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Global approximation theorems for general Gamma type Operators

Research Article

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Dedicated to Prof. P. N. Agrawal

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Abstract: In the present paper we obtained some direct approximation theorems for general Gamma type operators in polynomial weighted spaces.

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1. Introduction

For a measurable complex valued and locally bounded function defined on $[0,\infty)$, Lupas and M$\ddot{u}$ller [1] defined and studied some approximation properties of Gamma operators $\{G_n\}$ defined by

$$G_n(f;x)=\int_0^\infty g_n(x,u)f\Big(\frac{n}{u}\Big)du,$$

where

$$g_n(x,u)=\frac{x^{n+1}}{n!}e^{-xu}u^n,\quad x>0.$$

In [2], Mazhar gives an important modifications of the Gamma operators using the same $g_n(x,u)$

$$\begin{aligned}F_n(f;x)&=\int_0^\infty\int_0^\infty g_n(x,u)g_{n-1}(u,t)f(t)du\,dt\\&=\frac{(2n)!x^{n+1}}{n!(n-1)!}\int_0^\infty\frac{t^{n-1}}{(x+t)^{2n+1}}f(t)dt,\ n>1,\ x>0.\end{aligned}$$

Recently, by using the techniques due to Mazhar, İzgi and B$\ddot{u}$y$\ddot{u}$kyazici [3], Karsli [4] independently considered the following Gamma type linear and positive operators

$$\begin{aligned}L_n(f;x)&=\int_0^\infty\int_0^\infty g_{n+2}(x,u)g_n(u,t)f(t)du\,dt\\&=\frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty\frac{t^n}{(x+t)^{2n+4}}f(t)dt,\ x>0,\end{aligned}$$

and obtained some approximation results.

In [5], Karsli and Özarslan obtained some local and global approximation results for the operators $L_n(f;x)$. Global

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approximation results for different operators were examined in many papers, for example in [6], [7] and [8].
In 2007, Mao [9] define the following generalised Gamma type linear and positive operators

$$M_{n,k}(f;x) = \int_0^\infty \int_0^\infty g_n(x,u) g_{n-k}(u,t) f(t) du dt$$
$$= \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) dt, \ x > 0,$$

which includes the operators $F_n(f;x)$ for $k = 1$ and $L_{n-2}(f;x)$ for $k = 2$.
Some approximation properties of $M_{n,k}$ were studied in [10], [11] and [12].
We can rewrite the operators $M_{n,k}(f;x)$ as

$$M_{n,k}(f;x) = \int_0^\infty K_{n,k}(x,t) f(t) dt, \tag{1}$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x, t \in (0,\infty).$$

In this paper, we study some global approximation results of the operators $M_{n,k}$. Let $p \in N_0$(set of non-negative integers), $f \in C_p$, where C_p is a polynomial weighted space with the weight function w_p,

$$w_0(x) = 1, \ w_p(x) = \frac{1}{1+x^p}, \ p > 0, \tag{2}$$

and C_p is the set of all real valued functions f for which $w_p f$ is uniformly continuous and bounded on $[0,\infty)$.
The norm on C_p is defined by the formula

$$\|f\|_p = \sup_{x \in [0,\infty)} w_p(x)|f(x)|.$$

We also consider the following Lipschitz classes:

$$\omega_p^2(f;\delta) = \sup_{h \in (0,\delta]} \|\Delta_h^2 f\|_p,$$

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$

$$\omega_p^1(f;\delta) = \sup_{h \in (0,\delta]} \|\Delta_h f\|_p,$$

$$\Delta_h f(x) = f(x+h) - f(x),$$

$$Lip_p^2 \alpha = \{f \in C_p[0,\infty) : \omega_p^2(f;\delta) = O(\delta^\alpha) \, as \delta \to 0^+\},$$

where $h > 0$ and $\alpha \in (0,2]$.
From the above it follows that

$$\lim_{\delta \to 0^+} \omega_p^1(f;\delta) = 0, \ \lim_{\delta \to 0^+} \omega_p^2(f;\delta) = 0,$$

for every $f \in C_p[0,\infty)$.

2. Auxiliary results

In this section we give some preliminary results which will be used in the main part of this paper.
Let us consider

$$e_m(t) = t^m, \ \varphi_{x,m}(t) = (t-x)^m, \ m \in N_0, \ x, t \in (0,\infty).$$

Lemma 2.1 ([11]).
For any $m \in N_0$ (set of non-negative integers), $m \leq n-k$

$$M_{n,k}(t^m;x) = \frac{[n-k+m]_m}{[n]_m} x^m \tag{3}$$

where $n, k \in N$ and $[x]_m = x(x-1)...(x-m+1), [x]_0 = 1, x \in R$.
In particular for m = 0, 1, 2... in (3) we get

(i) $M_{n,k}(1;x) = 1$,

(ii) $M_{n,k}(t;x) = \dfrac{n-k+1}{n}x$,

(iii) $M_{n,k}(t^2;x) = \dfrac{(n-k+2)(n-k+1)}{n(n-1)}x^2$.

Lemma 2.2 ([11]).
Let $m \in N_0$ and fixed $x \in (0,\infty)$, then

$$M_{n,k}(\varphi_{x,m};x) = \left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{(n-m+j)!(n-k+m-j)!}{n!(n-k)!}\right)x^m.$$

Lemma 2.3.
For $m = 0,1,2,3,4$, one has

(i) $M_{n,k}(\varphi_{x,0};x) = 1$,

(ii) $M_{n,k}(\varphi_{x,1};x) = \dfrac{1-k}{n}x$,

(iii) $M_{n,k}(\varphi_{x,2};x) = \dfrac{k^2-5k+2n+4}{n(n-1)}x^2$,

(iv) $M_{n,k}(\varphi_{x,3};x) = \dfrac{-k^3+12k^2-17k+n(18-12k)+24}{n(n-1)(n-2)}x^3$,

(v) $M_{n,k}(\varphi_{x,4};x) = \dfrac{k^4-22k^3+k^2(143+12n)-k(314+108n)+12n^2+268n+192}{n(n-1)(n-2)(n-3)}x^4$,

(vi) $M_{n,k}(\varphi_{x,m};x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof. Using Lemma 2.2, we get Lemma 2.3. □

Theorem 2.1.
For the operators $M_{n,k}$ and for fixed $p \in N_0$, there exists a positive constant $N_{p,k}$ depending only on the parameters p and k such that

$$w_p(x)M_{n,k}\left(\frac{1}{w_p(t)};x\right) \le N_{p,k}. \tag{4}$$

Moreover for every $f \in C_p[0,\infty)$, we have

$$\|M_{n,k}(f;.)\|_p \le N_{p,k}\|f\|_p, \tag{5}$$

which shows that $M_{n,k}$ is a linear positive operator from the space $C_p[0,\infty)$ into $C_p[0,\infty)$.

Proof. For $p = 0$, (4) follows immediately. Using Lemma 2.1, we get

$$\begin{aligned} w_p(x)M_{n,k}\left(\frac{1}{w_p(t)};x\right) &= w_p(x)\left(M_{n,k}(e_0;x)+M_{n,k}(e_p;x)\right)\\ &= w_p(x)\left(1+\frac{(n-p)!(n-k+p)!}{n!(n-k)!}x^p\right)\\ &\le N_{p,k}w_p(x)(1+x^p) = N_{p,k}, \end{aligned}$$

where

$$N_{p,k} = \max\left\{\sup_n \frac{(n-p)!(n-k+p)!}{n!(n-k)!},1\right\}.$$

Observe that for all $f \in C_p$ and every $x \in (0,\infty)$, we get

$$\begin{aligned} w_p(x)\left|M_{n,k}(f;x)\right| &\le w_p(x)\frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}|f(t)|\frac{w_p(t)}{w_p(t)}dt\\ &\le \|f\|_p w_p(x)M_{n,k}\left(\frac{1}{w_p(t)};x\right)\\ &\le N_{p,k}\|f\|_p. \end{aligned}$$

Taking supremum over $x \in (0,\infty)$, we get (5). □

Lemma 2.4.
For the operators $M_{n,k}$ and fixed $p \in N_0$, there exists a positive constant $N_{p,k}$ depending only on the parameters p and k such that

$$w_p(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right) \leq N_{p,k}\frac{x^2}{n}.$$

Proof. Using Lemma 2.3, we can write

$$\begin{aligned} w_0(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_0(t)};x\right) &= \frac{k^2-5k+2n+4}{n(n-1)}x^2 \\ &\leq N_k\frac{x^2}{n}, \end{aligned}$$

which gives the result for $p=0$.
Let $p \geq 1$. Then using Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned} M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right) &= M_{n,k}(e_{p+2};x) - 2xM_{n,k}(e_{p+1};x) + x^2M_{n,k}(e_p;x) + M_{n,k}(\varphi_{x,2};x) \\ &= \frac{(n-p-2)!(n-k+p+2)!}{n!(n-k)!}x^{p+2} - 2\frac{(n-p-1)!(n-k+p+1)!}{n!(n-k)!}x^{p+2} \\ &+ \frac{(n-p)!(n-k+p)!}{n!(n-k)!}x^{p+2} + \frac{k^2-5k+2n+4}{n(n-1)}x^2 \\ &\leq N_{p,k}\frac{x^2}{n}(1+x^p), \end{aligned}$$

where $N_{p,k} = \left(1+\left(k^2-5k+6p+4p^2-4kp\right)\frac{(n-p-2)!(n-k+p)!}{(n-2)!(k^2-5k+2n+4)}x^p\right)\frac{k^2-5k+2n+4}{n-1}$ is a positive constant. This completes the proof. □

3. Direct results

The proof of direct theorems will follow from Jackson type inequality, the Steklov means and appropriate estimates of the moments of the operators.
Let $p \in N_0$. By $C_p^2[0,\infty)$, we denote the space of all functions $f \in C_p[0,\infty)$ such that $f', f'' \in C_p[0,\infty)$.

Theorem 3.1.
Let $p \in N_0$ and $f \in C_p^1[0,\infty)$, there exists a positive constant $N_{p,k}$ depending only on the parameters p and k such that

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq N_{p,k}\|f'\|_p\frac{x}{\sqrt{n}}$$

for all $x \in (0,\infty)$ and $n \in N$.

Proof. Let $x \in (0,\infty)$ be fixed. Then for $f \in C_p^1[0,\infty)$ and $t \in (0,\infty)$, we have

$$f(t) - f(x) = \int_x^t f'(v)\,dv.$$

By using linearity of $M_{n,k}$ we get

$$M_{n,k}(f;x) - f(x) = M_{n,k}\left(\int_x^t f'(v)\,dv;x\right). \tag{6}$$

Remark that

$$\left|\int_x^t f'(v)\,dv\right| \leq \|f'\|_p\left|\int_x^t \frac{dv}{w_p(v)}\right| \leq \|f'\|_p|t-x|\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right).$$

From (6) we obtain

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq \|f'\|_p\left\{M_{n,k}(|\varphi_{x,1}|;x) + w_p(x)M_{n,k}\left(\frac{|\varphi_{x,1}|}{w_p(t)};x\right)\right\}.$$

Using Cauchy-Schwarz inequality, we can write

$$M_{n,k}(|\varphi_{x,1}|;x) \le \left(M_{n,k}(\varphi_{x,2};x)\right)^{1/2} \times \left(M_{n,k}(\varphi_{x,0};x)\right)^{1/2},$$

$$M_{n,k}\left(\frac{|\varphi_{x,1}|}{w_p(t)};x\right) \le \left(M_{n,k}\left(\frac{1}{w_p(t)};x\right)\right)^{1/2} \times \left(M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right)\right)^{1/2}.$$

Using Lemma 2.3, Theorem 2.1 and Lemma 2.4, we obtain

$$w_p(x)|M_{n,k}(f;x) - f(x)| \le N_{p,k}\|f'\|_p \frac{x}{\sqrt{n}}.$$

□

Lemma 3.1.
Let $p \in N_0$. If

$$T_{n,k}(f;x) = M_{n,k}(f;x) - f\left(x + \frac{1-k}{n}x\right) + f(x), \tag{7}$$

then there exists a positive constant $N_{p,k}$ such that for all $x \in (0,\infty)$ and $n \in N$, we have

$$w_p(x)|T_{n,k}(g;x) - g(x)| \le N_{p,k}\|g''\|_p \frac{x^2}{n}$$

for any function $g \in C_p^2$.

Proof. From Lemma 2.1, we observe that the operators $T_{n,k}$ are linear and reproduce the linear functions. Hence

$$T_{n,k}(\varphi_{x,1};x) = 0.$$

Let $g \in C_p^2$. By the Taylor formula one can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-v)g''(v)\,dv, \ \ t \in (0,\infty).$$

Then,

$$\begin{aligned}|T_{n,k}(g;x) - g(x)| &= |T_{n,k}(g - g(x));x| = \left|T_{n,k}\left(\int_x^t (t-v)g''(v)\,dv;x\right)\right| \\ &= \left|M_{n,k}\left(\int_x^t (t-v)g''(v)\,dv;x\right) - \int_x^{x+\frac{1-k}{n}x}\left(x + \frac{1-k}{n}x - v\right)g''(v)\,dv\right|.\end{aligned}$$

Since

$$\left|\int_x^t (t-v)g''(v)\,dv\right| \le \frac{\|g''\|_p (t-x)^2}{2}\left(\frac{1}{w_p(x)} + \frac{1}{w_p(t)}\right)$$

and

$$\left|\int_x^{x+\frac{1-k}{n}x}\left(x + \frac{1-k}{n}x - v\right)g''(v)\,dv\right| \le \frac{\|g''\|_p}{2w_p(x)}\left(\frac{1-k}{n}x\right)^2,$$

we get

$$w_p(x)|T_{n,k}(g;x) - g(x)| \le \frac{\|g''\|_p}{2}\left(M_{n,k}(\varphi_{x,2};x) + w_p(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right)\right) + \frac{\|g''\|_p}{2}\left(\frac{1-k}{n}x\right)^2.$$

Hence by Lemma 2.4, we obtain

$$w_p(x)|T_{n,k}(g;x) - g(x)| \le N_{p,k}\|g''\|_p \frac{x^2}{n}$$

for any function $g \in C_p^2$. The Lemma is proved. □

Theorem 3.2.
Let $p \in N_0$, $n \in N$ and $f \in C_p[0,\infty)$, there exists a positive constant $N_{p,k}$ depending only on the parameters p and k such that

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le N_{p,k}\omega_p^2\left(f,\frac{x}{\sqrt{n}}\right)+\omega_p^1\left(f,\frac{1-k}{n}x\right).$$

Furthermore, if $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$, then

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le N_{p,k}\left(\frac{x^2}{n}\right)^{\alpha/2}+\omega_p^1\left(f,\frac{1-k}{n}x\right),$$

holds.

Proof. Let $p \in N_0$, $f \in C_p[0,\infty)$ and $x \in (0,\infty)$ be fixed. We consider the Steklov means of f by f_h and given by the formula

$$f_h(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(x+s+t)-f(x+2(s+t))\}dsdt,$$

for $h,x \in (0,\infty)$. We have

$$f(x)-f_h(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(x)dsdt,$$

which gives

$$\|f-f_h\|_p \le \omega_p^2(f,h). \tag{8}$$

Furthermore, we have

$$f_h^{''}(x)=\frac{1}{h^2}\left(8\Delta_{h/2}^2 f(x)-\Delta_h^2 f(x)\right),$$

and

$$\|f_h^{''}\|_p \le \frac{9}{h^2}\omega_p^2(f,h). \tag{9}$$

From (8) and (9) we conclude that $f_h \in C_p^2[0,\infty)$ if $f \in C_p[0,\infty)$.
Moreover

$$|M_{n,k}(f;x)-f(x)| \le T_{n,k}(|f(t)-f_h(t);x|)+|f(x)-f_h(x)|+|T_{n,k}(f_h;x)-f_h(x)|+\left|f\left(x+\frac{1-k}{n}x\right)-f(x)\right|,$$

where $T_{n,k}$ is defined in (7).
Since $f_h \in C_p^2[0,\infty)$ by the above, it follows from Theorem 2.1 and Lemma 3.1 that

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le (\mathcal{N}+1)\|f-f_h\|_p+N_{p,k}\|f_h^{''}\|_p\frac{x^2}{n}w_p(x)\left|f\left(x+\frac{1-k}{n}x\right)-f(x)\right|.$$

By (8) and (9), the last inequality yields that

$$w_p(x)\left|M_{n,k}(f;x)-f(x)\right| \le N_{p,k}\omega_p^2(f;h)\left(1+\frac{1}{h^2}\frac{x^2}{n}\right)+\omega_p^1\left(f,\frac{1-k}{n}x\right).$$

Thus, choosing $h=\dfrac{x}{\sqrt{n}}$, the first part of the proof is completed.
The proof of second part can be easily obtained from the definition of the space $Lip_p^2\alpha$. □

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