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Large advantage of herd behavior of prey in Prey−Predator dynamics with disease in predator

Research Article

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Abstract: In prey−predator system prey population take various mechanisms to save themselves from predators. In this work we consider a prey−predator model with a transmissible disease spreading in predator population and prey gathered in a group so it exhibits a herd behavior. We used Holling type−II response function for spreading of disease among predators and find the impact of herd behavior mechanism of prey population in the model system both analytically and numerically. Sufficient conditions are derived for disease−free equilibrium and the interior equilibrium points of the model system. Numerical simulations are carried out to support our analytical findings.

MSC: 92D25

Keywords: Prey−predator model • Transmissible diseases in predator • Herd behavior of prey and Holling type−II funcional response

1. Introduction

Predator−prey interaction is an important area in ecology and mathematical ecology for which many problems still remain open [1] . In the last few decades a number of prey−predator models have been studied extensively, where several possible dynamics have been considered, starting from the famous work of Lotka [2] and Volterra [3] to the recent works [4]-[23] which depart from classical assumptions. Many researchers have shown that interaction between prey−predator and corresponding response function plays an important role in the dynamics of prey−predator models. Till now several classical response function namely Holling type−I, type−II, type−III, and type−IV, Tanner ratio dependence [7] , Beddington−DeAngelis [8] has been introduced. Also an increasing number of works are devoted to study of the relationships between demographic processes among different populations diseases. Most of these works dealt with prey−predator models with disease in prey such as [9–12]. In [9] Chattopadhyay and Arino consider a predator−prey model with diseases in prey. Xiao and Chen [11] formulated and analyzed a three species eco−epidemiological system, where it was assumed that the disease spreads among the prey population only and disease is not genetically inherited. They consider the case where the predator mainly eats only the infected prey and infected populations do not recover or immune. Recently, attention has been paid to the modelling and analysis of eco−epidemiological predator−prey system by assuming that the predator population suffer a transmissible disease (see, for example [13–17]). Venturino [13] formulated two eco−epidemiological models with disease in the predators where mass action and standard incident rates were considered as interaction among the species. On the other hand in a recent work of Ajraldi et al. [6] took a different view of interaction among the species. They argued the interactions not just of individuals of two populations that intermingle on a common ground, but consider a more elaborated social model, in which the individuals of one population gather together in herds, to wander about in search of food

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sources and for defensive purposes. The concept of group defense has already been considered, via suitable assumptions on the form and type of functional responses of the prey modeled in very general terms. Specially, there is a threshold on the size of herd of the prey beyond which the predator's hunting capabilities begin to fall. In other words, the larger the prey population is, the smaller the success of hunting and the corresponding return rate are for predators. More recently, Matia at el. [21] consider a prey−predator model where prey population exhibits herd behaviour as their own defense mechanism and second prey population release toxin elements which gives them advantage of inter species competition. In their model formulation Matia at el. [21] corporated square root term of prey population in the response function to capture the herd behaviour of prey population as suggested by Braza [20] . In this article we consider a prey−predator model where prey species gather together to form a herd for their own defense mechanism and disease spreads among the predator species only by direct contact. The disease can not be transmitted vertically. The infected predator population do not recover or immune. The detailed assumptions and corresponding mathematical model has been shown in the next section.

2. Mathematical model formation

Here the following assumptions are made to formulate our mathematical model:

(A1) In the absence of predation, the prey population $x(t)$ grows logistically with intrinsic growth rate r and carrying capacity k. Here we assume that the prey population gathered in a group to form a herd (as a part of their defensive mechanism) to save themselves from predator population.

(A2) The predator population $y(t)$ is divided in two classes; namely susceptible predator $y_1(t)$ and infected predator $y_2(t)$.

(A3) The disease spreads among the predator species only by direct contact and Holling type−II response function has been adapted to describe the spread of disease among the predators. The disease can not be transmitted vertically and the infected predator population does not recover or immune.

(A4) Only susceptible predators have ability to capture prey species and a square root term of prey population has been adapted in the response function function as suggested by Braza [20] and Matia [21] . The infected predator are unable to capture the prey.

(A5) Here 'd' is the natural death rate of predator and 'μ' is the disease induced extra death rate of the predator.

Under the above assumptions, mathematically we write down the following model system:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - a\sqrt{x}y_1,\\
\frac{dy_1}{dt} &= a\sqrt{x}y_1 - dy_1 - \lambda\frac{y_1y_2}{y_1+y_2},\\
\frac{dy_2}{dt} &= \lambda\frac{y_1y_2}{y_1+y_2} - (d+\mu)y_2.
\end{aligned}
\tag{1}
$$

2.1. Positive invariance of the system

Let us put Eq. (1) in a vector form by setting

$Y = col(x, y_1, y_2) \in R^3.$

$$
F(Y) = \begin{pmatrix} F_1(Y)\\ F_2(Y)\\ F_3(Y)\end{pmatrix} = \begin{pmatrix} rx\left(1-\frac{x}{k}\right) - a\sqrt{x}y_1\\ a\sqrt{x}y_1 - dy_1 - \lambda\frac{y_1y_2}{y_1+y_2}\\ \lambda\frac{y_1y_2}{y_1+y_2} - (d+\mu)y_2 \end{pmatrix},
$$

where $F: C_+ \to R^3$ and $F \in C^{\infty}(R^3)$. The Eq. (1) becomes $\dot{Y} = F(Y)$, with $Y(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C_+$ and $\phi_i(\theta) > 0$(i=1, 2, 3). It is easy to cheek in the above equation that whenever choosing $Y(\theta) \in C_+$ such that $Y_i = 0$, then $F_i(y) \mid y_i(t) = 0, y(t) \in C_+ \geq 0, (i = 1, 2, 3)$. Due to lemma (Yang et al. [24]) any solution of the above equation with $Y(\theta) \in C_+$, say $Y(t) = Y(t, Y(\theta))$, such that $Y(\theta) \in R^3$ for all $t > 0$.

2.2. Boundedness of the system

Let us define,

$w = x + y_1 + y_2.$

The time derivatives,

$$\frac{dw}{dt}=\frac{dx}{dt}+\frac{dy_1}{dt}+\frac{dy_2}{dt}.$$

Therefore we obtained as,

$$\frac{dw}{dt}=rx(1-\frac{x}{k})-dy_1-(d+\mu)y_2.$$

Now,

$$\frac{dw}{dt}+qw=xr(1-\frac{x}{k})-dy_1-(d+\mu)y_2+qx+qy_1+qy_2,$$

$$\frac{dw}{dt}+qw=(rx-\frac{rx^2}{k}+qx)-y_1(d-q)-y_2(d+\mu-q).$$

Now if $q<\min(d,d+\mu)$, then

$$\frac{dw}{dt}+qw\leqslant(rx-\frac{rx^2}{k}+qx)$$

$$\frac{dW}{dt}+qw\leqslant k\frac{(r+q)^2}{4r}=B(say)$$

Applying the result of differential inequality [25] we obtain,

$$0\leqslant w\Big((x(t),y_1(t),y_2(t)\Big)\leqslant\frac{B}{q}(1-e^{-qt})+w\Big((x(0),y_1(0),y_2(0)\Big)e^{-qt}.$$

Which implies that $0\leqslant w\leqslant\frac{B}{q}$ as $t\to\infty$. Hence all the solution of (1) are bounded.

3. Qualitative analysis of the model system

3.1. Equilibria and existence

The system of Eq. (1) has four equilibrium points, namely

(a) $E_0(0,0,0)$,

(b) $E_1(k,0,0)$,

(c) $E_2\Big(\frac{d^2}{a^2},\frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}),0\Big)$ and

(d) $E_3\Big(\frac{(\lambda-\mu)^2}{a^2}=\alpha^2,\frac{r\alpha}{a}(1-\frac{\alpha^2}{k}),(\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\Big)$.

It is easy to see that equilibria E_0 and E_1 exist for all parametric values of the model system. The equilibria E_2 exist if $ka^2-d^2>0$ and E_3 will be exist if $\frac{\alpha^2}{k}<1$ and $\frac{\lambda}{d+\mu}>1$.

3.2. Stability analysis

In this section we perform stability analysis of our model system (1). Here we first calculate the jacobian matrix of this system (1)

$$J=\begin{pmatrix} r-\frac{2rx}{k}-\frac{ay_1}{2\sqrt{x}} & -a\sqrt{x} & 0\\ \frac{ay_1}{2\sqrt{x}} & -a\sqrt{x}-d-\lambda\frac{y_2^2}{(y_1+y_2)^2} & -\lambda\frac{y_1^2}{(y_1+y_2)^2}\\ 0 & -\lambda\frac{y_2^2}{(y_1+y_2)^2} & -(d+\mu)\end{pmatrix}$$

Lemma 3.1.
The equilibrium point $E_0(0,0,0)$ is always unstable.

Proof. It is to be noted that when we evaluate the jacobian matrix J at $E_0(0,0,0)$ then several element of it takes indeterminant form. Thus it has been evaluated in limiting sense. The jacobian matrix at $E_0(0,0,0)$ is given by

$$\begin{pmatrix} r & 0 & 0 \\ 0 & -(d+\lambda) & -\lambda \\ 0 & -\lambda & -(d+\mu) \end{pmatrix}.$$

Obviously, one of its eigen values is r, which is positive. So the equilibrium point $E_0(0,0,0)$ is always unstable. □

Lemma 3.2.
The equilibrium point $E_1(k,0,0)$ is stable if $k^2 > \frac{d+\lambda}{a}$.

The proof of the above lemma is obvious and hence omitted.

Lemma 3.3.
The equilibrium point $E_2\left(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\right)$ is always locally asymptotically stable without any restriction of the parameter values of our model system.

Proof. The jacobian matrix of the system around the equilibrium point $E_2\left(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\right)$ is as follows:

$$\begin{pmatrix} r - \frac{2rd^2}{ka^2} - \frac{r}{2}(\frac{ka^2-d^2}{ka^2}) & -d & 0 \\ \frac{r}{2}(\frac{ka^2-d^2}{ka^2}) & 0 & -\lambda \\ 0 & 0 & -(d+\mu) \end{pmatrix}.$$

Considering $\alpha = \frac{ka^2-d^2}{ka^2}$, the jacobian matrix takes the form

$$\begin{pmatrix} r - \frac{2rd^2}{ka^2} - \frac{r}{2}\alpha & -d & 0 \\ \frac{r}{2}\alpha & 0 & -\lambda \\ 0 & 0 & -(d+\mu) \end{pmatrix}.$$

The characteristic root of this equation are $-(d+\mu)$, $-\frac{A}{2} + \frac{\sqrt{A^2-2rd\alpha}}{2}$ and $-\frac{A}{2} - \frac{\sqrt{A^2-2rd\alpha}}{2}$, where $A = (r - \frac{3r\alpha}{2})$. We see that the parameter r, α, d are all positive that is why $\sqrt{A^2-2rd\alpha}$ is always less than A. So all the roots are negative and the equilibrium point $E_2\left(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\right)$ is asymptotically stable. □

Lemma 3.4.
The interior equilibrium point $E_3\left(\frac{(\lambda-\mu)^2}{a^2} = \alpha^2, \frac{r\alpha}{a}(1-\frac{\alpha^2}{k}), (\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\right)$ is asymptotically stable when $A>0$, $AB-C>0$ and $C>0$.

Proof. The jacobian matrix of the system around the equilibrium point $E_3\left(\frac{(\lambda-\mu)^2}{a^2} = \alpha^2, \frac{r\alpha}{a}(1-\frac{\alpha^2}{k}), (\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\right)$ is as follows:

$$\begin{pmatrix} \frac{r}{2} - \frac{3r\alpha^2}{2k} & -a\alpha & 0 \\ \frac{r}{2}(1-\frac{\alpha^2}{k}) & a\alpha - d - (\frac{(\lambda-d-\mu)^2}{\lambda}) & -\frac{(d+\mu)^2}{\lambda} \\ 0 & (-\frac{(\lambda-d-\mu)^2}{\lambda}) & -(d+\mu) \end{pmatrix}.$$

The characteristic equation is $x^3 + Ax^2 + Bx + C = 0$, where

$$A = \left((\beta - a\alpha + d + \gamma) - (\frac{r}{2} - \frac{3r^2\alpha}{2k})\right),$$

$$B = -\left((\beta + d + \gamma - a\alpha)(\frac{r}{2} - \frac{3r^2\alpha}{2k}) + (d\beta - a\beta\alpha + \gamma\beta - \frac{\gamma\beta^2}{\lambda^2}) + (\frac{a\alpha\gamma}{2}(\frac{\alpha^2}{k} - 1))\right)$$

and

$$C = -\left((d\beta - a\alpha\beta + \beta\gamma - \frac{\gamma\beta^2}{\lambda^2})(\frac{r}{2} - \frac{3r\alpha^2}{2k}) + (\frac{ar\alpha}{2})(\frac{\beta\alpha^2}{k} - \beta)\right).$$

Using Routh-Hurwitz criterion, each of the three roots are with negative real part if $A>0$, $AB-C>0$ and $C>0$. □

4. Numerical simulation and discussions

It is to be noted that in the aforesaid lemmas analytically we established the conditions for stability of different equilibrium points, which are not only complicated to understand but also very complex to interpret the system dynamics from it. Thus, we perform numerically simulation in Matlab to observe the stability of the equilibrium points $E_1(k,0,0)$, $E_2\Big(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\Big)$ and $E_3\Big(\frac{(\lambda-\mu)^2}{a^2} = \alpha^2, \frac{r\alpha}{a}(1-\frac{\alpha^2}{k}), (\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\Big)$. Figs. 1-3 shows the stability of axial equilibrium point $E_1(k,0,0)$, disease free equilibrium point $E_2\Big(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\Big)$ and interior equilibrium point $E_3\Big(\frac{(\lambda-\mu)^2}{a^2} = \alpha^2, \frac{r\alpha}{a}(1-\frac{\alpha^2}{k}), (\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\Big)$ respectively.

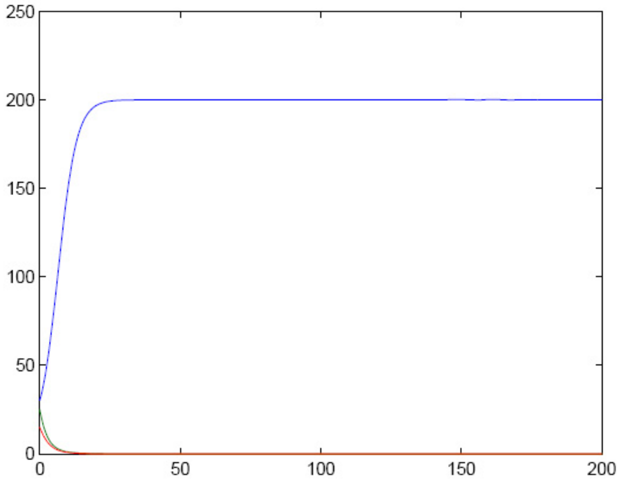

Fig. 1. Depict the stability of axial equilibrium point $E_1(k,0,0)$. The model system (1) has been solved using following set of values of parameters : $r = 0.3041$, $k = 200$, $a = 0.0155$, $d = 0.241$, $\lambda = 0.531$ and $\mu = 0.381$.

In the Lemma 3.4 we established the conditions for stability of the interior equilibrium point $E_3\Big(\frac{(\lambda-\mu)^2}{a^2} = \alpha^2, \frac{r\alpha}{a}(1-\frac{\alpha^2}{k}), (\frac{r\alpha}{a})(1-\frac{\alpha^2}{k})(\frac{\lambda}{d+\mu}-1)\Big)$. Numerically we also achieved the stable solutions for the interior equilibrium point under certain parametrical values as given in Fig. 3. But during numerical simulation we observed that the steady state of interior equilibrium point is not robust in the sense that slide change of these set of parametric values can drastically change the behaviour of the solution. We have simulated our model system by changing several parametric values and observe that natural death rate(d) and diseases induced extra death rate(μ) of predators and

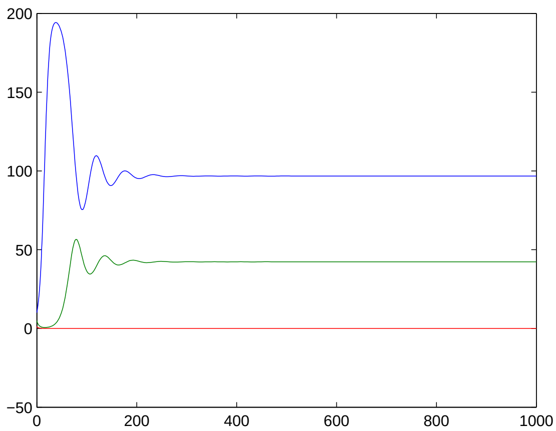

Fig. 2. Depict the stability of planar equilibrium point. The model system (1) has been solved using following set of values of parameters : $r = 0.2041$, $k = 200$, $a = 0.0245$, $d = 0.241$, $\lambda = 0.431$ and $\mu = 0.381$.

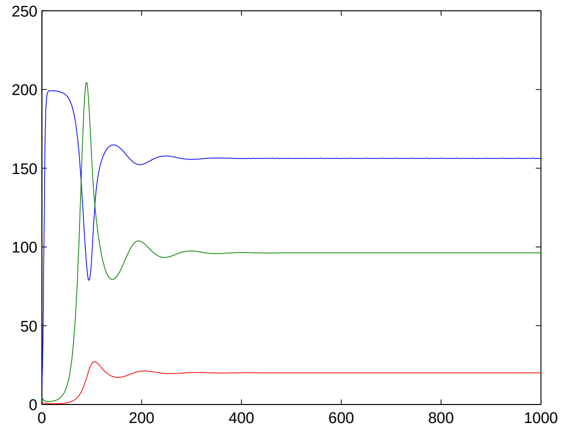

Fig. 3. Depict the stability of interior equilibrium point. The model system (1) has been solved using following set of values of parameters: $r = 0.7041$, $k = 200$, $a = 0.02$, $d = 0.141$, $\lambda = 0.631$ and $\mu = 0.381$.

rate of infection are relatively sensitive with respect to other model parameters.

Finally, we observe from analytical conditions (as given in Lemma 3.3) that disease free equilibrium point $E_2\left(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\right)$ is always asymptotically stable. During numerical simulation we observe that the stability of diseases free equilibrium point $E_2\left(\frac{d^2}{a^2}, \frac{rd}{a^2}(\frac{ka^2-d^2}{ka^2}), 0\right)$ is robust enough specially when the natural death rate(d) can be maintained in moderate level and diseases induced extra death rate(μ) exceeds a certain threshold level. If we observe all the equilibrium points of our model carefully then it is clear that there is no threat on prey species except at the trivial equilibrium point $E_0(0,0,0)$ which is fortunately always unstable. Thus overall analysis of our model system establish the fact that the prey species are always in a risk free zone except the extreme case where all the species goes to extinct [namely $E_0(0,0,0)$]. Thus we may conclude that herd behavior strategy is a good effective strategy for prey population as their own defensive mechanism specially when predator species suffers from diseases. On the other hand we noticed during numerical simulation of our model system that high death rate (both natural and disease induced) is a great threat for predator population. Thus, the scenario for the predator species will be more complicated for its survival if prey species release toxin elements (which increase the diseases induced death rate μ) along with their herd behavior strategy. In this case the predator species have no other option but have search for alternative food source and/or any other alternative strategy for its survival, which can be an interesting area for further research.

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