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On the bounds for the spetral norm of particular matrices with Fibonacci and Lucas numbers

Research Article

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Abstract: In this note we study a particular $n\times n$ matrix $F=[F_{k_{i,j}}]_{i,j=1}^{n}$ and its Hadamard exponential matrices $e^{\circ F}=[e^{F_k}]$, where $k_{i,j}=min(i,j)+1$ and F_k is the k^{th} Fibonacci number. Determinant , inversion and principal minors of these matries are considered. Then Some upper bounds and lower bounds for the spectral norm of these matrices are represented.

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Keywords: Fibonacci and Lucas numbers • Spectral norm • Determinant

1. Introduction

Due to the various applications of Fibonacci and Lucas numbers many authors have studied Fibonacci and Lucas numbers. In [1] Solak and Bahsi investigated on the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers. In [2] Akbulak studied Hadamard exponential Hankel matrix of the form $e^{H_n}=[e^{i+j}]_{i,j=0}^{n-1}$ and found ℓ_p norm and two upper bounds for the spectral norms of this matrix. In [3] Bozkort determined bounds for the spectral and ℓ_p norm of Cauchy-Hankel matrices of the form $H_n=[\frac{1}{g+kh}]_{i,j=0}^{n}$, where number k is defined by $i+j=k$ and g,h are any positive numbers. Civciv and Turkmen in [4] established a lower bound and upper bound for the ℓ_p norms of the Khatri-Rao product of Cauchy-Hankel matrix of the form $H_n=[\frac{1}{\frac{1}{2}+(i+j)}]$. The authors in [5] found some bounds for the spectral norm of particular matrix of the form $A=[a^{min(i,j)}]_{i,j=0}^{n-1}$ where a is a real positive number. For more information one can see [6]-[11].
In this paper we study a particular $n\times n$ matrix $F=[F_{k_{ij}}]_{i,j=1}^{n}$ and its Hadamard exponential Matrix $e^{\circ F}=[e^{F_k}]$, where $k_{i,j}=min(i,j)+1$ and F_k is the k^{th} Fibonacci number. In exact these matrices are as following

$$F=[F_{min(i,j)+1}]_{i,j=1}^{n}=\begin{bmatrix} F_2 & F_2 & F_2 & \cdots & F_2\\ F_2 & F_3 & F_3 & \cdots & F_3\\ F_2 & F_3 & F_4 & \cdots & F_4\\ \vdots & \vdots & \vdots & \vdots & \\ F_2 & F_3 & F_4 & \cdots & F_{n+1}\end{bmatrix}, \tag{1}$$

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and

$$e^{\circ F} = [e^{F_{min(i,j)+1}}]_{i,j=1}^{n} = \begin{bmatrix} e^{F_2} & e^{F_2} & e^{F_2} & \cdots & e^{F_2} \\ e^{F_2} & e^{F_3} & e^{F_3} & \cdots & e^{F_3} \\ e^{F_2} & e^{F_3} & e^{F_4} & \cdots & e^{F_4} \\ \vdots & \vdots & \vdots & \vdots & \\ e^{F_2} & e^{F_3} & e^{F_4} & \cdots & e^{F_{n+1}} \end{bmatrix}, \tag{2}$$

where F_k is the k^{th} Fibonacci number. Determinant, inversion and principal minors of these matrices are considered. Then we find some upper and lower bounds for the spectral norm of these matrices. Finally some other properties of these matrices also are represented.

All definitions and statements of this section are available in references [1]-[12].

Let $A = (a_{ij})$ is an $n \times n$ matrix, then we define the maximum column length norm $c_1(.)$ and maximum row length norm $r_1(.)$ of matrix A by

$$c_1(A) = max_j \sqrt{\sum_i |a_{ij}|^2} \quad , \quad r_1(A) = max_i \sqrt{\sum_j |a_{ij}|^2}. \tag{3}$$

Hadamard exponential and Hadamard inverse of this matrix is defined by $e^{\circ A} = (e^{a_{ij}})$ and $A^{\circ(-1)} = (\frac{1}{a_{ij}})$. The ℓ_p norm of A is defined by

$$\|A\|_p = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \quad . \tag{4}$$

For $p = 2$ this norm is called Frobenius or Euclidean norm and showed by $\|A\|_E$. Let $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices. Then Hadamard product of A and B is defined by $A \circ B = (a_{ij}b_{ij})$. Let A, B and C be $m \times n$ matrices and $A = B \circ C$, (Hadamard product of B and C) then we have

$$\|A\|_2 \le r_1(B)c_1(C). \tag{5}$$

The spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i}, \tag{6}$$

where λ_i is the eigenvalue of matrix AA^H and A^H is conjugate transpose of matrix A. There is a relation between Frobenius and spectral norm, that is

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E. \tag{7}$$

It is known that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1} \quad , \tag{8}$$

$$\sum_{k=1}^{n-1} kx^k = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}. \tag{9}$$

If we start from $n = 0$, then the Fibonacci numbers F_n and Lucas numbers L_n are given respectively by

$$F_n = F_{n-1} + F_{n-2}; \quad F_0 = 0, F_1 = 1, \qquad L_n = L_{n-1} + L_{n-2}; \quad L_0 = 2, L_1 = 1. \tag{10}$$

From [6]-[7] we have the following statements about Fibonacci and Lucas numbers.

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1, \quad \sum_{k=1}^{n-1} {F_k}^2 = F_n F_{n-1}, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{11}$$

$$L_n = \alpha^n + \beta^n, \quad \alpha\beta = -1, \tag{12}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}. \tag{13}$$

For mor information about Fibonacci numbers and Lucas numbers and some generalizations of them one can see [1] and [7]-[11].

2. Main results

Theorem 2.1.
Let F be a matrix as in (1), then $det(F) = \prod_{i=2}^{n} F_{i-1}$.

Proof. By using elementary row operations on (1)we have

$$det(F) = det \begin{bmatrix} F_2 & F_2 & F_2 & \cdots & F_2 \\ 0 & F_3 - F_2 & F_3 - F_2 & \cdots & F_3 - F_2 \\ 0 & 0 & F_4 - F_3 & \cdots & F_4 - F_3 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & F_{n+1} - F_n \end{bmatrix}.$$

So we get

$$det(F) = F_2 \prod_{i=2}^{n} (F_{i+1} - F_i) = \prod_{i=2}^{n} F_{i-1}.$$

□

Lemma 2.1.
Let A is an $n \times n$ nonsigular matrix and b is an $n \times 1$ matrix, also c is a real number.

$$If \quad M = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}$$

then the inversion of M is

$$N = \begin{bmatrix} A^{-1} + \frac{1}{l} A^{-1} b b^T A^{-1} & -\frac{1}{l} A^{-1} b \\ -\frac{1}{l} b^T A^{-1} & c \end{bmatrix},$$

where $l = c - b^T A^{-1} b$.
Proof. By definition of M and N we have $M.N = I_{n+1}$. Thus $M^{-1} = N$.

Theorem 2.2.
Let F be a matrix as in (1), then F is invertible and the inversion of F is a symmetric tridiagonal matrix of the form

$$F^{-1} = \begin{bmatrix} F_3 & \frac{-1}{F_1} & 0 & & & \cdots & & 0 \\ \frac{-1}{F_1} & F_3 & \frac{-1}{F_2} & 0 & & & \cdots & 0 \\ 0 & \frac{-1}{F_2} & \frac{F_4}{F_2 F_3} & \frac{-1}{F_3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-1}{F_3} & \frac{F_5}{F_3 F_4} & \frac{-1}{F_4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & & \cdots & 0 & \frac{-1}{F_{n-2}} & \frac{F_n}{F_{n-1} F_{n-2}} & \frac{-1}{F_{n-1}} \\ 0 & 0 & & \cdots & & 0 & \frac{-1}{F_{n-1}} & \frac{1}{F_{n-1}} \end{bmatrix}.$$

Proof. By Theorem 2.1 we know F is nonsingular so it is invertible. Now we prove the rest of theorem by mathematical induction on n. The result is true for $n = 2$, that is if

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_2 & F_2 \\ F_2 & F_3 \end{bmatrix}$$

then we have

$$F^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & \frac{-1}{F_2} \\ \frac{-1}{F_2} & \frac{1}{F_2} \end{bmatrix}.$$

Now assume that the result is true for n, that is
$A = F = [F_{K_{ij}}]_{n \times n}$, $A^{-1} = [F_{K_{ij}}]^{-1}_{n \times n}$.
Thus by taking

$$b = (F_2, F_3, \cdots, F_{n+1})^T, \quad b^T = (F_2, F_3, \cdots, F_{n+1})$$

and $c = F_{n+2}$ along with Lemma 2.1 the proof is completed for $n + 1$. So the result is true for each n. □

Theorem 2.3.
Let F be a matrix as in (1), then the Euclidean norm of F is

$$\|F\|_E = \frac{1}{\sqrt{5}}\sqrt{(2n+1)[L_{2n+3}-L_3]-2[nL_{2n+4}-(n+1)L_{2n+2}+L_2]-2n}.$$

Proof. By definition of the Euclidean norm we have

$$\|F\|_E^2 = \left[(\sum_{i=1}^{n}\sum_{j=1}^{n}|F_{K_{ij}}|^2)^{\frac{1}{2}} \right]^2.$$

Thus we have

$$\|F\|_E^2 = \sum_{k=1}^{n}(2n-2k+1)F_{k+1}^2 = (2n+1)\sum_{k=1}^{n}F_{k+1}^2 - 2\sum_{k=1}^{n}kF_{k+1}^2.$$

By using (11) we get

$$\|F\|_E^2 = (2n+1)\sum_{k=1}^{n}\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}\right)^2 - 2\sum_{k=1}^{n}k\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}\right)^2,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. According to(8),(9), (10), (12) and by some computations we obtain

$$\|F\|_E^2 = \frac{1}{5}\left((2n+1)[L_{2n+3}-L_3]-2[nL_{2n+4}-(n+1)L_{2n+2}+L_2]-2n\right).$$

If we take $\frac{1}{2}^{th}$ power from the both side of last eqaulity we get

$$\|F\|_E = \frac{1}{\sqrt{5}}\sqrt{(2n+1)[L_{2n+3}-L_3]-2[nL_{2n+4}-(n+1)L_{2n+2}+L_2]-2n}.$$

Thus the proof is completed. □

Theorem 2.4.
Let F be a matrix as in (1), then we have the following upper bound for the spectral norm of F.

$$\|F\|_2 \le \sqrt{F_nF_{n+1}(F_{n+1}F_{n+2}-1)}.$$

Proof. By definition of Hadamard product for matrix F we have

$$F = \begin{bmatrix} F_2 & 1 & 1 & \cdots & 1 \\ F_2 & F_3 & 1 & \cdots & 1 \\ F_2 & F_3 & F_4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ F_2 & F_3 & F_4 & \cdots & F_{n+1} \end{bmatrix} \circ \begin{bmatrix} 1 & F_2 & F_2 & \cdots & F_2 \\ 1 & 1 & F_3 & \cdots & F_3 \\ 1 & 1 & 1 & \cdots & F_4 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} = A \circ B.$$

By definition of maximum row length norm and maximum column length norm we have

$$r_1(A) = max_i\sqrt{\sum_j |a_{ij}|^2} = \sqrt{\sum_{i=2}^{n+1}F_i^2} = \sqrt{F_{n+2}F_{n+1}-1},$$

$$c_1(B) = max_j\sqrt{\sum_i |b_{ij}|^2} = \sqrt{\sum_{i=2}^{n}F_i^2+1} = \sqrt{F_nF_{n+1}}.$$

Therefore by (5) we get

$$\|F\|_2 \le \sqrt{F_nF_{n+1}(F_{n+1}F_{n+2}-1)}.$$

□

Theorem 2.5.
Let F be a matrix as in(1), then we have the following upper and lower bounds for the spectral norm of F.

$$\frac{1}{\sqrt{5n}}[(2n+1)[L_{2n+3}-L_3]-2[nL_{2n+4}-(n+1)L_{2n+2}+L_2]-2n]^{\frac{1}{2}} \leq \|F\|_2$$

$$\leq \frac{1}{\sqrt{5}}[(2n+1)[L_{2n+3}-L_3]-2[nL_{2n+4}-(n+1)L_{2n+2}+L_2]-2n]^{\frac{1}{2}}.$$

Proof. The proof is conculded from (7) and Theorem 2.3. □

Theorem 2.6.
Let F be a matrix as in (1), then F is a positive definite matrix and all eigenvalues of F are positive.

Proof. By Theorem 2.1 we know that for each $n \geq 1$, $det(F)$ is positive. So all leading principal minors of F are positive. Thus by [13], F is positive definite matrix. Consequently all eigenvalues of F are positive. □

Example 2.1.
Let F be a matrix as in (1). Determinants and eigenvalues of F for some values of n are represented in Table 1.

Theorem 2.7.
Let F be a matrix as in (1) and Δ_n denotes the leading principal minor of F, in exact $\Delta_1 = \Delta_2 = \Delta_3 = 1$, $\Delta_4 = 2, \cdots$, $\Delta_n = det[F_{min(i,j)+1}]_{i,j=1}^{n}$, then we have

(1) $\Delta_n \Delta_{n-2} \geq \Delta_{n-1}^2$.

(2) $\Delta_n \Delta_{n-2} F_{n-2} = \Delta_{n-1}^2 F_{n-1}$.

(3) $\Delta_1 \Delta_2 \Delta_3 \cdots \Delta_n = \prod_{i=1}^{n-1} (F_i)^{n-i}$.

Proof. By Theorem 2.1 we know
$\Delta_n = det[F_{min(i,j)+1}]_{i,j=1}^{n} = \prod_{i=2}^{n} F_{i-1}$.
So we have

$$\Delta_n \Delta_{n-2} = \prod_{i=2}^{n} F_{i-1} \prod_{i=2}^{n-2} F_{i-1} = \left(\prod_{i=2}^{n-2} F_{i-1}\right)^2 F_{n-1}F_{n-2}$$

$$\geq \left(\prod_{i=2}^{n-2} F_{i-1}\right)^2 F_{n-2}F_{n-2} = (\Delta_{n-1})^2.$$

Thus the proof of (1) is completed.
By (1) we know

$$\Delta_n \Delta_{n-2} = \left(\prod_{i=2}^{n-2} F_{i-1}\right)^2 F_{n-1}F_{n-2} = \frac{F_{n-1}}{F_{n-2}} \left(\prod_{i=2}^{n-2} F_{i-1}\right)^2 F_{n-2}F_{n-2}$$

$$= \frac{F_{n-1}}{F_{n-2}} \Delta_{n-1}^2.$$

So the proof of (2) is completed.
Equality (3) can be easily proved by mathematical induction on n. □

Theorem 2.8.
Let F be a matrix as in (1), then determinant of Hadamard inverse of F is

$$det(F^{\circ(-1)}) = \frac{(-1)^{n-1}}{F_n} \prod_{i=2}^{n} \frac{1}{F_{i+1}}.$$

Proof. Let F be a matrix as in (1) then by definition of Hadamard inverse we have

$$F^{\circ(-1)} = \begin{bmatrix} \frac{1}{F_2} & \frac{1}{F_2} & \frac{1}{F_2} & \cdots & \frac{1}{F_2} \\ \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_3} & \cdots & \frac{1}{F_3} \\ \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots & \frac{1}{F_4} \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{F_2} & \frac{1}{F_3} & \frac{1}{F_4} & \cdots & \frac{1}{F_{n+1}} \end{bmatrix}.$$

By using elementary row operations we get

$$det(F^{\circ(-1)}) = det \begin{bmatrix} \frac{1}{F_2} & \frac{1}{F_2} & \frac{1}{F_2} & \cdots & \frac{1}{F_2} \\ 0 & \frac{1}{F_3}-\frac{1}{F_2} & \frac{1}{F_3}-\frac{1}{F_2} & \cdots & \frac{1}{F_3}-\frac{1}{F_2} \\ 0 & 0 & \frac{1}{F_4}-\frac{1}{F_3} & \cdots & \frac{1}{F_4}-\frac{1}{F_3} \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \frac{1}{F_{n+1}}-\frac{1}{F_n} \end{bmatrix}.$$

So we get

$$det(F^{\circ(-1)}) = \frac{1}{F_2}\prod_{i=2}^{n}(\frac{1}{F_{i+1}}-\frac{1}{F_i}) = \prod_{i=2}^{n}\left(\frac{F_i - F_{i+1}}{F_i F_{i+1}}\right)$$

$$= \prod_{i=2}^{n}\left(\frac{F_{i-1}}{F_i F_{i+1}}\right) = \frac{(-1)^{n-1}}{F_n}\prod_{i=2}^{n}\frac{1}{F_{i+1}}.$$

□

Theorem 2.9.
Let $F^{\circ(-1)}$ be the Hadamard inverse of matrix F, then $F^{\circ(-1)}$ is invertible and the inversion of $F^{\circ(-1)}$ is $(F^{\circ(-1)})^{-1} =$

$$= \begin{bmatrix} -F_2 & \frac{F_1F_2}{F_1} & 0 & & \cdots & & & 0 \\ \frac{F_1F_2}{F_1} & -\frac{(F_3)^3}{F_1F_2} & \frac{F_3F_4}{F_2} & 0 & & & \cdots & 0 \\ 0 & \frac{F_3F_4}{F_2} & -\frac{(F_4)^3}{F_2F_3} & \frac{F_4F_5}{F_3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{F_4F_5}{F_3} & -\frac{(F_5)^3}{F_3F_4} & \frac{F_5F_6}{F_4} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & & \cdots & 0 & \frac{F_{n-1}F_n}{F_{n-2}} & -\frac{(F_{n+1})^3}{F_{n-1}F_{n-2}} & \frac{F_nF_{n+1}}{F_{n-1}} \\ 0 & 0 & & \cdots & & 0 & \frac{F_nF_{n+1}}{F_{n-1}} & -\frac{F_nF_{n+1}}{F_{n-1}} \end{bmatrix}.$$

Proof. We can prove this theorem by similar method which is used in Theorem 2.2. □

Theorem 2.10.
Let $e^{\circ F}$ be a matrix as in (2), then we have

$$det(e^{\circ F}) = e^{F_2}\prod_{i=2}^{n}\left(e^{F_{i+1}} - e^{F_i}\right).$$

Proof. The proof is similar to Theorem 2.1. □

Theorem 2.11.
Let $e^{\circ F}$ be a matrix as in (2), then $e^{\circ F}$ is invertible and the inversion of $e^{\circ F}$ is $(e^{\circ F})^{-1} =$

$$\begin{bmatrix} \frac{1}{e-1} & \frac{-1}{e(e-1)} & 0 & & & \cdots & & 0 \\ \frac{-1}{e(e-1)} & \frac{-(e+1)}{e(e-1)} & \frac{-1}{e^2(e-1)} & 0 & & & \cdots & 0 \\ 0 & \frac{-1}{e^2(e-1)} & \frac{-(e^2+e+1)}{e^3-e^5} & \frac{1}{e^3-e^5} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{e^3-e^5} & A & \frac{1}{e^5-e^8} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \cdots & 0 & \frac{1}{e^{F_{n-1}}-e^{F_n}} & B & \frac{1}{e^{F_n}-e^{F_{n+1}}} \\ 0 & 0 & & \cdots & & 0 & \frac{1}{e^{n-2}(e-1)} & \frac{-1}{e^{n-1}(e-1)} \end{bmatrix},$$

where

$$A=-\frac{(e^4+e^3+e^2+e+1)}{e^5(-e^4-e^3+e+1)},$$

and

$$B=-\frac{\sum_{k=0}^{F_n-1}e^k}{e^{F_n}(\sum_{k=F_{n-1}}^{F_n-1}e^k+\sum_{k=0}^{F_{n-2}-1}e^k)}.$$

Proof. We can prove this theorem by similar method which is used in Theorem 2.2. □

Theorem 2.12.
Let $e^{\circ F}$ be a matrix as in (2), then $e^{\circ F}$ is a positive definite matrix and all eigenvalues of $e^{\circ F}$ are positive.

Proof. By Theorem 2.10 we know that for each $n \geq 1$, determinant of $e^{\circ F}$ is positive. So all leading principal minors of $e^{\circ F}$ are positive. Thus by [13], $e^{\circ F}$ is a positive definite matrix. Consequently all eigenvalues of $e^{\circ F}$ are positive. □

Theorem 2.13.
Let $e^{\circ F}$ be a matrix as in (2), then we have the following upper bound for the spectral norm of $e^{\circ F}$.

$$\|e^{\circ F}\|_2 \leq \sqrt{(e^{2F_n}+n-1)(e^{2F_{n+1}}+n-1)}.$$

Proof. By definition of Hadamard product for $e^{\circ F}$ we have

$$e^{\circ F}=\begin{bmatrix} 1 & e^{F_2} & e^{F_2} & \cdots & e^{F_2} \\ 1 & 1 & e^{F_3} & \cdots & e^{F_3} \\ 1 & 1 & 1 & \cdots & e^{F_4} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1e^{F_{n+1}} \end{bmatrix} \circ \begin{bmatrix} e^{F_2} & 1 & 1 & \cdots & 1 \\ e^{F_2} & e^{F_3} & 1 & \cdots & 1 \\ e^{F_2} & e^{F_3} & e^{F_4}1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ e^{F_2} & e^{F_3} & e^{F_4} & \cdots & e^{F_n}1 \end{bmatrix}$$

$$= A \circ B.$$

By definition of maximum row length norm and maximum column length norm we have

$$r_1(A)=max_i\sqrt{\sum_j |a_{ij}|^2}=\sqrt{e^{2F_{n+1}}+n-1}.$$

Also we have

$$c_1(B)=max_j\sqrt{\sum_i |b_{ij}|^2}=\sqrt{e^{2F_n}+n-1}.$$

Thus according to (5) we obtain

$$\|e^{\circ F}\|_2 \leq \sqrt{(e^{2F_n}+n-1)(e^{2F_{n+1}}+n-1)}.$$

□

Theorem 2.14.
Let $e^{\circ F}$ be a matrix as in (2) and D_n denotes the leading principal minor of $e^{\circ F}$, in exact $D_1=e^{F_2}=e$ and $D_2=e^{F_2}(e^{F_3}-e^{F_2})=e^2(e-1),\cdots,D_n=det[e^{\circ(F_{min(i,j)+1})}]_{i,j=1}^n$, then we have

(1) $D_nD_{n-2} \geq D_{n-1}^2$.
(2) $D_nD_{n-2}=e^{F_{n-2}}D_{n-1}^2$.

Proof. By Theorem 2.10 we have

$$D_n=det[e^{\circ(F_{min(i,j)+1})}]_{i,j=1}^n=e^{F_2}\prod_{i=2}^n(e^{F_{i+1}}-e^{F_i}).$$

So we obtain

$$D_nD_{n-2}=e^{2F_2}(e^{F_{n+1}}-e^{F_n})(e^{F_n}-e^{F_{n-1}})\left(\prod_{i=2}^{n-2}(e^{F_{i+1}}-e^{F_i})\right)^2$$

$$\geq e^{2F_2}(e^{F_n}-e^{F_{n-1}})(e^{F_n}-e^{F_{n-1}})\left(\prod_{i=2}^{n-2}\left(e^{F_{i+1}}-e^{F_i}\right)\right)^2 = D_{n-1}^2.$$

Thus the proof of (1) is completed.
Now from (1) we know

$$D_nD_{n-2} = e^{2F_2}(e^{F_{n+1}}-e^{F_n})(e^{F_n}-e^{F_{n-1}})\left(\prod_{i=2}^{n-2}\left(e^{F_{i+1}}-e^{F_i}\right)\right)^2.$$

Thus we get

$$D_nD_{n-2} = \frac{e^{F_{n+1}}-e^{F_n}}{e^{F_n}-e^{F_{n-1}}}D_{n-1}^2 = \frac{e^{F_n}}{e^{F_{n-1}}}D_{n-1}^2 = e^{F_n-F_{n-1}}D_{n-1}^2 = e^{F_{n-2}}D_{n-1}^2.$$

So the proof of (2) is completed.

□

Table 1. Determinant and eigenvalues of F for some values of n

n	det(F)	Eigenvalues of F (is rounded of to three decimal places)
2	1	0.382, 2.618
3	1	0.308, 0.643, 5.049
4	2	0.301, 0.542, 1.403, 8.755
5	6	0.300, 0.532, 1.000, 2.573, 14.595
6	30	0.300, 0.532, 0.954, 1.844, 4.466, 23.903
7	240	0.300, 0.533, 0.952, 1.733, 3.134, 7.478, 38.871
8	3120	0.300, 0.533, 0.952, 1.725, 2.920, 5.257, 12.294, 63.019
9	65520	0.300, 0.533, 0.952, 1.725, 2.907, 4.872, 8.635, 20.031, 102.048
10	2227680	0.300, 0.533, 0.952, 1.725, 2.904, 4.841, 7.988, 14.084, 32.507, 165.167

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