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On the solution of unsteady power law flow near a moving wall by new similarity analysis method

Research Article

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Abstract: New similarity analysis method is employed to derive solution of non-linear partial differential equation with physical boundary conditions governing the unsteady power law flow near a moving wall. The resulting second order non-linear similarity equation which is essentially an ordinary differential equation of boundary value type. The obtained ordinary differential boundary value problem is transformed to an initial value problem by the application of one parameter scaling group transformation.

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1. Introduction

Fluids are categorized as liquids or gases. The electrical conductivity of the fluid plays, its role, only when, fluid is moving in an electromagnetic field, otherwise it may be considered as an ordinary fluid (non-electrically conducting fluid.) Magneto fluid dynamics (MFD) is the study of the motion of an electrically conducting fluid in the presence of a magnetic field. When the fluid is incompressible and its other properties such as viscosity, thermal conductivity and electrical conductivity are regarded as constant then it is known as magneto hydrodynamics (MHD) or hydromagnetic. The partial differential equations governing the motion of fluid flow problems are usually non linear, hence its solution is not easy. Some time, exact solution or some closed form solution of such equation is possible if it can be transformed in to ordinary differential equation with constant boundary conditions. The method of transformation of PDE into ODE (reduction of number of variable at least by one) is called similarity method.

A new similarity analysis method with a new set of dimensionless similarity variables are derived systematically. The present new similarity analysis method has following advantages:

1. More convenient for consideration and treatment of the variable physical properties
2. More convenient for analysis and investigation of the two-dimensional velocity field
3. More convenient for satisfaction of the interfacial mass transfer matching conditions in the numerical calculation and for rigorous investigation of mass transfer for two-phase film flows with three-point boundary problem.

In the present paper, a new similarity analysis method is reported for extensive investigation of similarity solution to unsteady power law flow near a moving wall. First, a system of dimensionless similarity variables, dimensionless coordinate variable and dimensionless velocity components, is derived and determined through the analysis with the typical basis conservation equation. In derivation of the dimensionless similarity variables, it is never necessary to introduce the stream function ψ, intermediate variable $f(\eta)$ and its derivatives. In this way, we attempt the new similarity analysis method in determining the similarity solution for the problem of unsteady power law flow near a moving wall.

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2. Formulation of the problem

Analytic solution for the power-law non-Newtonian fluid model for the case of pseudo plastic fluids was studied by [4,8,9] for various flow geometries. These solutions were limited for the steady flow near a wall was probably first time studied by Bird [2] . The pseudo plastic fluids past semi infinite body bounded by $y = 0, y \to \infty$ and with one side by a solid surface imbedded in the XZ-plane. Here, initially the fluid is at rest and for time $t \geq 0$ the solid surface moves with constant velocity in the X-direction. It is found that $x-$component of velocity u as a function of the distance from the solid surface and the time for a pseudo-plastic fluid obeys the following relation for the components of momentum flux: [4]

$$\tau_{ij} = -m\left(\frac{1}{2}\sum_k\sum_l \Delta_{kl}^2\right)^{\frac{(n-1)}{2}} \Delta_{ij} \tag{1}$$

In this expression $\Delta_{ij} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the ij component of the rate-of-strain tensor. The relation (1), for the two-dimensional flow can be written as [6]

$$\tau_{xy} = -m\left|\frac{\partial u}{\partial y}\right|^{n-1}\left(\frac{\partial u}{\partial y}\right) \tag{2}$$

for the xy component of the momentum flux.
The equation of motion for the system is

$$\rho\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \tau_{xy}}{\partial y}\right) \tag{3}$$

Substituting Eq. (2) into Eq. (3), taking into account the fact that $\frac{\partial u}{\partial y}$ is everywhere negative, one obtains the partial differential equation for the velocity distribution:

$$\rho\left(\frac{\partial u}{\partial t}\right) = -m\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^n \tag{4}$$

This is to be solved with the boundary conditions $u = U$ at $y = 0$, $u = 0$ at $y \to \infty$, and the initial condition that $u = 0$ for $t > 0$. According to the method of combination of variables y and t are combined into a new dimensionless variable:

$$r = (n+1)^{-1} y\left(\frac{\rho}{mtU^{n-1}}\right)^{1/(n+1)} \tag{5}$$

In this present paper, we consider a new similarity analysis method of the equation of motion of a semi-infinite body of pseudo plastic (power law) fluid occupying the half- space $y > 0$ [2,7], i.e.

$$\frac{\rho}{m}\frac{\partial u}{\partial t} + \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^n = 0 \tag{6}$$

where $u(x,t)$ is the x-component of velocity in a cartesian system. The fluid is set in motion at time t=0 by imparting a constant velocity U on the boundary $y = 0$; since the effects of this velocity is expected to decay and eventually vanish as we move into the body of the fluid, Eq. (1) is to be solved with the boundary conditions

$$u(0,t) = U, \quad t > 0; \quad u(\infty,t) = 0, \quad t > 0 \tag{7}$$

In Eq. (6), $\frac{\rho}{m}$ is the ratio of the fluid density ρ to the nempirical constant m, characteristic of the fluid, defined via Eq. (2) of [2] ; the parameter n also is characteristic of the fluid [2] .The solution process proposed here for the partial differential Eq. (6) with boundary conditions (7), and where the novelty lies, involves the complete reduction of the original problem (6) and (7) to an initial value problem entirely by stretching group transformations.

3. Select whole physical independent variable dominating the physical phenomenon

Observing governing Eqs. (6) and (7), it is seen that the whole physical independent variables dominating the physical phenomenon can be obtained and expressed as the following form:

$$F_0\left(u, y, t, \frac{\rho}{m}, U\right) = 0 \tag{8}$$

The above five physical variable (i.e. $n = 5$) are whole independent physical variables.

3.1. Select basic dimension system

For investigation of the given problem the following three physical dimensions are taken as basic dimensions: time[s], length [m], mass [kg]. While the dimensions of the above independent physical variables$u, y, t, \frac{\rho}{m}$ and Ucan be described by the basic dimensions, that is, $\left[\frac{m}{s}\right], [m], [s], \left[\frac{kg}{m^3}\right]$ and $\left[\frac{m}{s}\right]$ respectively. The above basic three dimensions are $[s], [m], [kg]$, (i.e. $r = 3$).

Here the number of the related dimensionless similarity physical parameters should be $n - r = 5 - 3 = 2$. According to the Buckingham's π-theorem, the dimensional analysis thus yield the result

$$F_0 = f(\pi_1, \pi_2) \tag{9}$$

where π_1 and π_2 are the dimensionless similarity parameters and F_0 is the suitable dimensionless physical phenomenon variable.

3.1.1. Determine the dimensionless similarity parameters π_1 and π_2

Before determination of the dimensionless similarity physical parameters of the physical phenomenon, we should select the physical variables. Each physical parameter is constituted by $r + 1 = 3 + 1 = 4$ physical variables. Then the dimensionless similarity parameters π_1 and π_2 can be expressed as the following equations, respectively:

$$\pi_1 = y^{a_1} \times t^{b_1} \times \left(\frac{\rho}{m}\right)^{c_1} \times u = 0 \tag{10}$$

$$\pi_2 = y^{a_2} \times t^{b_2} \times \left(\frac{\rho}{m}\right)^{c_2} \times U = 0 \tag{11}$$

By using dimensional analysis, the following dimensional equation is obtained for the dimensionless similarity parameters π_1:

$$[m]^{a_1} \cdot [s]^{b_1} \cdot \left[\frac{kg}{m^3}\right]^{c_1} \cdot \left[\frac{m}{s}\right] = 0$$

Obviously, the indexes a_1 to c_1 are suitable to the following equations:
For dimension $[kg]$ balance: $c_1 = 0$
For dimension $[m]$ balance: $a_1 - 3c_1 + 1 = 0$
For dimension $[s]$ balance: $b_1 - 1 = 0$
The solution is $a_1 = -1, b_1 = 1, c_1 = -1$
Then the dimensionless similarity parameter π_1 is

$$\pi_1 = \frac{tu}{y} \tag{12}$$

Similarly for dimensionless similarity parameter π_2 we have:

$$[m]^{a_2} \cdot [s]^{b_2} \cdot \left[\frac{kg}{m^3}\right]^{c_2} \cdot \left[\frac{m}{s}\right] = 0$$

These yields, $a_2 = -1, b_2 = 1, c_2 = 0$
Hence the dimensionless similarity parameters π_2 will be given by,

$$\pi_2 = \frac{tU}{y} \tag{13}$$

3.2. Investigation of the dimensionless similarity variables on the velocity field

3.2.1. Derivation of dimensionless coordinate variable

$$u \propto U \tag{14}$$

Hence, with quantity grade analysis, (6) can be approximately rewritten as the following form:

$$\frac{\rho}{m}\left(\frac{U}{t}\right) - \frac{U^n}{y^{n+1}} \propto 0$$

Then $y \propto \left(\frac{mtU^{n-1}}{\rho}\right)^{\frac{1}{n+1}}$ and, hence

$$\eta = y\left(\frac{\rho}{mtU^{n-1}}\right)^{\frac{1}{n+1}} \tag{15}$$

3.2.2. Derivation for dimensionless velocity components

From Eqs. (12) and (13), we have

$$\frac{\frac{tu}{y}}{\frac{tU}{y}} = \frac{\pi_1}{\pi_2}$$

$$\frac{u}{U} = \frac{\pi_1}{\pi_2}$$

$$u = Uf(\eta) \qquad \left(\because \quad \text{consider} \quad \frac{\pi_1}{\pi_2} = f(\eta)\right) \tag{16}$$

From (15) we have,

$$\frac{\partial \eta}{\partial t} = t^{-1}\eta \tag{17}$$

$$\frac{\partial \eta}{\partial y} = \left(\frac{\rho}{mtU^{n-1}}\right)^{\frac{1}{n+1}} \tag{18}$$

From (16)-(18) immediately we can find $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}$ etc. and substituting all these in Eq. (6) and simplifying, we get

$$\left(f'\right)^{n-1} f'' + \frac{\eta}{n(n+1)} f' = 0; \quad f(0) = 1, \quad f(\infty) = 0 \tag{19}$$

where the dashes denote differentiation with respect to η.

Hence we can reduce the original partial differential Eq. (6), with boundary conditions (7), to a boundary value problem in one variable only. It means that an ordinary differential equation with appropriately transformed boundary conditions under the similarity transformation (15) and (16), Eqs. (6) and (7) are transformed into the ordinary differential boundary value problem.

The relations (15), (16) and (19) are essentially those obtained by Bird [2] from ad hoc dimensional considerations. In fact, $\eta = (n+1)r$, where r is defined by Eq. (**??**)of reference [2] . In what follows, it proves convenient to transform the system (19) into the form obtained by Bird [2] by making the transformation $\eta^* = \frac{\eta}{(n+1)}$ to get (the dash denotes differentiation with respect to 'new' η)

$$(f')^{(n-1)} f'' + \frac{(n+1)^n}{n}\eta^* f' = 0; \qquad f(0) = 1, \quad f(\eta) = 0 \tag{20}$$

4. Reduction to an initial value problem: further group analysis

Equation (20) is a two point BVP. We need to solve this Eq. (20) to find the dimensionless velocity f, from which we can get the $x-$ component of the velocity, u, from Uf. Bird solved (20) using a semi-analytic method [2]. In what follows, we solve the system (20) for f, following Phan-Thien [7] , by turning (20)into an initial value problem. To do this we introduce, first, the change of variable $g = 1 - f$, when the system (20) becomes (the variable g is not to be confused with the transformation group

$$\left(g'\right)^{n-1} g'' - \frac{(n+1)^n}{n}\eta^* g' = 0; \qquad g(0) = 0, \quad g(\infty) = 1 \tag{21}$$

The boundary value problem (21) may be transformed into an initial value problem by a one-parameter stretching group transformation (see, for example, the books by Na, Bluman and Cole [3] and Ames 1). Introducing following one-parameter group of transformation (with parameter e) in Eq. (21)

$$\bar{g} = e^{\varepsilon\alpha_1} g, \qquad \bar{\eta} = e^{\varepsilon\alpha_2}\eta \tag{22}$$

where $\varepsilon > 0$, α_1, α_2 are constants then the equation for $g = 1 - f$ in (21), we get

$$e^{\varepsilon\{(n-1)\alpha_2 - (n-1)\alpha_1\}}\left(\bar{g}'\right)^{n-1} e^{\varepsilon(2\alpha_2 - \alpha_1)}\left(\bar{g}''\right) - \frac{(n+1)^n}{n} e^{-\varepsilon\alpha_2}\bar{\eta} e^{\varepsilon(\alpha_2 - \alpha_1)}\left(\bar{g}'\right) = 0 \tag{23}$$

The invariance of Eq. (23) yields,

$$\varepsilon(n-1)\alpha_1 - \varepsilon(n+1)\alpha_2 = 0 \tag{24}$$

To determine α_1 and α_2, we require another equation, and we obtain this from the missing initial condition $g'(0)$. For that here we have:

$$g'(0) = e^{\varepsilon} \tag{25}$$

then, using (22) in (25), we will have

$$e^{\varepsilon(\alpha_2 - \alpha_1)}\bar{g}'(0) = e^{\varepsilon} \tag{26}$$

Again invariance of (26) gives:

$$\varepsilon\alpha_2 - \varepsilon\alpha_1 = 1 \tag{27}$$

Equation (26) becomes, $\bar{g}'(0) = 1$ and the simultaneous Eqs. (24) and (27) may be solved, for α_1 and α_2 , to give

$$\alpha_1 = -\frac{n+1}{2}, \quad \alpha_2 = -\frac{n-1}{2} \tag{28}$$

It remains only to find the group parameter e^{ε}. To find e^{ε}, we consider the boundary condition at infinity, $g(\infty) = 1$. Transforming the boundary condition at infinity, we find that

$$e^{\varepsilon} = \left[\frac{1}{\bar{g}(\infty)}\right]^{\frac{2}{n+1}} \tag{29}$$

With everything now in place, the solution procedure is as follows. First, for any n, we solve the initial value problem

$$\left(\bar{g}'\right)^{n-1}\bar{g}'' - \frac{(n+1)^n}{n}\eta^*\bar{g}' = 0; \qquad \bar{g}(0) = 0, \quad \bar{g}'(0) = 1 \tag{30}$$

To find (an approximation) to $\bar{g}'(\infty)$. Next, we use the information from the solution to the initial value problem (28), that means, $\bar{g}'(\infty)$. To solve the required system for f, from (21) with $f = 1 - g$, that is

$$\left(f'\right)^{n-1}f'' - \frac{(n+1)^n}{n}\eta^* f' = 0; \qquad f(0) = 1, \quad f'(0) = -\left[\frac{1}{\bar{g}(\infty)}\right]^{\frac{2}{n+1}} \tag{31}$$

The initial value problems (30) and (31) can be solved using a standard fourth-order Runge-Kutta method. The entire solution technique is discussed in detail by Na [5] .

5. Conclusion

- The new similarity analysis method is successfully employed to similarity solution of derive solution of non-linear partial differential equation with physical boundary conditions governing the unsteady power law flow near a moving wall. Similarity transformations and similarity equation are derived systematically.
- The derived dimensionless velocity components $f(\eta)$ based on the present new similarity analysis method is directly proportional to the related velocity component u.
- The obtained boundary value problem is successively transformed to an initial value problem.
- The obtained similarity equation, which is second order ordinary differential equation with related boundary conditions, can be solved by some suitable numerical method.

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