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A note on the upper bound of the energy of a connected graph

Research Note

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Abstract: A new upper bound for the energy of a connected graph is presented in this note.

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1. Introduction

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. The independence number, denoted $\alpha(G)$, of G is defined as the size of the largest independent set in G. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G. The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^{n} |\mu_i(G)|$ (see [2]).

Several authors have obtained the upper bounds for the energy of a graph (see [3], [4], [5], [6], and [7]). In this note, we will present a new upper bound for the energy of a connected graph. The main result is as follows.

Theorem 1.1.
Let G be a connected graph of order $n \geq 2$ with e edges. Then

$$Eng(G) \leq 2\sqrt{(n-\alpha)e}$$

with equality if and only if G is $K_{1,n-1}$, where α is the independence number of G.

2. Proofs of the main result

In order to prove Theorem 1.1, we need the following Lemma 2.1 which is Theorem 3.14 on Pages 88 and 89 in [8].

Lemma 2.1.
Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p, q, and r, respectively, then

$$\alpha \leq r + \min\{p, q\},$$

where α is the independence number of G.

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Next, we will present the proof of Theorem 1.1.

Proof. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_p$ be the p positive eigenvalues of G and let $\rho_q \geq \rho_{q-1} \geq ... \geq \rho_1$ be the q negative eigenvalues of G. Then G has $n-p-q$ eigenvalues which are equal to zero. From Lemma 1, we have

$$\alpha \leq (n-p-q) + \min\{p, q\}.$$

Thus $\alpha \leq (n-p-q)+q$ and $\alpha \leq (n-p-q)+p$. Namely, $p \leq n-\alpha$ and $q \leq n-\alpha$. Since $\sum_{i=1}^{p}\mu_i + \sum_{i=1}^{q}\rho_i = 0$, we have that

$$Eng(G) = 2\sum_{i=1}^{p}\mu_i = 2\sum_{i=1}^{q}|\rho_i|.$$

From Cauchy - Schwarz inequality, we have that

$$Eng(G) = 2\sum_{i=1}^{p}\mu_i \leq 2\sqrt{p\sum_{i=1}^{p}\mu_i^2}.$$

Similarly, we have that

$$Eng(G) = 2\sum_{i=1}^{q}|\rho_i| \leq 2\sqrt{q\sum_{i=1}^{q}\rho_i^2}.$$

Therefore

$$\frac{Eng^2(G)}{2} = \frac{Eng^2(G)}{4} + \frac{Eng^2(G)}{4} \leq p\sum_{i=1}^{p}\mu_i^2 + q\sum_{i=1}^{q}\rho_i^2$$

$$\leq (n-\alpha)\sum_{i=1}^{p}\mu_i^2 + (n-\alpha)\sum_{i=1}^{q}\rho_i^2 = (n-\alpha)\left(\sum_{i=1}^{p}\mu_i^2 + \sum_{i=1}^{q}\rho_i^2\right) = 2(n-\alpha)e.$$

Hence

$$Eng(G) \leq 2\sqrt{(n-\alpha)e}.$$

If G is $K_{1,n-1}$, then $e = (n-1)$, $\alpha = (n-1)$, and the eigenvalues of G are $\sqrt{n-1}$, 0, 0, ..., 0, and -$\sqrt{n-1}$. Thus $Eng(G) = 2\sqrt{n-1} = 2\sqrt{(n-\alpha)e}$.

If $Eng(G) = 2\sqrt{(n-\alpha)e}$, then, from the proofs above, we have that $p = n-\alpha$, $q = n-\alpha$, $Eng(G) = 2\sum_{i=1}^{p}\mu_i = 2\sqrt{p\sum_{i=1}^{p}\mu_i^2}$, and $Eng(G) = 2\sum_{i=1}^{q}|\rho_i| = 2\sqrt{q\sum_{i=1}^{q}\rho_i^2}$. Thus, from the conditions for a Cauchy - Schwarz inequality becoming an equality, we have that $\mu_1 = \mu_2 = ... = \mu_p$ and $\rho_q = \rho_{q-1} = ... = \rho_1$. Therefore $Eng(G) = 2\sqrt{p\sum_{i=1}^{p}\mu_i^2} = 2\sqrt{(n-\alpha)^2\mu_1^2}$ and $Eng(G) = 2\sqrt{q\sum_{i=1}^{q}\rho_i^2} = 2\sqrt{(n-\alpha)^2\rho_1^2}$. So $2\sqrt{(n-\alpha)^2\mu_1^2} = 2\sqrt{(n-\alpha)^2\rho_1^2}$, Therefore $\mu_1 = -\rho_1$. Since G is connected and the largest eigenvalue of G is equal to the negation of the smallest eigenvalue of G, G is a bipartite graph. Again, since G is connected, its adjacency matrix is irreducible. From Perron - Frobenius theorem, we have that $p = 1$. Thus $\alpha = n-1$. Hence G must be $K_{1,n-1}$. □

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