Int. J. Adv. Appl. Math. and Mech. 3(3) (2016) 92 – 99 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Further results on relaxed mean labeling

Research Article

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Received 19 December 2015; accepted (in revised version) 28 January 2016

Abstract: In this paper further results on relaxed mean labeling is discussed. The condition for a graph to be relaxed mean is that $p = q + 1$. We prove, the following theorems to study path, star and the characterization for the relaxed mean labeling of two star. We prove that the disjoint union of any path with $n-1$ edges joining the pendent vertices of distinct paths is a relaxed mean graph and $K_{1,m}$ is not a relaxed mean graph for $m \geqslant 5$. Also, we prove that the two star $G = (K_{1,m} \cup K_{1,n})$ with an edge in common is a relaxed mean graph if and only if $|m-n| \leqslant 5$..

MSC: 05C78

Keywords: Relaxed mean graph • Path and Star

1. Introduction

All graphs in this paper are finite, simple an undirected. Terms not defined here are used in the sense of Harary [1]. In 1966, Rosa [2] introduced β - valuation of a graph. Golomb subsequently called such a labeling graceful. In 1980, Graham and Sloane [3] introduced the harmonious labeling of a graph. In 2003, Somasundaram and Ponraj [4, 5] introduced the mean labeling for some standard graphs. In 2015, Maheswari and Ramesh [6] proved the two star $G = (K_{1,m} \cup K_{1,n})$ with an edge in common is a mean graph if and only if $|m-n| \leqslant 4$. From [7–9] , we investigated a relaxed mean labeling for some standard graphs. Let $G = (V, E)$ be a simple graph of order p and size q. Then G is said to be a relaxed mean graph if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $\{0, 1, 2, ...q-1, q+1\}$ in such a way that when each edge $e = uv$ is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd, then the resulting edge labels $\{1, 2, 3...., q\}$ are distinct. Then f is called a relaxed mean labeling of G. Also, we investigate any path is a *relaxed mean graph*; If $m \geq 5$, $K_{1,m}$ is not a relaxed mean graph; The bistar $B_{m,n}$ is a relaxed mean graph if and only if $|m - n| \leqslant 3$; The subdivision of the central edge of $B_{n,n}$ is a mean and relaxed mean graphs; Comb is a relaxed mean graph; $C_m \cup P_n$ is a relaxed mean graph for $m = 3, n \geq 2$. Also, we prove the some basic theorems on relaxed mean graphs. We prove that cycle related graphs are relaxed mean labeling; If $n > 4$, K_n is not a relaxed mean graph; $K_{2,n}$ is a relaxed mean graph for all n; Any Triangular snake is a relaxed mean graph; Any Quadrilateral snake is a relaxed mean graph; P_n^2 is a relaxed mean graph; $C_3^{(t)}$ is called a friendship graph or Dutch $t-$windmill and $C_3^{(t)}$ is a relaxed mean graph and $L_n \Theta K_1$ is a relaxed mean graph. Also, we consider the $K_n^c + 2K_2$ is a relaxed mean graph for all n; W_4 is a relaxed mean graph; $K_2 + mK_1$ is a relaxed mean graph for all m; If G_1 and G_2 are trees, then $G = G_1 \cup G_2$ is not a relaxed mean graph; The planar grid $P_m \times P_n$ is a relaxed mean graph for $m \geqslant 2, n \geqslant 2$ and the prism $P_m \times C_n$ is a relaxed mean graph for m $\geqslant 2$, $n \geqslant 3$. Next, we study the identification of Graphs, let $G_1 = (p_1, q_1)$ and $G_1 = (p_1, q_1)$ are any two relaxed mean graphs with labeling f and g respectively, then the graph

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$(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is a relaxed mean graph; The caterpillar $S(x_1, x_1)$, $x_1 > 0$ is a relaxed mean graph; We prove that, let $x_1 \leqslant x_2 \leqslant x_3 \ldots \leqslant x_n$ and

$$k = \begin{cases} x_2 + x_4 + \ldots + x_n - (x_1 + x_2 + \ldots + x_{n-1}) & \text{if n is even} \\ x_1 + x_3 + \ldots + x_n - (x_2 + x_4 + \ldots + x_{n-1}) & \text{if n is odd.} \end{cases}$$

Then $S(x_1, x_2, \ldots, x_n, k)$ is a relaxed mean graph; Dragon $C_n @ P_m$ is a relaxed mean graph; Arbitrary super subdivision $AS(u_1, u_2)$ of the path $P_2 = u_1 u_2$ is a relaxed mean graph; Arbitrary super subdivision $AS(u_1, u_2, \ldots, u_n)$ of the path $P_n = u_1 u_2 \ldots u_n$ is a relaxed mean graph and the graph $C_n \hat{o} K_{1,1}, C_n \hat{o} K_{1,2}$, is a relaxed mean graph.

2. Relaxed mean labeling

Definition 2.1.
A graph $G = (V, E)$ with p vertices and q edges is said to be a relaxed mean graph if there exists a function f from the vertex set of G to $0, 1, 2, \ldots, q-1, q+1$ such that the induced map f^* from the edge set of G to $1, 2, 3, \ldots, q$ defined by

$$f*(e = uv) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd,} \end{cases}$$

then the edges get distinct labels from $1, 2, 3, \ldots, q$.

Remark 2.1.
The only graphs satisfying the condition $p = q + 1$ are paths and stars.

In the following theorem we prove that any path is a relaxed mean graph.

Theorem 2.1.
Any path $P_m = (P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_n)$ for all n with $(n-1)$ edges joining the pendent vertices of distinct paths is a relaxed mean graph.Any path $P_m = (P_1 \cup P_2 \cup P_3$âĂę$\cup P_n)$ for all n with $(n-1)$ edges joining the pendent vertices of distinct paths is a relaxed mean graph.

Proof. Let P_m be the path with edges $u_1 u_2, u_2 u_3 \ldots u_{m-1} u_m$.
Define a map $f : V(P_m) \to 0, 1, 2, \ldots, q-1, q+1$ by $f(u_i) = i - 1$ for $1 \leqslant i \leqslant m-1$.

Then, the corresponding label of the edge $u_i u_{i+1}$ is i for $1 \leqslant i \leqslant m-2$. Also the edge label of $u_{m-1} u_m$ is $m-1$. Hence P_m is a relaxed mean graph.

Next our aim is to prove that $K_{1,m}$is a relaxed mean graph. As the star $K_{1,1}$is P_2 and $K_{1,2}$ is P_3 are relaxed mean graphs by Theorem 2.1 and we give the relaxed mean labeling for $K_{1,3}$ and $K_{1,4}$ below:

- Let$G(V, E) = K_{1,3}$, Then $|V| = p = 4$ and $|E| = q = 3$ also, $q + 1 = 4$. Fig. 1
- Let $K_{1,4}$, Then $|V| = p = 5$ and $|E| = q = 4$ also, $q + 1 = 5$. Fig. 2

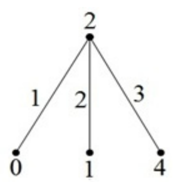

Fig. 1. $K_{1,3}$

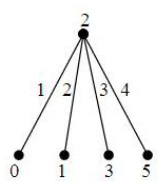

Fig. 2. $K_{1,4}$

□

Result 2.1.
$K_{1,4}$ is a relaxed mean graph but not a mean graph.

Proof. By Fig. 2 is a relaxed mean graph. By [4], $K_{1,m}$ is not a mean graph for $m \geqslant 4$. Hence $K_{1,4}$ is a relaxed mean graph but not a mean graph. □

Theorem 2.2.
If $m \geqslant 5$ then $K_{1,m}$ is not a relaxed mean graph.

Proof. Suppose that $G = K_{1,m}$ is a relaxed mean graph. Then the distinct edge labels are $1,2,3,...,m$. Let (V_1, V_2) be the bipartition of $K_{1,m}$ with $V_1 = u$. To get the edge label m, we must have $m-1$ and $m+1$ or $m-2$ and $m+1$ as the vertex labels of adjacent vertices. Clearly one of $m-2, m-1$ and $m+1$ must be the label of u. In both cases, since $m \geqslant 5$, there will be no edge with label 1. This contradiction proves that $K_{1,m}$ is not a relaxed mean graph. Hence the theorem. □

We define two stars and find the characterization for the relaxed mean labeling of two stars.

Definition 2.2.
The two stars is the disjoint union of $K_{1,m}$ and $K_{1,n}$. It is denoted by $K_{1,m} \cup K_{1,n}$

Theorem 2.3.
The two star $G = K_{1,m} \cup K_{1,n}$ with an edge in common is a relaxed mean graph if and only if $|m-n| \leqslant 5$.

Proof. Without loss of generality, we assume that $m \leqslant n$. Let us first take the case that $|m-n| \leqslant 5$. There are five cases viz. $n = m, n = m+1, n = m+2, n = m+3, n = m+4$ and $n = m+5$. In each case we have to prove that G is a relaxed mean graph.

Case 1: Let $n = m$

Consider the graph $G = 2(K_{1,m})$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m\}$ be the vertex set of first and second copies of $K_{1,m}$ respectively. Then G has $2m+1$ edges and $2m+2$ vertices.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{v_j : 1 \leqslant j \leqslant m\}$.

The required vertex labeling $f : V(G) \to \{0,1,2,...,2m,2m+2\}$ is defined as follows:

$$f(u) = 2; \; f(v) = 2m-1;$$
$$f(u_1) = 0$$
$$f(u_i) = 2i-3 \quad \text{for } 2 \leqslant i \leqslant m;$$
$$f(v_j) = 2j+2 \quad \text{for } 1 \leqslant j \leqslant m.$$

The corresponding edge labels are as follows:

The edge label of uu_1 is 1 and uu_i is i for $2 \leqslant i \leqslant m$. The edge label of vv_j is $m+j+1$ for $1 \leqslant j \leqslant m$.

Also, the edge label of $u_i v_j$ is $m+1$ for all u_i and v_j such that $\dfrac{f(u_i) + f(v_j) + 1}{2} = m+1$ Therefore, the edge labels of $G = \{1,2,3,...,m,m+1,m+2,...,2m+1\}$ and has $2m+1$ distinct edges. Hence the induced edge labels of G are distinct.

Case 2: Let $n = m+1$

Consider the graph $G = (K_{1,m} \cup K_{1,m+1}$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ be the vertices of $K_{1,m}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m+1\}$ be those of $K_{1,m+1}$. Then G has $2m+3$ vertices and $2m+2$ edges.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{vj : 1 \leqslant j \leqslant m+1\}$.

The required vertex labeling $f : V(G) \to \{0,1,2,...,2m+1,2m+3\}$ is defined as follows :

$$f(u) = 1; \; f(v) = 2m;$$
$$f(u_i) = 2i-2 \quad \text{for } 1 \leqslant i \leqslant m \text{ and}$$
$$f(v_j) = 2j+1 \quad \text{for } 1 \leqslant j \leqslant m+1.$$

The edge label of uu_i is i for $1 \leqslant i \leqslant m$ and vv_j is $m+j+1$ for $1 \leqslant j \leqslant m+1$. Also, the edge label of $u_i v_j$ is $m+1$ for all u_i and v_j such that $\dfrac{f(u_i) + f(v_j) + 1}{2} = m+1$ Therefore, the required edge labels of $G = \{1,2,3,...,m,m+1,m+2,...,2m+2\}$ and has $2m+2$ distinct edges. Hence the induced edge labels of G are distinct.

case 3: Let $n = m+2$

Consider the graph $G = (K_{1,m} \cup K_{1,m+2})$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ be the vertices of $K_{1,m}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m+2\}$ be those of $K_{1,m+2}$. Then G has $2m+4$ vertices and $2m+3$ edges.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{v_j : 1 \leqslant j \leqslant m+2\}$.

The required vertex labeling $f : V(G) \to \{0,1,2,...,2m+2,2m+4\}$ is defined as follows :

$f(u) = 0$; $f(v) = 2m + 1$;
$f(u_i) = 2i - 1$ for $1 \leqslant i \leqslant m$ and
$f(v_j) = 2j$ for $1 \leqslant j \leqslant m + 2$.

The corresponding edge labels are as follows:
The edge label of uu_i is i for $1 \leqslant i \leqslant m$ and vv_j is $m + j + 1$ for $1 \leqslant j \leqslant m+2$.

Also, the edge label of uiv_j is $m+1$ for all u_i and v_j such that $\frac{f(u_i) + f(v_j) + 1}{2} = m+1$. Therefore, the required edge labels of $G = \{1,2,3,...,m,m+1,m+2,...,2m+3\}$ and has $2m+3$ distinct edges. Hence the induced edge labels of G are distinct.

Case 4: Let $n = m+3$
Consider the graph $G = (K_{1,m} \cup K_{1,m+3})$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ be the vertices of $K_{1,m}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m+3\}$ be those of $K_{1,m+3}$. Then G has $2m+5$ vertices and $2m+4$ edges.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{v_j : 1 \leqslant j \leqslant m+3\}$.
The required vertex labeling $f : V(G) \to \{0,1,2,...,2m+3,2m+5\}$ is defined as follows:

$f(u) = 0$; $f(v) = 2m + 2$
$f(u_i) = 2i$ for $1 \leqslant i \leqslant m$ and
$f(v_j) = 2j - 1$ for $1 \leqslant j \leqslant m + 3$.

The corresponding edge labels are as follows:
The edge label of uu_i is i for $1 \leqslant i \leqslant m$ and vv_j is $m + j + 1$ for $1 \leqslant j \leqslant m+3$.

Also, the edge label of u_iv_j is $m+1$ for all u_i and v_j such that $\frac{f(u_i) + f(v_j) + 1}{2} = m + 1$ Therefore, the required edge labels of $G = \{1,2,3,...,m,m+1,m+2,...,2m+4\}$ is $2m+4$ distinct edges. Hence the induced edge labels of G are distinct.

Case 5: Let $n = m+4$
Consider the graph $G = K_{1,m} \cup K_{1,m+4})$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ be the vertices of $K_{1,m}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m+4\}$ be those of $K_{1,m+4}$. Then G has $2m+6$ vertices and $2m+5$ edges.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{v_j : 1 \leqslant j \leqslant m+4\}$.
The required vertex labeling $f : V(G) \to \{0,1,2,...,2m+4,2m+6\}$ is defined as follows :

$f(u) = 1$; $f(v) = 2m + 3$;
$f(u_i) = 2i + 1$ for $1 \leqslant i \leqslant m$ and
$f(v_j) = 2j - 2$ for $1 \leqslant j \leqslant m + 4$

The corresponding edge labels are as follows:
The edge label of uu_i is $i+1$ for $1 \leqslant i \leqslant m$ and vv_j is $m + j + 1$ for $1 \leqslant j \leqslant m+4$.
Also, the edge label of v_1u is 1. Therefore, the required edge labels of $G = \{1,2,3,$âĂę$,m,m+1,m+2,$âĂę$,2m+5\}$ and has $2m+5$ distinct edges. Hence the induced edge labels of G are distinct.

Case 6: Let $n = m+5$
Consider the graph $G = (K_{1,m} \cup K_{1,m+5})$ with an edge in common. Let $\{u\} \cup \{u_i : 1 \leqslant i \leqslant m\}$ be the vertices of $K_{1,m}$ and $\{v\} \cup \{v_j : 1 \leqslant j \leqslant m+5\}$ be those of $K_{1,m+5}$. Then G has $2m+7$ vertices and $2m+6$ edges.
We have $V(G) = \{u, v\} \cup \{u_i : 1 \leqslant i \leqslant m\} \cup \{v_j : 1 \leqslant j \leqslant m+5\}$.
The required vertex labeling $f : V(G) \to \{0,1,2,...,2m+5,2m+7\}$ is defined as follows :

$f(u) = 1$; $f(v) = 2m + 4$;
$f(u_i) = 2i + 1$ for $1 \leqslant i \leqslant m$
$f(v_j) = 2j - 2$ for $1 \leqslant j \leqslant m + 2$
$f(v_k) = 2k - 3$ for $m+3 \leqslant i \leqslant m + 5$

The corresponding edge labels are as follows:
The edge label of uu_i is $i+1$ for $1 \leqslant i \leqslant m$; vv_j is $m+j+1$ for $1 \leqslant j \leqslant m+2$ and vv_k is $m+k+1$ for $m+3 \leqslant j \leqslant m+5$.
Also, the edge label of v_1u is 1. Therefore, the required edge labels of $G = \{1,2,3,...,m,m+1,m+2,...,2m+5\}$ and has $2m+5$ distinct edges. Hence the induced edge labels of G are distinct. Hence the graph G is a relaxed mean graph if $|m-n| \leqslant 5$.

Conversely, Let us take the case that $|m-n|>5$.
Suppose that $G=K_{1,m}\cup K_{1,n}$ with an edge in common for $n=m+r$ for is a relaxed mean graph.
Let us assume that $G=G_1\cup G_2$ with an edge in common for $G_1=K1,m+r$ and $G_2=K_{1,m}$.
Let us now consider the case when $r=6$ and $m=1$. Then the graph $G = K_{1,7}\cup K_{1,1}$ With an edge in common, have 10 vertices and 9 edges.

Let $V(G) = \{v_{1,j}:0\leqslant j\leqslant 1\}\cup\{v_{2,j}: 0\leqslant j\leqslant 7\}$ and $E(G) = \{v_{1,0}v_{1,j}: j=1\}\cup\{v_{2,0}v_{2,j}:1\leqslant j\leqslant 7\}\cup\{v_{1,1}v_{2,j}:$ for any one of vertex $v_{2,j}\ for\ 1\leqslant j\leqslant 7\}$.
Suppose G is a relaxed mean graph. Let $p=|V|=10$ and $q=|E|=9$. Then there exists a function f from the vertex set of G to $\{0,1,...,q-1,q+1\}$ such that the induced map f^* from the edge set of G to $\{1,2,...,q\}$ defined by,

$$f*(e=uv)=\begin{cases}\dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even}\\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.}\end{cases}$$

Then the edges get distinct labels from $\{1,2,...,q\}$.

Let $t_{i,j}$ be the label given to the vertex $v_{1,j}$ for $0\leqslant j\leqslant 1$ and $v_{2,j}$ for $0\leqslant j\leqslant 7$ and $x_{i,j}$ be the corresponding edge label of the edge $v_{1,0}v_{1,1}$ and $v_{2,0}v_{2,j}$ for $1\leqslant j\leqslant 7$.

Case (a): Let us first consider the case that $t_{2,0}=10$.

If .$t_{2,j}=2n-1$. and $t_{2,k}=2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j})=\left\lceil\frac{10+2n-1}{2}\right\rceil=n+5=\left\lceil\frac{10+2n}{2}\right\rceil=f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), (5 or 6) and (7 or 8). These five labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1\leqslant j\leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0}=10$.

Case (b): $t_{2,0}=8$.

If $t_{2,j}=2n-1$ and $t_{2,k}=2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j})=\left\lceil\frac{8+2n-1}{2}\right\rceil=n+4=\left\lceil\frac{8+2n}{2}\right\rceil=f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), (5 or 6), 7 and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1\leqslant j\leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0}=8$.

Case (c): Let us next consider the case that $t_{2,0}=7$.

If $t_{2,j}=2n$ and $t_{2,k}=2n+1$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j})=\left\lceil\frac{7+2n}{2}\right\rceil=n+4=\left\lceil\frac{7+2n+1}{2}\right\rceil=f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are (0 or 1), (2 or 3), (4 or 5), 6, 8 and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1\leqslant j\leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0}=7$.

Case (d): Let us next consider the case that $t_{2,0}=6$.

If $t_{2,j}=2n-1$ and $t_{2,k}=2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j})=\left\lceil\frac{6+2n-1}{2}\right\rceil=n+3=\left\lceil\frac{6+2n}{2}\right\rceil=f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), (3 or 4), 5, (7 or 8) and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1\leqslant j\leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0}=6$.

Case (e): Let us next consider the case that $t_{2,0}=5$.

If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{5 + 2n + 1}{2} \right\rceil = n + 3 = \left\lceil \frac{5 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are (0 or 1), (2 or 3), 4, (6 or 7), 8 and 10. These six labels are not sufficient to label the seven vertices $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 5$.

Case (f): Let us next consider the case that $t_{2,0} = 4$.
If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{4 + 2n - 1}{2} \right\rceil = n + 2 = \left\lceil \frac{4 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, (1 or 2), 3, (5 or 6), (7 or 8) and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 4$.

Case (g): Let us next consider the case that $t_{2,0} = 3$.
If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{3 + 2n + 1}{2} \right\rceil = n + 2 = \left\lceil \frac{3 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are (0 or 1), 2, (4 or 5), (6 or 7), 8 and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 3$.

Case (h): Let us next consider the case that $t_{2,0} = 2$.
If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{2 + 2n - 1}{2} \right\rceil = n + 1 = \left\lceil \frac{2 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, 1, (3 or 4), (5 or 6), (7 or 8) and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 2$.

Case (i): Let us next consider the case that $t_{2,0} = 1$.
If $t_{2,j} = 2n + 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{1 + 2n + 1}{2} \right\rceil = n + 1 = \left\lceil \frac{1 + 2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are 0, (2 or 3), (4 or 5), (6 or 7), 8 and 10. These six labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 1$.

Case (j): Let us next consider the case that $t_{2,0} = 0$.
If $t_{2,j} = 2n - 1$ and $t_{2,k} = 2n$ for some n, j and k then

$$f^*(v_{2,0}v_{2,j}) = \left\lceil \frac{2n - 1}{2} \right\rceil = n = \left\lceil \frac{2n}{2} \right\rceil = f^*(v_{2,0}v_{2,k}).$$

This is not possible as f^* is a bijection.

Therefore, the possible vertex labels are (1 or 2), (3 or 4), (5 or 6), (7 or 8) and 10. These five labels are not sufficient to label the seven vertices, $t_{2,j}$ for $1 \leqslant j \leqslant 7$.
Therefore, G is a not a relaxed mean graph when $t_{2,0} = 0$.

Therefore, $G = K_{1,7} \cup K_{1,1}$ with an edge in common is not a relaxed mean graph for any possible the values i.e., $G = (K_{1,m} \cup K_{1,n})$ with an edge in common is not a relaxed mean graph when $|m - n| = 6$.

Similarly, we can prove that $G = K_{1,8} \cup K_{1,1}$ with an edge in common is not a relaxed mean graph when $|m - n| = 7$.

Hence, $G = K_{1,n} \cup K_{1,m}$ with an edge in common is not a relaxed mean graph if $|m - n| \geqslant 6$. Hence the theorem. □

3. Application of graph labeling in communication networks

The Graph Theory plays a vital role in various fields. One of the important area is Graph (Relaxed mean) Labeling, used in many applications like coding theory, X - ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management. Applications of labeling (Relaxed Mean) of graphs extends to heterogeneous fields but here we mainly focus on the communication networks. Communication network is of two types 'Wired Communication' and 'Wireless Communication'. Day by day wireless networks have been developed to ease communication between any two systems, results more efficient communication. To explore the role of labeling in expanding the utility of this channel assignment process in communication networks. Also, graph labeling has been observed and identified its usage towards communication networks. We address how the concept of graph labeling can be applied to network security, network addressing, channel assignment process and social networks.

Network representations play an important role in many domains of computer science, ranging from data structures and graph algorithms, to parallel and communication networks.

Geometric representation of the graph structure imposed on these data sets provides a powerful aid to visualizing and understanding the data. The graph labeling is one of the most widely used labeling methods of graphs. While the labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, it serves as models in a wide range of applications as listed below.

- The coding theory.
- The x-ray crystallography.
- The communication network addressing.
- Fast Communication in Sensor Networks Using Graph Labeling.
- Automatic Channel Allocation for Small Wireless Local Area Network.
- Graph Labeling in Communication Relevant to Adhoc Networks.
- Effective Communication in Social Networks by Using Graphs.
- Secure Communication in Graphs.

4. Conclusion

Researchers may get some information related to graph labeling and its applications in communication field and work on some ideas related to their field of research.

- **Objectives:** To analyze the problem of finding the communication system or network improved by a particular graph is controlled using graph labeling. To evaluate the communication system or network for different cases by changing the graphs and comparison study of graphs using graph labeling techniques.
- **Methodology:** We will apply the graph labeling concepts, failure communications to different systems or networks and situations, we will be arriving the results to achieve the mentioned objective.
- **Plan of Work and Target to be achieved:** We will focus on the applications of communication systems or networks. We investigate the graph labeling system and also deal with the communication system or network in application; since the application has to be characterized by highest communication systems or networks in order to perfect the graph labeling.

Acknowledgement

One of the authors (Dr. V. Balaji) acknowledges University Grants Commission, SERO, Hyderabad and India for financial assistance [No. F MRP 5766/15 (SERO/UGC)].

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