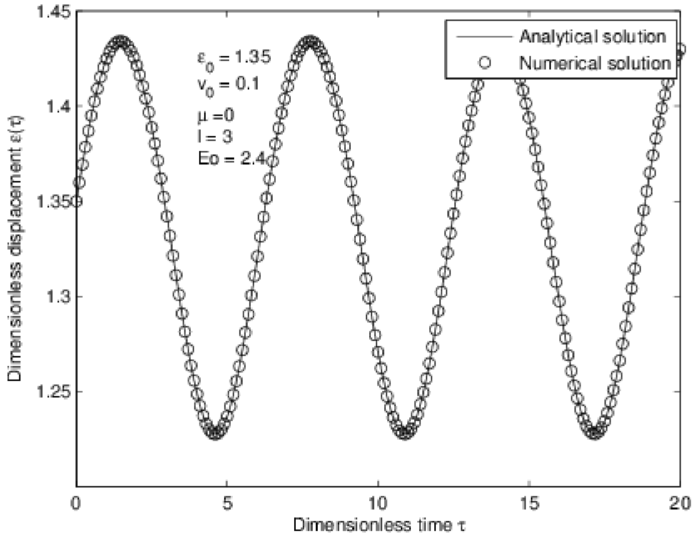

Fig. 7. Comparison between analytical (solid line) and numerical (circles) solutions for non-viscously damped nonlinear forced response

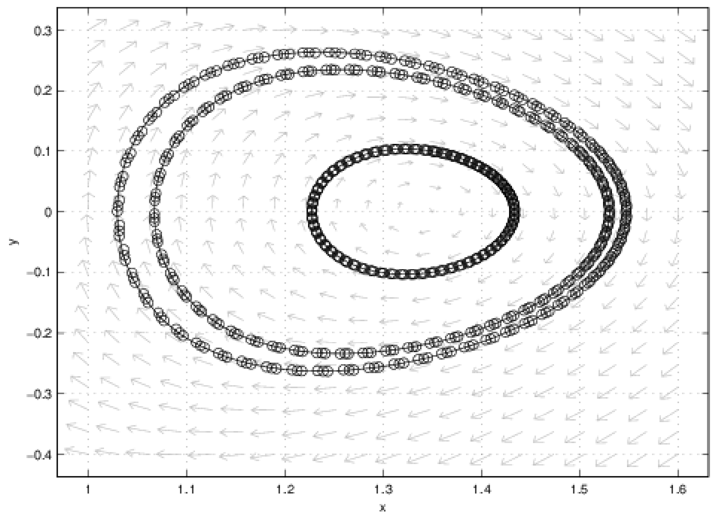

Fig. 8. Phase orbits of the analytical (solid line) and numerical (circles) solutions for non-viscously damped nonlinear forced response

3.3. Comparison of analytical and numerical results for viscously under-damped nonlinear forced response

The model performance is tested, here, against numerical experiment, under the condition that $\mu < 1$. The time history of deformation and its phase portraits are then graphically illustrated in the $(\tau, \varepsilon(\tau))$ plane and (x, y) phase plane respectively.

3.3.1. Time variation of deformation

The typical behavior of deformation $\varepsilon(\tau)$ as a function of a dimensionless time τ is plotted in Fig. 5. The exact analytical solution (10) is plotted in solid curve while the numerical result of equation (3) is represented in circle line for a set of parameters $\mu = 0.25$, $l = 3$ and $E_0 = 2.4$, over a time range from $\tau = 0$ to $\tau = 20$. Initial conditions are then $\varepsilon_0 = 1.35$, and $v_0 = 0.1$. As indicated by the Matlab's mean squared error $mse = 1.9856e-10$, the graphical evaluation of results shows a satisfactory agreement between exact analytical theory and numerical experiment.

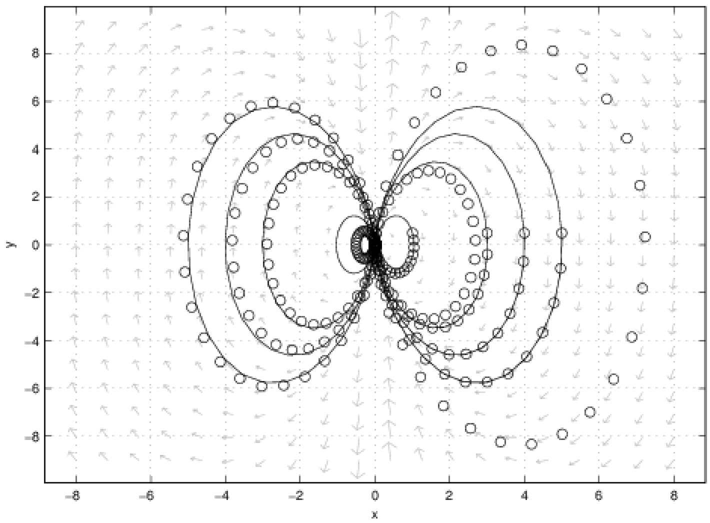

Fig. 9. Phase orbits of the analytical (solid line) and numerical (circles) solutions for non-viscously damped nonlinear free vibration response

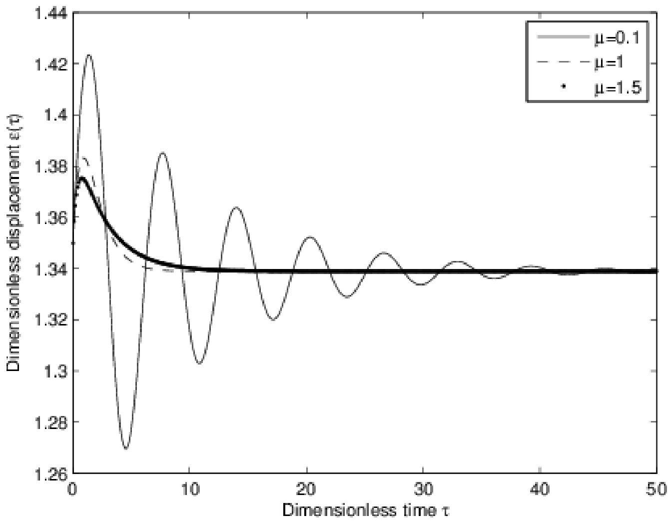

Fig. 10. Effect of viscous damping on the steady state response

3.3.2. Phase portraits

The graphical comparison of phase portraits of exact analytical solution (10) and equation (3) is shown in Fig. 6. The simulations in the phase plane are performed under the conditions $\mu = 0.25$, $l = 3$ and $E_0 = 2.4$. The phase orbits are drawn for several initial deformations $\varepsilon_0 = 1.35, 1.5, 1.52$ and for initial dimensionless velocity $v_0 = 0.1$. The solid curves represent the phase paths of the exact solution (10) and the circle lines designate the phase trajectories of the numerical solution to (3).

3.4. Comparison of analytical and numerical results for non-viscously damped nonlinear forced response

The analytical model (12) is tested, here, against numerical experiment by setting the limiting value $\mu = 0$, into equation (3). So, the other parameter values used in the previous section of under-damped nonlinear forced response are yet valid in this section. The time variation of deformation is shown in Fig. 7. Periodic oscillations are then observed. The Matlab's mean squared error $mse = 1.2331e-08$, reveals a consistent agreement between analytical and numerical results.

The phase portraits of this special case $\mu = 0$, are illustrated in Fig. 8. The simulation shows closed orbits surrounding the equilibrium position, as predicted by the analytical approach. The simulations are run under the same conditions as in the under-damped nonlinear forced response, except $\mu = 0$.

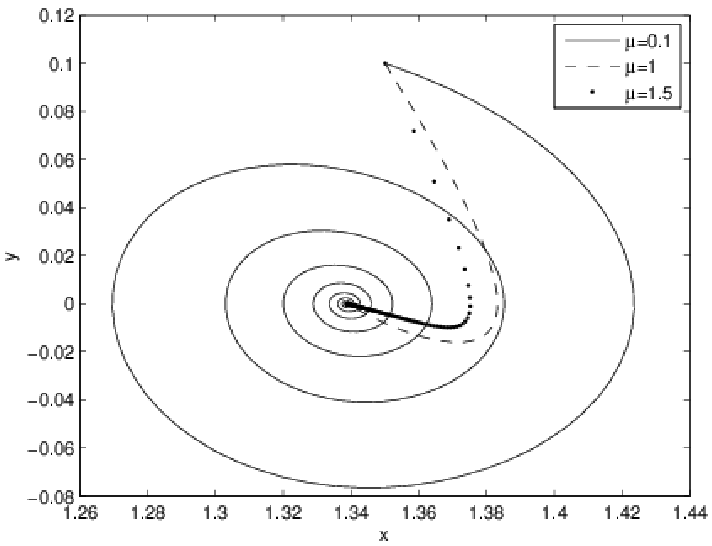

Fig. 11. Effect of viscous damping on the steady state response

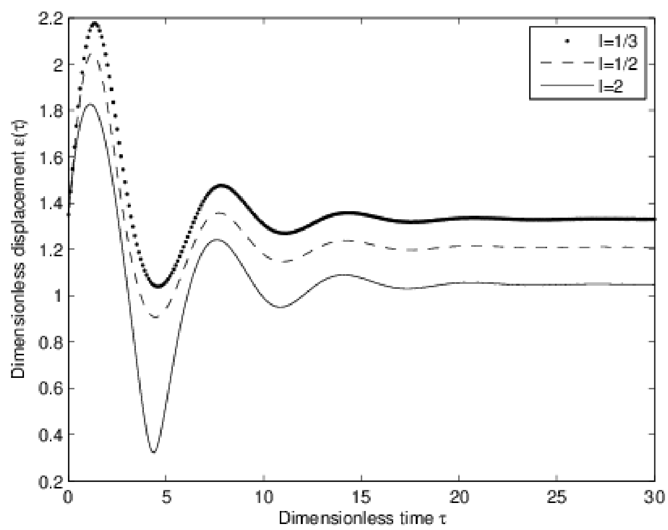

Fig. 12. Effect of hardening nonlinearity on the steady state deformation

Fig. 9 shows the phase portraits of the special case $\mu = 0$, and $E_0 = 0$, characterizing the non-viscously damped nonlinear response of the structural system under free vibration experiment for several values of initial deformation $\varepsilon_0 = 1, 3, 4, 5$. The parameter $l = \dfrac{1}{3}$ and initial velocity is $\upsilon_0 = 0.5$.

So with that, to perform a complete numerical evaluation of the structural model performance, it is suitable to assess numerically the effects of viscous damping (Paragraph 3.5) and stiffening control parameter (Paragraph 3.6) on the steady state deformation response.

3.5. Effect of viscous damping on the steady state response

The effect of viscous damping on the steady state deformation is analyzed by varying μ whereas other parameters are kept constant. On Fig. 10 the time variations of deformation for viscously over-damped ($\mu = 1.5$), critically damped ($\mu = 1$) and under-damped ($\mu = 0.1$) regimes of the nonlinear dynamical system are superposed under the conditions that $E_0 = 2.4$, $l = 3$, $\varepsilon_0 = 1.35$, and $\upsilon_0 = 0.1$, to show the effect of viscous damping. It is clearly observed that the stationary deformation does not depend on the viscous damping properties as also indicated by the phase

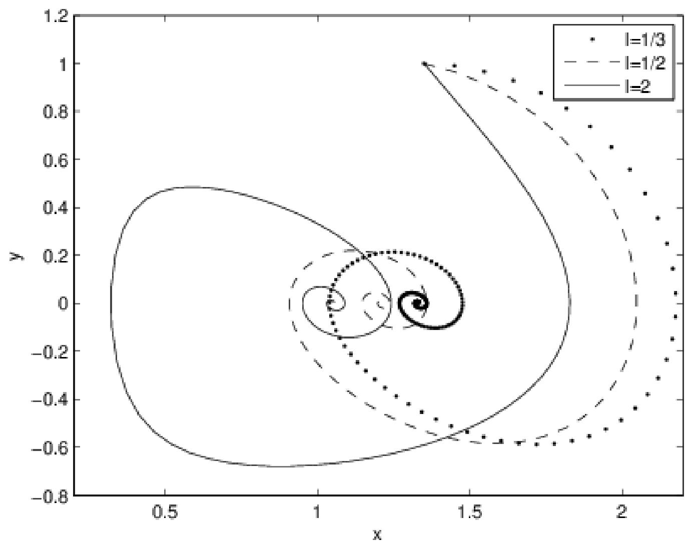

Fig. 13. Effect of hardening nonlinearity on the steady state deformation simulated in the phase plane

portraits (Fig. 11). However as μ increases, the first peak amplitude decreases. Let us, now, investigate the effect of stiffening nonlinearity on the equilibrium deformation.

3.6. Effect of stiffening on the steady state deformation

This effect is investigated by varying the coefficient l while other parameters are held constant. So, the oscillatory deformation of the system for three different values of $l = 1/3, 1/2, 2$, are shown in Fig. 12. The corresponding phase diagrams are simulated on Fig. 13 under the initial conditions that $\varepsilon_0 = 1.35$, and $v_0 = 1$. Other parameters are chosen to be $\mu = 0.25$ and $E_0 = 1.1$.

Fig. 12 and Fig. 13 clearly show the strong dependence of the steady state deformation on l. It is found that as l increases, the stationary deformation value decreases, as the first peak amplitude, meaning that the viscoelastic nonlinear dynamical system becomes more stiffened, under the condition that $E_0 > 1$. For $E_0 = 1$, it will be easy to note that l has no effect on the equilibrium deformation. So, it may be said that the stiffening control coefficient as well as the excitation affect strongly the steady state response of the dynamical system. In this regard the excitation may cause stiffening or softening effect within the system following its magnitude for a given l. Conversely, for a given excitation E_0, the value l may also be adjusted for the stiffening control [3, 4]. In other words, both stiffening and excitation effects are closely interrelated, as reported by several research works. So with that, it now becomes possible to perform a reasonable analysis and discussion on the accuracy and reliability of the predictive performance of the viscoelastic structural model under study.

4. Discussion

It is usually question in structural design of prevention and control of vibration in some engineering structures subject to dynamical perturbation field. Under dynamical excitation, the most important response of mechanical systems is the steady state motion. So, vibration analysis or control reduces essentially to the system stationary response control. One of the most important analytical tools to assure this objective is the representation of system response in terms of differential equation which relates mathematically the system state variables to external applied excitation field. The dynamical response of the system depends then on the governing differential equation coefficients. Accordingly, the dynamical behavior of the system response may then be predicted or specified concurrently from excitation magnitude or values assigned to model parameters viewed as system inputs. In this regard, it has been observed that the system external applied excitation may induce stiffening effect which is responsible of changes in system properties such as equilibrium position, resonance frequency, etc. This effect could not be explained by the well established viscously damped linear harmonic oscillator which predicts a linear relationship between excitation amplitude and equilibrium deformation under forced step excitation experiment. In non-dimensional form, the viscous damped linear harmonic oscillator equation shows that steady state value and excitation magnitude are equal. In other words, the viscous damped linear harmonic oscillator states that decrease of equilibrium deformation requires decrease of excitation magnitude, and vice versa, increase of equilibrium deformation requires increase of excitation magnitude. In this sense, the nonlinear theory is required for the control of steady state response of mechanical systems characterized essentially by stiffening or softening, to say a nonlinear viscoelastic response. In that, there is no yet universal structural model valid for each material system in all engineering and science situations. In this regard the present work aims to show that the structural model (3) has the ability to give a means for the control of the steady state response of mechanical systems in the context of closely interrelated stiffness and damping nonlinearity properties, that is to say, in the context of nonlinear viscoelastic behavior. More precisely, the objective was to show that change in the hardening nonlinearity parameter l, changes the system equilibrium deformation. To that end, exact analytical solutions to equation (3) depending on the viscous damping value are established. The reliability and accuracy of these results are verified against numerical evaluations. It is found that there exists a satisfactory agreement, as shown by the computed mean squared errors, between exact analytical and numerical results. The local stability behavior of these solutions is also demonstrated according to the linear stability analysis. The research work shows that the steady state response does not depend on the viscous damping factor μ, so that the equilibrium deformation has the same value for the over, critically and under-damped forced responses of the viscoelastic nonlinear dynamical system to step excitation field. It is noted, as a major finding of this work, that the current structural model can predict the stiffening effect on the equilibrium deformation of the system. In other words, it shows that a change in the hardening exponent l produces a change in the equilibrium deformation. So, it is found that the steady state deformation is a power law of excitation with the strain hardening nonlinearity parameter l as exponent. It is also found that the greater the l, the lesser the equilibrium deformation produced by a given excitation, meaning that the material system becomes more stiff. Conversely, the lesser the l, the greater the equilibrium deformation produced by a given excitation, meaning that the system becomes more soft. This result is of great importance, since it means that the steady state deformation of the viscoelastic nonlinear dynamical system to step excitation field can be controlled by means of the hardening nonlinearity exponent. In this perspective, making $l = 1$, into the power law describing the variation of the equilibrium deformation of the proposed structural system, gives the expression of equilibrium deformation of the viscously damped linear harmonic oscillator to step loading. By doing so, the current structural model incorporates the viscously damped linear harmonic oscillator as a special case. In this context, it may be stated with reason that, the model predictive performance under evaluation is verified, to say, the expected result has been attained.

5. Conclusions

It has been noted that the viscously damped linear harmonic equation could not be used in the control of equilibrium deformation to step excitation field of real mechanical systems characterized essentially by softening or hardening nonlinearity. Also, real mechanical systems exhibit viscoelastic response to say, closely interrelated stiffness and damping nonlinearity properties. In this context, a structural model developed within the framework of the Bauer's nonlinear dynamic theory of viscoelasticity is analyzed under forced vibration experiment. More precisely, the nonlinear forced response of the structural model to step excitation field is considered. Such a testing of mechanical systems is of great interest, since in many practical situations structural components undergo creep deformation due to their viscoelastic properties, which may induce failure of engineering as well as living structures. It is demonstrated that the current model has the ability to simulate and predict, depending on the viscous damping value, aperiodic, critic aperiodic, oscillatory and periodic deformation regimes in nonlinear mechanical systems under step loading. It has been observed a satisfactory agreement between exact analytical and numerical results. As a major finding, the work has shown that the steady state response of the structural model to step excitation does not depend on the viscous damping factor, but that a change in the hardening nonlinearity exponent produces a change in the equilibrium deformation without altering the local asymptotic stability behavior of the system. The steady state deformation varies as a power law of excitation with the hardening nonlinearity parameter as exponent, so that the prediction of the viscously damped linear harmonic oscillator has become a special case of the present structural model. This finding is of primary importance as it means that the strain hardening nonlinearity exponent may be used for the control of steady state deformation, in particular, and vibration in general, in viscoelastic mechanical systems subject to step loading. It is found, in addition, that the structural system natural frequency depends on the hardening exponent, enabling then its control from the latter. The present work suggests reasonably to examine, as future work, the predictive performance of the current structural model in forced vibration experiment under harmonic excitation field, and in particular, the strain hardening nonlinearity exponent effect, since this parameter can give the ability to control some nonlinear system characteristic phenomena such as frequency dependent amplitude, jump and bifurcation.

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