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A study on fractional order magneto-thermoelasticity with three-phase-lag

Research Article

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Abstract: In this work, we consider a two-dimensional fractional order generalized thermoelastic problem in a homogeneous, isotropic and perfectly conducting thermoelastic half-space subjected to a moving load. The surface of half-space is initially placed in an external magnetic field with constant intensity and therefore, Maxwell's theory of electrodynamics has been effectively introduced. The basic governing equations of the problem are derived in the context of fractional order three phase lag model of generalized thermoelasticity (Ezzat et al. [35]). The formulation is solved by using Laplace-Fourier transform technique and inversion is carried out numerically. Numerical results are computed and represented graphically for the displacement, temperature and stress distributions. Effects of fractional order parameter and magnetic field on the different thermoelastic fields are analyzed on the basis of analytical and numerical results. Some special cases have also been deduced from the present investigation.

MSC: 74A15 • 80A20

Keywords: Fractional order three-phase-lag model • Generalized thermoelasticity • Magnetic field • Laplace-Fourier transforms • Moving load

1. Introduction

Based on thermodynamical principle of irreversible processes, Biot [1] proposed the coupled theory of thermoelasticity by including the coupling between thermal and strain fields. This theory removes the first drawback but shares the second defect of the uncoupled theory due to the presence of parabolic type heat conduction equation. Many approaches were introduced to overcome this unacceptable prediction of the classical uncoupled and coupled theories. The general notion of relaxing the heat flux in the classical Fourier heat conduction equation is the basis of these approaches.

Lord and Shulman [2] introduced the theory of generalized thermoelasticity for an isotropic body, referred to as L-S theory or extended thermoelasticity theory. The L-S theory postulated a wave-type heat conduction law to replace the classical Fourier's law and ensures finite speed of propagation of the thermal and elastic waves. Green and Lindsay [3] proposed another generalized thermoelasticity theory termed as G-L theory or temperature-rate dependent theory of thermoelasticity. G-L theory modified both the energy equation and the Duhamel-Neumann relation and incorporates two relaxation times. The extension of the L-S theory to the anisotropic case has been established by Dhaliwal and Sherief [4]. The development of the generalized theories has been reviewed by Chandrasekharaiah [5]. Green and Naghdi [6–8] formulated three models of thermoelasticity for homogeneous and isotropic materials referred to as G-N models of type I, II and III. Sur and Kanoria [9] presented an analysis of thermoelastic response in a functionally graded infinite space subjected to a Mode-I crack under Green-Naghdi generalized thermoelasticity theory using normal mode analysis.

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Subsequently, Tzou [10] proposed a dual-phase-lag heat conduction model to incorporate the effects of microscopic interactions in the fast-transient process of heat transport mechanism in a macroscopic formulation. Two different phase-lags τ_T and τ_q have been introduced in the constitutive relation between heat flux vector and the temperature gradient. The delay time τ_T is interpreted as that caused by the micro-structural interactions and is called the phase-lag of the temperature gradient. The other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of the thermal inertia, which is called the phase-lag of the heat flux.

Roy Choudhuri [11] proposed a generalized mathematical model of a coupled thermoelasticity theory that includes three-phase-lag τ_T, τ_q and τ_v for the heat flux vector, the temperature gradient and the thermal displacement gradient respectively. Quintanilla and Racke [12] discussed the stability in the three-phase-lag heat conduction equation and the relations among the three different phase lags τ_T, τ_q and τ_v to be satisfied. Deswal and Kalkal [13] solved a three-dimensional problem by employing normal mode analysis under two-temperature thermo-viscoelasticity theory with three-phase-lag heat transfer. Sur and Kanoria [14] investigated the elasto-thermodiffusive interactions in a homogeneous and isotropic half-space under initial hydrostatic stress in the context of three-phase-lag theory, Green-Naghdi type II theory and Green-Naghdi type III theory of generalized thermoelasticity.

Kaliski and Nowacki [15] investigated the magneto-thermoelastic disturbances generated by a thermal shock in an elastic half-space, having a finite conductivity. A survey of relevant magneto-thermoelasticity theories was studied by Willson [16], Purushothama [17], Paria [18] etc., in the second half of the last century. Nayfeh and Nemat-Nasser [19] studied the propagation of plane waves under generalized magneto-thermoelasticity for a solid medium. Deswal and Kalkal [20] considered Green and Lindsay theory to study a problem on generalized magneto-thermo-viscoelasticity with diffusion in an elastic half-space subjected to different loadings. Deswal *et al.* [21] illustrated a magneto-thermoelastic problem with laser pulse heating in the context of Green-Naghdi theory of type III.

In recent years, there has been a growing interest in the area of fractional calculus and its applications, which includes mechanics of solids, field theory and control theory etc. By applying fractional calculus, several interesting models have been established successfully to study the physical processes particularly in electricity, heat conduction, dielectrics and semiconductors, glasses, porous, polymer chains, biological systems, diffusion problems, viscoelasticity etc. There are some materials (e.g. porous materials, polymers and colloids, glassy etc.) and physical situations (like low-temperature, amorphous media and transient loading etc.) where the classical thermoelasticity theory based on the classical Fourier's law is inconsistent. In such cases, one needs to use a generalized thermoelastic (more generally thermo-viscoelastic) model based on time-fractional anomalous diffusion equation. The most important advantage of using fractional differential equations in these and other applications is their non-local property.

The first application of the fractional derivative was given by Abel [22], who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. A survey of many emerging applications of the fractional calculus in the area of science and engineering has appeared in the text of Podlubny [23]. The review article by Rossikhin and Shitikova [24] is devoted to the analysis of new trends and results in the field of fractional calculus and its application to dynamic problems of solid mechanics. The results obtained are critically estimated in view of the role of fractional calculus in engineering problems. A quasi-static uncoupled theory of thermoelasticity based on a fractional heat conduction equation was introduced by Povstenko [25]. Many applications have been solved in the context of the fractional order generalized thermoelasticity by Povstenko [26], who investigated stresses due to the fractional heat conduction law in an infinite body with a circular cylindrical hole.

Youssef [27] established a theory of generalized thermoelasticity in the context of a new consideration of the heat conduction equation with fractional order time derivatives, proved uniqueness theorem and presented one-dimensional application. Sherief *et al.* [28] introduced a new model of thermoelasticity using fractional calculus, proved a uniqueness theorem and derived a reciprocity relation and a variational principle. Ezzat [29] introduced another new model of fractional heat conduction equation by using the new Taylor's series expansion of time-fractional order, developed by Jumarie [30]. Several authors studied a number of problems by employing the above mentioned theories of fractional order generalized thermoelasticity in solid medium. Some of them are found in the references [31–34].

Ezzat *et al.* [35] proposed a new model of thermoelasticity with three-phase-lag heat conduction in the context of a new consideration of time-fractional order Fourier's law of heat conduction, proved uniqueness and reciprocity theorems, and solved one-dimensional problem for an elastic half-space in the presence of heat sources. In this model, time-fractional order heat conduction equation takes the form

$$\left(1+\frac{\tau_q^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\vec{q}=-\left[\left(\tau_v^*+k\frac{\tau_T^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\vec{\nabla}\theta+k^*\vec{\nabla}v\right],$$

where $\tau_v^*=k+k^*\dfrac{\tau_v^{\alpha}}{\Gamma(\alpha+1)}\dfrac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}$, $0<\alpha\le 1$. Ezzat *et al.* [36] solved one dimensional problem in a homogeneous isotropic infinitely extended thermoelastic medium with a periodically varying heat source.

The present article is an attempt to study a problem on magneto-thermo-elastic interactions in a homogeneous, isotropic and perfectly conducting solid half-space under fractional order theory of thermoelasticity with three-phase-lag. The solution of the problem is first obtained in the Laplace-Fourier transforms space and then in the physical domain by using a numerical inversion technique of integral transforms. The derived expressions are computed numerically for copper material and the results are presented in graphical form. Effects of fractional order parameter and magnetic field on the variation of different field quantities are also analyzed.

2. Formulation of problem and solution

The constitutive equations and field equations for a homogeneous, isotropic and perfectly conducting elastic solid under the fractional order theory of generalized magneto-thermoelasticity can be written as (Ezzat *et al.* [35]):
(i) Equation of motion:

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i, \tag{1}$$

where u_i are the components of displacement vector, ρ is the density of the medium, σ_{ij} are the components of the stress tensor and F_i are the components of the Lorentz body force vector $\vec{F}$ given by $\vec{F} = \mu_0 \vec{J} \times \vec{H}$, μ_0 is the magnetic permeability, $\vec{H}$ is the applied magnetic field, $\vec{J}$ is the current density vector, a comma followed by a suffix denotes spatial derivative and a superposed dot denotes the derivative with respect to time t.
(ii) Time-fractional heat conduction equation with three-phase-lag

$$\left[k^*\left(1+\frac{\tau_v^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right)+k\frac{\partial}{\partial t}\left(1+\frac{\tau_T^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right)\right]\theta_{,ii}=\left(1+\frac{\tau_q^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_E\ddot{\theta}+\beta_1 T_0\ddot{e}\right), \tag{2}$$

where $\theta = T - T_0$, T is the absolute temperature, T_0 is reference temperature assumed to obey the inequality $|\theta/T_0| \ll 1$, C_E is the specific heat, k^* is material constant given by $k^* = \dfrac{C_E(\lambda+2\mu)}{4}$, k is the thermal conductivity, $\beta_1 = (3\lambda+2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, λ and μ are the Lame's constants, τ_T, τ_v and τ_q are relaxation times, $e = e_{kk}$ is the cubical dilatation.

(iii) The stress-strain-temperature relations

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \beta_1\theta)\delta_{ij}, \tag{3}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

where e_{ij} are the components of strain tensor, δ_{ij} is the Kronecker delta function.

The linearized equations of Maxwell's theory of elctromagnatism are

$$\text{curl}\,\vec{h} = \vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}, \tag{5}$$

$$\text{curl}\,\vec{E} = -\mu_0\frac{\partial\vec{h}}{\partial t}, \tag{6}$$

$$\vec{E} = -\mu_0\left(\frac{\partial\vec{u}}{\partial t}\times\vec{H}\right), \tag{7}$$

$$\text{div}\,\vec{h} = 0, \tag{8}$$

where

$$\vec{h} = \text{curl}(\vec{u}\times\vec{H}), \tag{9}$$

and magnetic field $\vec{H}$ produces an induced magnetic field $\vec{h}$ and an induced electric field $\vec{E}$, ε_0 is the electric permittivity.

We have considered a two-dimensional problem in xz-plane with x-axis is along the direction of the bounding plane and z-axis is vertically downward in the medium such that the half-space occupies the region $z \geq 0$. A uniform magnetic field $\vec{H} = (0, H_0, 0)$ is acting on the xz-plane, which produces an induced magnetic field $\vec{h} = (0, h_2, 0)$ and induced electric field $\vec{E} = (E_1, 0, E_3)$. The displacement vector $\vec{u}$ has the components

$$u = u_x = u(x,z,t),\;\; v = u_y = 0,\;\; w = u_z = w(x,z,t). \tag{10}$$

Using Eqs. (5)-(9) in the relation $\vec{F} = \mu_0\vec{J}\times\vec{H}$, we get the components of Lorentz's force as

$$\vec{F} = (F_x, F_y, F_z) = -\varepsilon_0\mu_0^2 H_0^2\left(\frac{\partial^2 u}{\partial t^2}, 0, \frac{\partial^2 w}{\partial t^2}\right). \tag{11}$$

It is more convenient to introduce non-dimensional variables as follows:

$$(x', z', u', w') = c_0\eta_0(x, z, u, w),\;\; (t', \tau'_T, \tau'_q, \tau'_v) = c_0^2\eta_0(t, \tau_T, \tau_q, \tau_v),$$
$$\sigma'_{ij} = \frac{1}{\mu}\sigma_{ij},\;\; \theta' = \frac{\beta_1}{(\lambda+2\mu)}\theta,\;\; F'_0 = \frac{F_0}{\mu}, \tag{12}$$

where $c_0^2 = \frac{\lambda + 2\mu}{\rho}$, $\eta_0 = \frac{\rho C_E}{k}$.

By introducing the above dimensionless quantities, Eqs. (1)-(3) assume the form (where the primes are suppressed for convenience)

$$\alpha_1 \frac{\partial^2 u}{\partial t^2} = \frac{1}{\beta^2}\nabla^2 u + \frac{\beta^2 - 1}{\beta^2}\frac{\partial e}{\partial x} - \frac{\partial \theta}{\partial x}, \tag{13}$$

$$\alpha_1 \frac{\partial^2 w}{\partial t^2} = \frac{1}{\beta^2}\nabla^2 w + \frac{\beta^2 - 1}{\beta^2}\frac{\partial e}{\partial z} - \frac{\partial \theta}{\partial z}, \tag{14}$$

$$\left[k_1\left(1 + \frac{\tau_v^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right) + \frac{\partial}{\partial t}\left(1 + \frac{\tau_T^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right)\right]\nabla^2\theta = \left(1 + \frac{\tau_q^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\ddot{\theta} + \varepsilon \ddot{e}\right), \tag{15}$$

$$\sigma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \tag{16}$$

$$\sigma_{zz} = 2\frac{\partial w}{\partial z} + (\beta^2 - 2)e - \beta^2\theta, \tag{17}$$

where

$$\alpha_1 = 1 + \frac{\varepsilon_0 \mu_0^2 H_0^2}{\rho},\ \beta^2 = \frac{\lambda + 2\mu}{\mu},\ k_1 = \frac{k^*}{C_E(\lambda + 2\mu)},\ \varepsilon = \frac{\beta_1^2 T_0}{\rho C_E(\lambda + 2\mu)}.$$

By using the Helmholtz decomposition $\vec{u} = \nabla\phi + \nabla \times \vec{\psi}$, $\vec{\psi} = (0, -\psi, 0)$, the displacement components can be written in the form

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}. \tag{18}$$

where the two functions ϕ and $\vec{\psi}$ are known in the theory of elasticity as Lame's potentials, resulting in irrotational and rotational parts respectively of the displacement vector $\vec{u}$.

Simplifying Eqs. (13)-(17) using the relations in Eq. (18), we obtain

$$\left(\alpha_1\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \theta = 0, \tag{19}$$

$$\left(\alpha_1\frac{\partial^2}{\partial t^2} - \frac{1}{\beta^2}\nabla^2\right)\psi = 0, \tag{20}$$

$$\left[k_1\left(1 + \frac{\tau_v^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right) + \frac{\partial}{\partial t}\left(1 + \frac{\tau_T^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right)\right]\nabla^2\theta = \left(1 + \frac{\tau_q^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\ddot{\theta} + \varepsilon\nabla^2\ddot{\phi}\right), \tag{21}$$

$$\sigma_{zx} = 2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x^2}, \tag{22}$$

$$\sigma_{zz} = 2\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + (\beta^2 - 2)\nabla^2\phi - \beta^2\theta. \tag{23}$$

Now, we define the Laplace transform of the function $f(x,z,t)$ with respect to variable t as

$$L[f(x,z,t)] = \bar{f}(x,\ z,\ s) = \int_0^\infty f(x,z,t)e^{-st}dt, \tag{24}$$

and the Fourier transform of the function $\bar{f}(x,\ z,\ s)$ with respect to variable x as

$$F[\bar{f}(x,z,s)] = \hat{f}(\xi,z,s) = \int_{-\infty}^{\infty} \bar{f}(x,z,s)e^{\iota\xi s}dx, \tag{25}$$

where s and ξ are the Laplace and Fourier transform parameters respectively.

Introducing Laplace and Fourier transforms into Eqs. (19)-(21) and employing elimination procedure, we get the following differential equations

$$\left(\frac{d^4}{dz^4} + L\frac{d^2}{dz^2} + M\right)(\hat{\phi}, \hat{\theta}) = 0, \tag{26}$$

and

$$\left(\frac{d^2}{dz^2} - \lambda_3^2\right)\hat{\psi} = 0, \tag{27}$$

where

$$L = -\frac{2\varepsilon_1\xi^2 + (\alpha_1\varepsilon_1 + \varepsilon_2(1+\varepsilon))s^2}{\varepsilon_1},$$
$$M = \frac{\varepsilon_1\xi^4 + (\alpha_1\varepsilon_1 + \varepsilon_2(1+\varepsilon))s^2\xi^2 + \alpha_1\varepsilon_2 s^4}{\varepsilon_1},$$
$$\lambda_3^2 = \xi^2 + \alpha_1\beta^2 s^2.$$

Now, Eq. (26) can be factorized as

$$\left(\frac{d^2}{dz^2} - \lambda_1^2\right)\left(\frac{d^2}{dz^2} - \lambda_2^2\right)(\hat{\phi}, \hat{\theta}) = 0, \tag{28}$$

where $\lambda_1^2 = \dfrac{-L + \sqrt{L^2 - 4M}}{2}$ and $\lambda_2^2 = \dfrac{-L - \sqrt{L^2 - 4M}}{2}$.

Following the regularity condition that the solution is bounded at infinity, the solution of Eqs. (27) and (28) can be expressed as

$$\hat{\phi} = A_1 e^{-\lambda_1 z} + A_2 e^{-\lambda_2 z}, \tag{29}$$
$$\hat{\theta} = B_1 e^{-\lambda_1 z} + B_2 e^{-\lambda_2 z}, \tag{30}$$
$$\hat{\psi} = A_3 e^{-\lambda_3 z}, \tag{31}$$

where A_i, B_i $(i = 1, 2)$ and A_3 are the unknown coefficients dependent on s and ξ such that $B_i = b_i A_i$ with $b_i = \varepsilon\varepsilon_2 s^2 \left(\dfrac{\lambda_i^2 - \xi^2}{\varepsilon(\lambda_i^2 - \xi^2) - \varepsilon_2 s^2}\right)$ $(i = 1, 2)$.

3. Application

We have considered a fractional order generalized thermoelastic half-space with magnetic field. The bounding surface $z = 0$ is subjected to moving load with velocity v. Mathematically, the boundary conditions of the problem can be expressed as:

$$\sigma_{zz}(x, z, t) = -F_0\delta(x - vt)H(t), \tag{32}$$
$$\sigma_{zx}(x, z, t) = 0, \tag{33}$$
$$\theta(x, z, t) = 0, \tag{34}$$

where $\delta(.)$ is a Dirac delta function and $H(.)$ is a Heaviside unit step function.

Applying Laplace-Fourier transform to Eqs. (32), (33) and using the relevant expressions of Eqs. (22), (23) and (29)-(31), we obtain the following linear system of equations

$$P_1A_1 + P_2A_2 + P_3A_3 = 0, \tag{35}$$
$$Q_1A_1 + Q_2A_2 + Q_3A_3 = 0, \tag{36}$$
$$R_1A_1 + R_2A_2 = 0, \tag{37}$$

where

$$P_i = \left[\left(\beta^2 - 2\right)(\lambda_i^2 - \xi^2) + 2\iota\xi\lambda_i - \beta^2 b_i\right],\ Q_i = 2\iota\xi\lambda_i,\ R_i = b_i\ (i = 1, 2),$$
$$P_3 = 2\xi^2,\quad Q_3 = \lambda_3^2 + \xi^2,\quad P = -\frac{F_0}{s - \iota\xi v}. \tag{38}$$

Solving Eqs. (35)-(37), we obtain

$$A_1 = \frac{\Delta_1}{\Delta},\quad A_2 = \frac{\Delta_2}{\Delta},\quad A_3 = \frac{\Delta_3}{\Delta}, \tag{39}$$

where

$$\Delta = R_1(P_2Q_3 - Q_2P_3) - R_2(P_1Q_3 - Q_1P_3),$$
$$\Delta_1 = -PR_2Q_3,\ \Delta_2 = PR_1Q_3,\ \Delta_3 = P(Q_1R_2 - R_1Q_2), \tag{40}$$

Now, using the solution obtained in Eq. (39) along with Eqs. (18), (22)-(25) and (29)- (31) simultaneously, we obtain the expressions of the fields in the Laplace-Fourier transform domain as

$$\hat{u} = -\frac{1}{\Delta}\left[\iota\xi\left(\Delta_1 e^{-\lambda_1 z} + \Delta_2 e^{-\lambda_2 z}\right) + \lambda_3\Delta_3 e^{-\lambda_3 z}\right], \tag{41}$$

$$\hat{w} = -\frac{1}{\Delta}\left[\lambda_1\Delta_1 e^{-\lambda_1 z} + \lambda_2\Delta_2 e^{-\lambda_2 z} + \iota\xi\Delta_3 e^{-\lambda_3 z}\right], \tag{42}$$

$$\hat{\theta} = \frac{1}{\Delta}\left[b_1\Delta_1 e^{-\lambda_1 z} + b_2\Delta_2 e^{-\lambda_2 z}\right], \tag{43}$$

$$\hat{\sigma}_{zz} = \frac{1}{\Delta}\left[C_1\Delta_1 e^{-\lambda_1 z} + C_2\Delta_2 e^{-\lambda_2 z} - C_3\Delta_3 e^{-\lambda_3 z}\right], \tag{44}$$

$$\hat{\sigma}_{zx} = \frac{1}{\Delta}\left[D_1\Delta_1 e^{-\lambda_1 z} + D_2\Delta_2 e^{-\lambda_2 z} + D_3\Delta_3 e^{-\lambda_3 z}\right], \tag{45}$$

where

$$C_i = (\beta^2 - 2)(\lambda_i^2 - \xi^2) + 2\iota\xi\lambda_i - \beta^2 b_i \;\; (i = 1,2), \;\; C_3 = 2\xi^2,$$
$$D_i = 2\iota\xi\lambda_i \;\; (i = 1,2), \;\; D_3 = \left(\lambda_3^2 + \xi^2\right).$$

4. Special cases

4.1. Neglecting fractional order effect

To discuss the phenomenon of wave propagation in the context of generalized magneto-thermoelasticity with three-phase-lag, we substitute $\alpha = 1$ in the heat conduction equation. Hence, we get the following modifications:

$$\eta_1 = 1 + \tau_v\omega, \;\; \eta_2 = 1 + \tau_T\omega, \;\; \varepsilon_2 = 1 + \tau_q\omega + \frac{\tau_q^2}{2}\omega^2. \tag{46}$$

4.2. Neglecting magnetic effect

If we assume that $H_0 = 0$ i.e. half-space is free from magnetic field. In this case, the Maxwell's equations of electromagnetism will disappear from the formulation and therefore, the components of Lorentz's force vector $\vec{F}$ will vanish. Now, considering above modifications in the equation of motion, we shall be left with the relevant problem in fractional order generalized tharmoelasticity with three-phase-lag and the corresponding expressions for displacements, temperature and stresses can be obtained from Eqs. (41)-(45).

5. Inversion of integral transforms

We shall now outline the numerical inversion method used to find the solution in the physical domain. The fields in the Laplace-Fourier transform domain are the functions of the form $\hat{f}(\xi,\, z,\, s)$. To find the functions in the physical domain i.e. in the form $f(x,\, z,\, t)$, we first invert the Fourier transform by using

$$F^{-1}[\hat{f}(\xi,z,s)] = \bar{f}(x,z,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi,z,s)e^{-\iota\xi s}d\xi. \tag{47}$$

Now, for fixed values of ξ, x and z, the function $\bar{f}(x,\, z,\, s)$ in Eq. (47) can be considered as the Laplace transform $\bar{f}(s)$ of the function $f(t)$. In order to find Laplace inverse transform of the function $\bar{f}(s)$ to get the function $f(t)$, we apply a numerical inversion method based on Fourier series expansion explained by Honig and Hirdes [37]. The integral in Eq. (47) is evaluated by following the procedure given in Rakshit and Mukhopadhyay [38].

6. Numerical results and discussion

To illustrate the analytical procedure presented earlier, we now consider a numerical example and give computational results, which depict the variation of temperature, displacement and stress fields in the context of fractional order magneto-thermoelasticity with three-phase-lag. The copper material is chosen for the purposes of numerical evaluations and the constants of the problem are taken as

$$k = 386\,Wm^{-1}K^{-1},\; \rho = 8954\,kgm^{-3},\; C_E = 383.1\,Jkg^{-1}K^{-1},$$
$$\alpha_t = 1.78\times10^{-5}\,K^{-1},\; \lambda = 7.76\times10^{10}\,kgm^{-1}s^{-2},\; T_0 = 293\,K,$$
$$\mu = 3.86\times10^{10}\,kgm^{-1}s^{-2},\; \varepsilon_0 = 10^{-9}/36\pi\,Fm^{-1},\; H_0 = 10^7/4\pi\,Am^{-1},$$
$$\mu_0 = 4\pi/10^7\,Hm^{-1}.$$

The relaxation time parameters are taken as $\tau_v = 0.001\ s$, $\tau_T = 0.0015\ s$, $\tau_q = 0.002\ s$, which agree with stability condition in [12]. Also, we have taken $F_0 = 10$, $\upsilon = 1.0$, $x = 1.0$ and $t = 0.1$ for computational purpose.

Figs. 1-3 are plotted to show the variations of displacement w, temperature θ and normal stress σ_{zz} against z for three different values of fractional order parameter α i.e. $\alpha = 0.25,\ 0.5,\ 1.0$. In figures 4-6, we have demonstrated the effect of magnetic field on the spatial variation of the physical fields w, θ and σ_{zz} at $\alpha = 0.5$. In figures 4-6, the following abbreviations are used: (i) FOTPLM (Fractional order three-phase-lag thermoelasticity with magnetic field) i.e. $H_0 \neq 0$ and (ii) FOTPL (Fractional order three-phase-lag thermoelasticity) i.e. $H_0 = 0$.

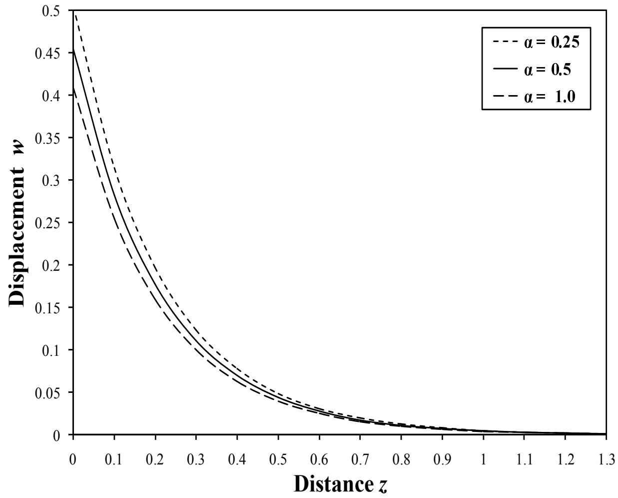

Fig. 1. Effect of fractional parameter α on the distribution of displacement w at $t = 0.1$

In Fig. 1, we have observed that the variations of displacement w corresponding to different values of fractional order parameter α follow almost similar pattern with difference in magnitudes and restricted to a bounded region. Outside this region, the variations vanish identically. Also, increase in the values of fractional parameter α acts to decrease displacement significantly.

It is evident from Fig. 2 that the profiles of temperature θ for $\alpha = 0.25,\ 0.5,\ 1.0$ have coincident initial point with zero value, which is consistent with the boundary conditions applied, afterward jump to attain maximum values and finally decrease gradually to diminish to zero value. It is also clear from the figure that the fractional order parameter α significantly affects the temperature distribution.

It is apparent from Fig. 3 that normal stress σ_{zz} begins with maximum modulus values for the cases $\alpha = 0.25,\ 0.5,\ 1.0$, afterward decreases gradually with an increase in distance z until reaches to the steady state. Moreover, fractional order parameter α has prominent effect on this fields.

It is revealed from the Figs. 4-6 that the magnetic field shows significant dominance on displacement component w, temperature θ and normal stress σ_{zz}. It is also concluded from these figures that all the variations show similar pattern for FOTPLM and FOTPL, and as the medium is made free from magnetic field (i.e. for FOTPL), the variations

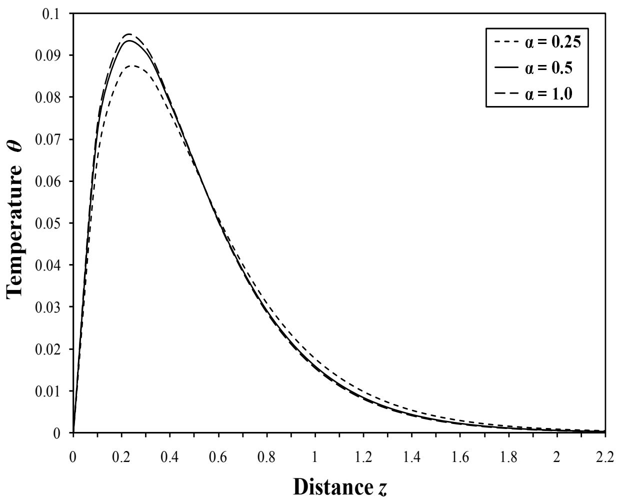

Fig. 2. Effect of fractional parameter α on the distribution of temperature θ at $t = 0.1$

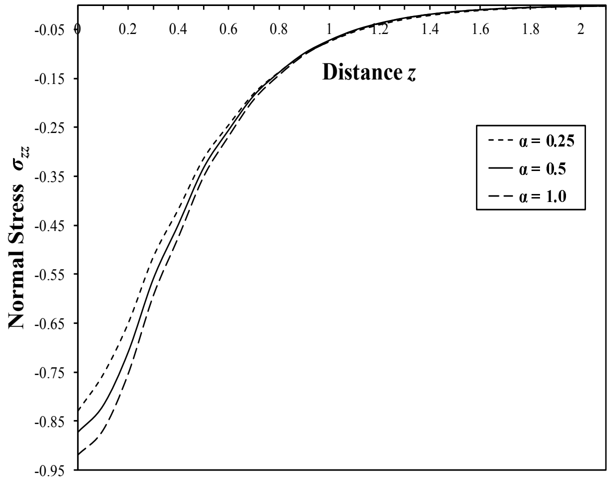

Fig. 3. Effect of fractional parameter α on the distribution of normal stress σ_{zz} at $t = 0.1$

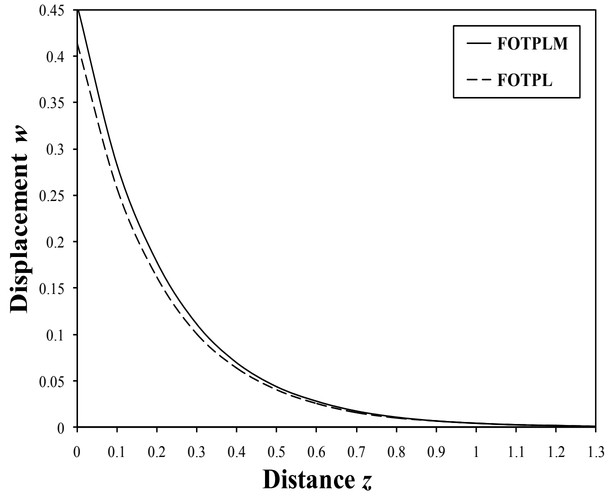

Fig. 4. Effect of magnetic field on the distribution of displacement w at $t = 0.1$ and $\alpha = 0.5$

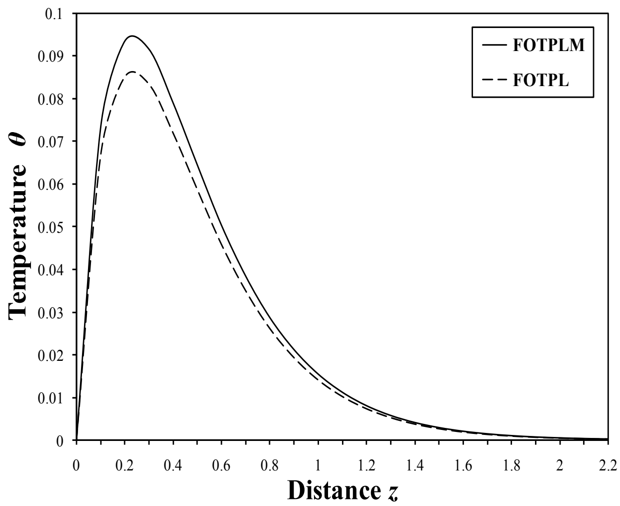

Fig. 5. Effect of magnetic field on the distribution of temperature θ at $t = 0.1$ and $\alpha = 0.5$

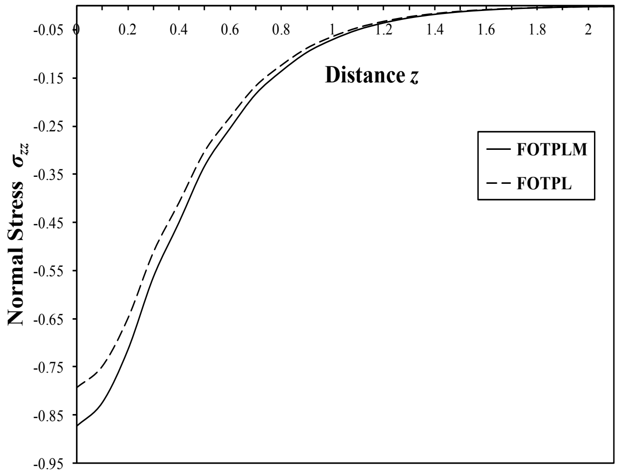

Fig. 6. Effect of magnetic field on the distribution of normal stress σ_{zz} at $t = 0.1$ and $\alpha = 0.5$

of the fields w, θ and σ_{zz} decrease numerically. Moreover, the phenomenon of finite wave speed is also predicted by these fields.

7. Conclusions

In the present paper we have constructed a model of fractional order generalized magneto-thermoelasticity with three-phase-lag for an elastic half-space and the general solutions are applied to a specific problem of a half-space subjected to moving load with a constant velocity.

It is clear from all the figures that all the field variables have nonzero values only in the bounded region of space. Outside this region, the values vanish identically. Hence, all the results are in agreement with the generalized theory of thermoelasticity.

We have observed from the Figs. 1-3 that the value of fractional order parameter plays a vital role on the wave propagation phenomenon of the displacement, temperature and stress fields. Also, the fractional order parameter has significant effect on all the studying fields.

It is concluded from the Fig. 4-6 that magnetic field has significant effect on all the field variables and in the absence of magnetic field the magnitude of the field variables decreases.

The profiles of temperature θ and normal stress σ_{zz} at the surface $z = 0$ are in accordance with the boundary conditions applied.

Theory of thermoelasticity with fractional order time derivatives is a new branch of research. The results presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists, as well as for those working on the development of a theory of hyperbolic thermoelasticity with fractional order.

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