Global approximation theorems for general Gamma type Operators

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Dedicated to Prof. P. N. Agrawal

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Abstract: In the present paper we obtained some direct approximation theorems for general Gamma type operators in polynomial weighted spaces.

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1. Introduction

For a measurable complex valued and locally bounded function defined on $[0, \infty)$, Lupas and Müller [1] defined and studied some approximation properties of Gamma operators $\{G_n\}$ defined by

$$G_n(f; x) = \int_0^\infty g_n(x, u) f\left(\frac{n}{u}\right) du,$$

where

$$g_n(x, u) = \frac{x^{n+1}}{n!} e^{-x u} u^n, \quad x > 0.$$

In [2], Mazhar gives an important modifications of the Gamma operators using the same $g_n(x, u)$

$$F_n(f; x) = \int_0^\infty \int_0^\infty g_n(x, u) g_{n-1}(u, t) f(t) dudt$$

$$= \frac{(2n)! x^{n+1}}{n!(n-1)!} \int_0^\infty t^{n-1} \left(\frac{x + t}{2n+1}\right)^n f(t) dt, \quad n > 1, \quad x > 0.$$

Recently, by using the techniques due to Mazhar, İzgi and Büyükyazıcı [3], Karsli [4] independently considered the following Gamma type linear and positive operators

$$L_n(f; x) = \int_0^\infty \int_0^\infty g_{n+2}(x, u) g_n(u, t) f(t) dudt$$

$$= \frac{(2n+3)! x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x + t)^{2n+4}} f(t) dt, \quad x > 0,$$

and obtained some approximation results.

In [5], Karsli and Özarslan obtained some local and global approximation results for the operators $L_n(f; x)$. Global

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approximation results for different operators were examined in many papers, for example in [6], [7] and [8]. In 2007, Mao [9] define the following generalised Gamma type linear and positive operators

\[ M_{n,k}(f; x) = \int_0^\infty \int_0^\infty g_n(x, u) g_{n-k}(u, t) f(t) du \, dt \]

\[ = \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) \, dt, \quad x > 0, \]

which includes the operators \( F_n(f; x) \) for \( k = 1 \) and \( L_{n-2}(f; x) \) for \( k = 2 \).

Some approximation properties of \( M_{n,k} \) were studied in [10], [11] and [12]. We can rewrite the operators \( M_{n,k}(f; x) \) as

\[ M_{n,k}(f; x) = \int_0^\infty K_{n,k}(x, t) f(t) \, dt, \tag{1} \]

where

\[ K_{n,k}(x, t) = \frac{(2n-k+1)! x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x, t \in (0, \infty). \]

In this paper, we study some global approximation results of the operators \( M_{n,k} \). Let \( p \in \mathbb{N}_0(\text{set of non-negative integers}), f \in C_p \), where \( C_p \) is a polynomial weighted space with the weight function \( w_p \)

\[ w_0(x) = 1, \quad w_p(x) = \frac{1}{1 + x^p}, \quad p > 0, \tag{2} \]

and \( C_p \) is the set of all real valued functions \( f \) for which \( w_p f \) is uniformly continuous and bounded on \([0, \infty)\).

The norm on \( C_p \) is defined by the formula

\[ \| f \|_p = \sup_{x \in (0, \infty)} w_p(x) |f(x)|. \]

We also consider the following Lipschitz classes:

\[ \omega^2_p(f; \delta) = \sup_{h \in (0, \delta)} \| \Delta^2_h f \|_p, \]

\[ \Delta^2_h f(x) = f(x + 2h) - 2f(x + h) + f(x), \]

\[ \omega^1_p(f; \delta) = \sup_{h \in (0, \delta)} \| \Delta_h f \|_p, \]

\[ \Delta_h f(x) = f(x + h) - f(x), \]

\[ Lip^2_p = \{ f \in C_p[0, \infty) : \omega^2_p(f; \delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+ \}, \]

where \( h > 0 \) and \( \alpha \in (0, 2) \).

From the above it follows that

\[ \lim_{\delta \to 0^+} \omega^1_p(f; \delta) = 0, \quad \lim_{\delta \to 0^+} \omega^2_p(f; \delta) = 0, \]

for every \( f \in C_p[0, \infty) \).

### 2. Auxiliary results

In this section we give some preliminary results which will be used in the main part of this paper.

Let us consider

\[ e_m(t) = t^m, \quad \varphi_{x,m}(t) = (t - x)^m, \quad m \in \mathbb{N}_0, \quad x, t \in (0, \infty). \]

**Lemma 2.1 ([11]).**

For any \( m \in \mathbb{N}_0(\text{set of non-negative integers}), \) \( m \leq n - k \)

\[ M_{n,k}(t^m; x) = \frac{[n - k + m]_m}{[n]_m} x^m \tag{3} \]

where \( n, k \in \mathbb{N} \) and \([x]_m = x(x-1) \ldots (x-m+1), [x]_0 = 1, x \in \mathbb{R} \).

In particular for \( m = 0, 1, 2 \ldots \) in (3) we get
(i) \( M_{n,k}(1; x) = 1 \),

(ii) \( M_{n,k}(t; x) = \frac{n - k + 1}{n} x \),

(iii) \( M_{n,k}(t^2; x) = \frac{(n - k + 2)(n - k + 1)}{n(n - 1)} x^2 \).

\textbf{Lemma 2.2 ([11]).}

Let \( m \in \mathbb{N}_0 \) and fixed \( x \in (0, \infty) \), then

\[
M_{n,k}(\varphi_{x,m}; x) = \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n-m+j)!(n-k+m-j)!}{n!(n-k)!} \right) x^m.
\]

\textbf{Lemma 2.3.}

For \( m = 0, 1, 2, 3, 4 \), one has

(i) \( M_{n,k}(\varphi_{x,0}; x) = 1 \),

(ii) \( M_{n,k}(\varphi_{x,1}; x) = \frac{1 - k}{n} x \),

(iii) \( M_{n,k}(\varphi_{x,2}; x) = \frac{k^2 - 5k + 2n + 4}{n(n - 1)} x^2 \),

(iv) \( M_{n,k}(\varphi_{x,3}; x) = \frac{-k^3 + 12k^2 - 17k + n(18 - 12k) + 24}{n(n - 1)(n - 2)} x^3 \),

(v) \( M_{n,k}(\varphi_{x,4}; x) = \frac{k^4 - 22k^3 + k^2(143 + 12n) - k(314 + 108n) + 12n^2 + 268n + 192}{n(n - 1)(n - 2)(n - 3)} x^4 \),

(vi) \( M_{n,k}(\varphi_{x,m}; x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right) \).

\textbf{Proof.} Using Lemma 2.2, we get Lemma 2.3. \qed

\textbf{Theorem 2.1.}

For the operators \( M_{n,k} \) and for fixed \( p \in \mathbb{N}_0 \), there exists a positive constant \( N_{p,k} \) depending only on the parameters \( p \) and \( k \) such that

\[
w_p(x) M_{n,k} \left( \frac{1}{w_p(t)} ; x \right) \leq N_{p,k}.
\] (4)

Moreover for every \( f \in C_p[0, \infty) \), we have

\[
\| M_{n,k}(f; \cdot) \|_p \leq N_{p,k} \| f \|_p,
\] (5)

which shows that \( M_{n,k} \) is a linear positive operator from the space \( C_p(0, \infty) \) into \( C_p(0, \infty) \).

\textbf{Proof.} For \( p = 0 \), (4) follows immediately. Using Lemma 2.1, we get

\[
w_p(x) M_{n,k} \left( \frac{1}{w_p(t)} ; x \right) = w_p(x) \left( M_{n,k}(e_0; x) + M_{n,k}(e_p; x) \right)
\]

\[
= w_p(x) \left( 1 + \frac{(n-p)!(n-k+p)!}{n!(n-k)!} x^p \right)
\]

\[
\leq N_{p,k} w_p(x)(1 + x^p) = N_{p,k},
\]

where

\[
N_{p,k} = \max \left\{ \frac{\sup_{n} (n-p)!(n-k+p)!}{n!(n-k)!} , 1 \right\}.
\]

Observe that for all \( f \in C_p \) and every \( x \in (0, \infty) \), we get

\[
w_p(x) |M_{n,k}(f; x)| \leq w_p(x) \frac{(2n - k + 1)! x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x + t)^{2n-k+2}} |f(t)| \frac{w_p(t)}{w_p(t)} dt
\]

\[
\leq \| f \|_p w_p(x) \left( \frac{1}{w_p(t)} ; x \right)
\]

\[
\leq N_{p,k} \| f \|_p.
\]

Taking supremum over \( x \in (0, \infty) \), we get (5). \qed
Lemma 2.4.
For the operators $M_{n,k}$ and fixed $p \in \mathbb{N}_0$, there exists a positive constant $N_{p,k}$ depending only on the parameters $p$ and $k$ such that

$$w_p(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right) \leq N_{p,k}\frac{x^2}{n}.$$ 

Proof. Using Lemma 2.3, we can write

$$w_0(x)M_{n,k}\left(\frac{\varphi_{x,2}}{w_0(t)};x\right) = \frac{k^2 - 5k + 2n + 4}{n(n-1)}x^2 \leq N_k\frac{x^2}{n},$$

which gives the result for $p = 0$. Let $p \geq 1$. Then using Lemma 2.1 and Lemma 2.3, we get

$$M_{n,k}\left(\frac{\varphi_{x,2}}{w_p(t)};x\right) = M_{n,k}(e_{p+2};x) - 2xM_{n,k}(e_{p+1};x) + x^2M_{n,k}(e_{p};x) + M_{n,k}(\varphi_{x,2};x) = (n-p-2)(n-k+p+2)!\frac{n!}{n!}x^{p+2} - 2(n-p-1)(n-k+p+1)!\frac{n!}{n!}x^{p+2} + \frac{(n-p)!}{n!}x^{p+2} + \frac{k^2 - 5k + 2n + 4}{n(n-1)}x^2 \leq N_{p,k}\frac{x^2}{n(1 + x^p)},$$

where $N_{p,k} = \left(1 + \left(k^2 - 5k + 6p + 4p^2 - 4kp\right)\frac{(n-p-2)(n-k+p+1)!}{(n-2)!}\frac{k^2 - 5k + 2n + 4}{n(n-1)}\right)$ is a positive constant. This completes the proof. \(\square\)

3. Direct results

The proof of direct theorems will follow from Jackson type inequality, the Steklov means and appropriate estimates of the moments of the operators.

Let $p \in \mathbb{N}_0$. By $C^p_{\infty}[0,\infty)$, we denote the space of all functions $f \in C_p[0,\infty)$ such that $f',f'' \in C_p[0,\infty)$.

Theorem 3.1.
Let $p \in \mathbb{N}_0$ and $f \in C^p_{\infty}[0,\infty)$, there exists a positive constant $N_{p,k}$ depending only on the parameters $p$ and $k$ such that

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq N_{p,k}\|f''\|_p\frac{x}{\sqrt{n}}$$

for all $x \in (0,\infty)$ and $n \in \mathbb{N}$.

Proof. Let $x \in (0,\infty)$ be fixed. Then for $f \in C^p_{\infty}[0,\infty)$ and $t \in (0,\infty)$, we have

$$f(t) - f(x) = \int_x^t f'(v)dv.$$ 

By using linearity of $M_{n,k}$ we get

$$M_{n,k}(f;x) - f(x) = M_{n,k}\left(\int_x^t f'(v)dv;x\right).$$

(6)

Remark that

$$\left|\int_x^t f'(v)dv\right| \leq \|f''\|_p \left|\int_x^t \frac{dv}{w_p(v)}\right| \leq \|f''\|_p|t-x|\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right).$$

From (6) we obtain

$$w_p(x)|M_{n,k}(f;x) - f(x)| \leq \|f''\|_p\left\{M_{n,k}(|\varphi_{x,1}|;x) + w_p(x)M_{n,k}\left(\frac{|\varphi_{x,1}|}{w_p(t)};x\right)\right\}.$$
Using Cauchy-Schwarz inequality, we can write
\[ M_{n,k}(\varphi_{x,1}; x) \leq \left( M_{n,k}(\varphi_{x,2}; x) \right)^{1/2} \times \left( M_{n,k}(\varphi_{x,0}; x) \right)^{1/2}, \]
\[ M_{n,k}\left( \frac{\varphi_{x,1}}{w_p(t)}; x \right) \leq \left( M_{n,k}\left( \frac{1}{w_p(t)}; x \right) \right)^{1/2} \times \left( M_{n,k}\left( \frac{\varphi_{x,2}}{w_p(t)}; x \right) \right)^{1/2}. \]

Using Lemma 2.3, Theorem 2.1 and Lemma 2.4, we obtain
\[ w_p(x)M_{n,k}(f; x) - f(x) \leq N_{p,k}\|f\|_p \frac{x^2}{\sqrt{n}}. \]

**Lemma 3.1.**

Let \( p \in N_0. \) If
\[ T_{n,k}(f; x) = M_{n,k}(f; x) - f\left(x + \frac{1 - k}{n} x\right) + f(x), \]
then there exists a positive constant \( N_{p,k} \) such that for all \( x \in (0, \infty) \) and \( n \in N, \) we have
\[ w_p(x)|T_{n,k}(g; x) - g(x)| \leq N_{p,k}\|g''\|_p \frac{x^2}{n} \]
for any function \( g \in C^2_p. \)

**Proof.** From Lemma 2.1, we observe that the operators \( T_{n,k} \) are linear and reproduce the linear functions. Hence
\[ T_{n,k}(\varphi_{x,1}; x) = 0. \]

Let \( g \in C^2_p. \) By the Taylor formula one can write
\[ g(t) - g(x) = (t - x)g'(x) + \int_x^t (t - v)g''(v)dv, \quad t \in (0, \infty). \]

Then,
\[ |T_{n,k}(g; x) - g(x)| = |T_{n,k}(g - g(x); x)| = \left| T_{n,k}\left( \int_x^t (t - v)g''(v)dv; x \right) \right| \]
\[ = \left| M_{n,k}\left( \int_x^t (t - v)g''(v)dv; x \right) - \int_x^{x + \frac{1 - k}{n} x} \left( x + \frac{1 - k}{n} x - v \right)g''(v)dv \right|. \]

Since
\[ \left| \int_x^t (t - v)g''(v)dv \right| \leq \frac{\|g''\|_p}{2} \left( \frac{1}{w_p(x)} + \frac{1}{w_p(t)} \right) \]
and
\[ \left| \int_x^{x + \frac{1 - k}{n} x} \left( x + \frac{1 - k}{n} x - v \right)g''(v)dv \right| \leq \frac{\|g''\|_p}{2w_p(x)} \left( \frac{1 - k}{n} x \right)^2, \]
we get
\[ w_p(x)|T_{n,k}(g; x) - g(x)| \leq \frac{\|g''\|_p}{2} \left( M_{n,k}(\varphi_{x,2}; x) + w_p(x)M_{n,k}\left( \frac{\varphi_{x,2}}{w_p(t)}; x \right) \right) + \frac{\|g''\|_p}{2} \left( \frac{1 - k}{n} x \right)^2. \]

Hence by Lemma 2.4, we obtain
\[ w_p(x)|T_{n,k}(g; x) - g(x)| \leq N_{p,k}\|g''\|_p \frac{x^2}{n} \]
for any function \( g \in C^2_p. \) The Lemma is proved.
Theorem 3.2. 
Let \( p \in N_0, n \in N \) and \( f \in C_{p}(0, \infty) \), there exists a positive constant \( N_{p,k} \) depending only on the parameters \( p \) and \( k \) such that 
\[
\omega_{p}(f; x) \leq N_{p,k} \omega_{p}^{2} f \left( \frac{x}{\sqrt{n}} \right) + \omega_{p}^{1} \left( f, \frac{1-k}{n} x \right).
\]
Furthermore, if \( f \in Li_{p}^2 \alpha \) for some \( \alpha \in (0, 2] \), then
\[
\omega_{p}(f; x) \leq N_{p,k} \left( \frac{x^{2}}{n} \right)^{a/2} + \omega_{p}^{1} \left( f, \frac{1-k}{n} x \right),
\]
holds.

Proof. Let \( p \in N_0, f \in C_{p}(0, \infty) \) and \( x \in (0, \infty) \) be fixed. We consider the Steklov means of \( f \) by \( f_{h} \) and given by the formula
\[
f_{h}(x) = \frac{4}{h^{2}} \int_{0}^{h/2} \int_{0}^{h/2} (2f(x+s+t) - f(x+2(s+t))) dsdt,
\]
for \( h, x \in (0, \infty) \). We have
\[
f(x) - f_{h}(x) = \frac{4}{h^{2}} \int_{0}^{h/2} \int_{0}^{h/2} \Delta^{2}_{s+t} f(x) dsdt,
\]
which gives
\[
\|f - f_{h}\|_{p} \leq \omega_{p}^{2}(f, h). \tag{8}
\]
Furthermore, we have
\[
f_{h}^\omega(x) = \frac{1}{h^{2}} \left( 8\Delta^{2}_{h/2} f(x) - \Delta^{2}_{h} f(x) \right),
\]
and
\[
\|f_{h}^\omega\|_{p} \leq \frac{9}{h^{2}} \omega_{p}^{2}(f, h). \tag{9}
\]
From (8) and (9) we conclude that \( f_{h} \in C_{p}^{2}(0, \infty) \) if \( f \in C_{p}[0, \infty) \).
Moreover
\[
|M_{n,k}(f; x) - f(x)| \leq T_{n,k}(1)f(t) - f_{h}(t; x) + |f(x) - f_{h}(x)| + |T_{n,k}(f_{h}; x) - f_{h}(x)| + \left| f \left( x + \frac{1-k}{n} x \right) - f(x) \right|,
\]
where \( T_{n,k} \) is defined in (7).
Since \( f_{h} \in C_{p}^{2}(0, \infty) \) by the above, it follows from Theorem 2.1 and Lemma 3.1 that
\[
w_{p}(f; x) \left| M_{n,k}(f; x) - f(x) \right| \leq (N + 1)\|f - f_{h}\|_{p} + N_{p,k}\|f_{h}^{\omega}\|_{p} \frac{x^{2}}{n} w_{p}(f) \left| f \left( x + \frac{1-k}{n} x \right) - f(x) \right|.
\]
By (8) and (9), the last inequality yields that
\[
w_{p}(f; x) \left| M_{n,k}(f; x) - f(x) \right| \leq N_{p,k} \omega_{p}^{2}(f; h) \left( 1 + \frac{1}{h^{2}} \right) \omega_{p}^{1} \left( f, \frac{1-k}{n} x \right).
\]
Thus, choosing \( h = \frac{x}{\sqrt{n}} \), the first part of the proof is completed.
The proof of second part can be easily obtained from the definition of the space \( Li_{p}^2 \alpha \).

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