A note on the upper bound of the energy of a connected graph

Rao Li *

Department of Mathematical Sciences, University of South Carolina Aiken, Aiken, SC 29801, USA

Received 08 October 2015; accepted (in revised version) 18 December 2015

Abstract: A new upper bound for the energy of a connected graph is presented in this note.

MSC: 05C50

Keywords: Upper bound • Energy • Eigenvalue

1. Introduction

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. The independence number, denoted α(G), of G is defined as the size of the largest independent set in G. The eigenvalues µ1(G) ≥ µ2(G) ≥ ... ≥ µn(G) of the adjacency matrix A(G) of G are called the eigenvalues of G. The energy, denoted Eng(G), of G is defined as \( \sum_{i=1}^{n} |\mu_i(G)| \) (see [2]).

Several authors have obtained the upper bounds for the energy of a graph (see [3], [4], [5], [6], and [7]). In this note, we will present a new upper bound for the energy of a connected graph. The main result is as follows.

**Theorem 1.1.**

Let G be a connected graph of order \( n \geq 2 \) with e edges. Then

\[ \text{Eng}(G) \leq 2\sqrt{(n - \alpha)e} \]

with equality if and only if G is \( K_{1, n-1} \), where \( \alpha \) is the independence number of G.

2. Proofs of the main result

In order to prove **Theorem 1.1**, we need the following **Lemma 2.1** which is Theorem 3.14 on Pages 88 and 89 in [8].

**Lemma 2.1.**

Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p, q, and r, respectively, then

\[ \alpha \leq r + \min\{p, q\}, \]

where \( \alpha \) is the independence number of G.

* E-mail address: raol@usca.edu
Next, we will present the proof of Theorem 1.1.

**Proof.** Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_p$ be the $p$ positive eigenvalues of $G$ and let $\rho_q \geq \rho_{q-1} \geq ... \geq \rho_1$ be the $q$ negative eigenvalues of $G$. Then $G$ has $n - p - q$ eigenvalues which are equal to zero. From Lemma 1, we have

$$\alpha \leq (n - p - q) + \min\{p, q\}.$$  

Thus $\alpha \leq (n - p - q) + q$ and $\alpha \leq (n - p - q) + p$. Namely, $p \leq n - \alpha$ and $q \leq n - \alpha$. Since $\sum_{i=1}^{p} \mu_i + \sum_{i=1}^{q} \rho_i = 0$, we have that

$$\text{Eng}(G) = 2 \sum_{i=1}^{p} \mu_i = 2 \sum_{i=1}^{q} |\rho_i|.$$  

From Cauchy - Schwarz inequality, we have that

$$\text{Eng}(G) = 2 \sum_{i=1}^{p} \mu_i \leq 2 \sqrt{p \sum_{i=1}^{p} \mu_i^2}.$$  

Similarly, we have that

$$\text{Eng}(G) = 2 \sum_{i=1}^{q} |\rho_i| \leq 2 \sqrt{q \sum_{i=1}^{q} \rho_i^2}.$$  

Therefore

$$\frac{\text{Eng}^2(G)}{2} = \frac{\text{Eng}^2(G)}{4} + \frac{\text{Eng}^2(G)}{4} \leq p \sum_{i=1}^{p} \mu_i^2 + q \sum_{i=1}^{q} \rho_i^2$$

$$\leq (n - \alpha) \sum_{i=1}^{p} \mu_i^2 + (n - \alpha) \sum_{i=1}^{q} \rho_i^2 = (n - \alpha) \left( \sum_{i=1}^{p} \mu_i^2 + \sum_{i=1}^{q} \rho_i^2 \right) = 2(n - \alpha)e.$$  

Hence

$$\text{Eng}(G) \leq 2\sqrt{(n - \alpha)e}.$$  

If $G$ is $K_{1,n-1}$, then $e = (n - 1)$, $\alpha = (n - 1)$, and the eigenvalues of $G$ are $\sqrt{n-1}$, $0$, ..., $0$, and $-\sqrt{n-1}$. Thus $\text{Eng}(G) = 2\sqrt{n-1} = 2\sqrt{(n - \alpha)e}.$

If $\text{Eng}(G) = 2\sqrt{(n - \alpha)e}$, then, from the proofs above, we have that $p = n - \alpha$, $q = n - \alpha$, $\text{Eng}(G) = 2 \sum_{i=1}^{p} \mu_i = 2 \sqrt{p \sum_{i=1}^{p} \mu_i^2}$, and $\text{Eng}(G) = 2 \sum_{i=1}^{q} |\rho_i| = 2 \sqrt{q \sum_{i=1}^{q} \rho_i^2}$. Thus, from the conditions for a Cauchy - Schwarz inequality becoming an equality, we have that $\mu_1 = \mu_2 = ... = \mu_p$ and $\rho_q = \rho_{q-1} = ... = \rho_1$. Therefore $\text{Eng}(G) = 2 \sqrt{p \sum_{i=1}^{p} \mu_i^2} = 2 \sqrt{(n - \alpha)^2 \mu_1^2}$ and $\text{Eng}(G) = 2 \sqrt{q \sum_{i=1}^{q} \rho_i^2} = 2 \sqrt{(n - \alpha)^2 \rho_1^2}$. So $2 \sqrt{(n - \alpha)^2 \mu_1^2} = 2 \sqrt{(n - \alpha)^2 \rho_1^2}$. Therefore $\mu_1 = \rho_1$. Since $G$ is connected and the largest eigenvalue of $G$ is equal to the negation of the smallest eigenvalue of $G$, $G$ is a bipartite graph. Again, since $G$ is connected, its adjacency matrix is irreducible. From Perron - Frobenius theorem, we have that $p = 1$. Thus $\alpha = n - 1$. Hence $G$ must be $K_{1,n-1}$.  

$\blacksquare$
References