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On heat equation and its comparative solutions

Research Article

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Abstract: The comparative study is conducted between the Adomian decomposition method and the separation of variables method for solving heat equation. One dimensional heat equation with homogeneous boundary conditions and polynomial initial condition is solved using modified Adomian decomposition method and compared the solution by the separation of variables method.

MSC: 35K05 • 49M27

Keywords: Heat equations • Adomian decomposition method • Separation of variables method

1. Comparison of solutions of heat equations

The heat equations are solved using traditional separation of variables method. The method is very well-known for solving heat, wave and Laplace equations. However, the method is less efficient to solve some problems like non-homogeneous equation with non-homogeneous boundary conditions or variable coefficients equation. Adomian decomposition method is more efficient to solve ordinary and partial, non-homogeneous or homogeneous differential equations [1, 2]. Some recent work on Adomian decomposition method and Laplace decomposition method found in the literature (see [3, 4]). Also Adomian method gives relatively simple expression compared to the separation of variables method. The comparative study is conducted by Alice Gorguis and W.K. Benny Chan in [5]. However, we see that, in one example the solutions do not match using both the methods. We correct the solution of the heat equation which is solved by Adomian method in [5]. First we consider the example of heat equation given in [5].

Example 1.1.
The heat equation is

$$U_t = U_{xx}\ ,\ 0 < x < \pi,\ t > 0 \tag{1}$$

with boundary conditions

$$U(0,t) = U(\pi,t) = 0,\ t > 0 \tag{2}$$

and initial condition

$$U(x,0) = x^2\ ,\ 0 < x < \pi. \tag{3}$$

A solution of the heat equation using separation of variables method is given as follows [5]:

$$U(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t}\left\{\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}((-1)^n - 1)\right\}\sin(nx). \tag{4}$$

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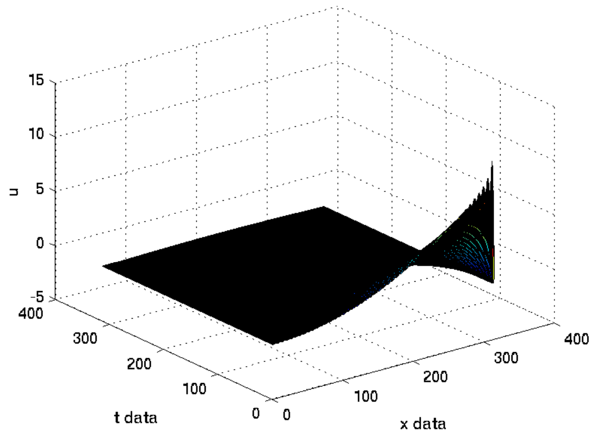

Fig. 1. Solution of heat equation using separation of variables method

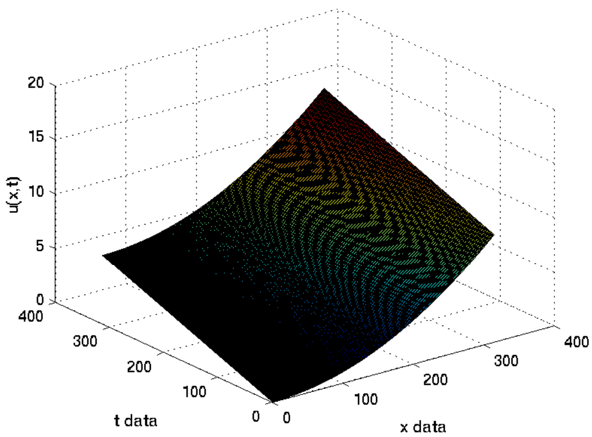

Fig. 2. Solution of heat equation using usual Adomian decomposition method

The solution (4) is shown in the Fig. 1. The same problem is solved using Adomian decomposition method in [5]. The solution obtained is as follows:

$$U(x,t) = x^2 + 2t \tag{5}$$

The solution (5) is shown in Fig. 2.

Solution of heat equation (1) have different solutions shown in Fig. 1 and Fig. 2. This shows that solution using Adomian method is not correct as it does not satisfy the specified boundary conditions. The heat on boundaries are $U(0,t) = 2t$ and $U(\pi,t) = \pi^2 + 2t$ using the solution (5), which are different from boundary conditions (2) in the given problem.

We modify the approach of Adomian method to find the correct solution of problem (1) with conditions (2) and (3). This approach is given in [6]. We express the initial condition $U(x,0) = x^2$ by Fourier series. That is, by half-range sine series

$$U(x,0) = x^2 = \sum_{n=1}^{\infty} B_n \sin(nx), \tag{6}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\,dx. \tag{7}$$

Integrating by parts, we get

$$B_n = \left\{\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}((-1)^n - 1)\right\} \tag{8}$$

Now we apply usual Adomian method. Equation (1) can be re-written in operator form as

$$L_t U(x,t) = L_{xx} U(x,t),\ 0 < x < \pi,\ t > 0 \tag{9}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$ are partial derivative operators. We assume that L_t and and L_{xx} are invertible, therefore their inverse operators are given as

$$L_t^{-1}(.) = \int_0^t (.)dt, \tag{10}$$

and

$$L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)dxdx. \tag{11}$$

Applying inverse operator $L_t^{-1}(.)$ to the both sides of equation (9), we get

$$U(x,t) - U(x,0) = L_t^{-1}(L_{xx}U(x,t)). \tag{12}$$

Using the initial condition (3),

$$U(x,t) = x^2 + L_t^{-1}(L_{xx}U(x,t)). \tag{13}$$

The decomposition method defines the unknown function $U(x,t)$ into a sum of components defined by the series

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) \tag{14}$$

where the components $U_0(x,t)$, $U_1(x,t)$, $U_2(x,t)$,... are to be determined. Substituting (14) into (13) yields

$$\sum_{n=1}^{\infty} U_n(x,t) = x^2 + L_t^{-1}(L_{xx} \sum_{n=1}^{\infty} U_n(x,t)). \tag{15}$$

The decomposition method suggests that the zeroth component $U_0(x,t)$ is identified by the terms arising from the initial, or boundary conditions and from the source terms. The remaining components of $U(x,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$U_0(x,t) = x^2, \qquad U_{k+1}(x,t) = L_t^{-1}(L_{xx}U_k(x,t))\ k \geq 0. \tag{16}$$

Now using equation (6), we express $U_0(x,t) = x^2$ by half-range Fourier sine series. This gives

$$U_0(x,t) = x^2 = \sum_{n=1}^{\infty} B_n \sin(nx), \tag{17}$$

where B_n are given by equation (8). $U_1(x,t)$, $U_2(x,t)$,... are determined individually by,

$$U_1(x,t) = L_t^{-1}(L_{xx}U_0(x,t)) = L_t^{-1}(L_{xx} \sum_{n=1}^{\infty} B_n \sin(nx)). \tag{18}$$

Therefore,

$$U_1(x,t) = -t \sum_{n=1}^{\infty} B_n n^2 \sin(nx). \tag{19}$$

$$U_2(x,t) = L_t^{-1}(L_{xx}U_1(x,t)) = L_t^{-1}(L_{xx}(-t \sum_{n=1}^{\infty} B_n n^2 \sin(nx))). \tag{20}$$

Therefore,

$$U_2(x,t) = \frac{t^2}{2!} \sum_{n=1}^{\infty} B_n n^4 \sin(nx). \tag{21}$$

Consequently, the solution $U(x,t)$ in a series form is given by

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \ldots. \tag{22}$$

Therefore,

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\Big(1 + (-tn^2) + (\frac{t^2}{2!}n^4) + \ldots\Big), \tag{23}$$

which is equivalent to

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-tn^2}, \tag{24}$$

where B_n are given by equation (8). Solution is shown in the Fig. 3. We see that the solution (24) of heat equation (1) same as (4) satisfies boundary conditions (2) and initial condition (3).

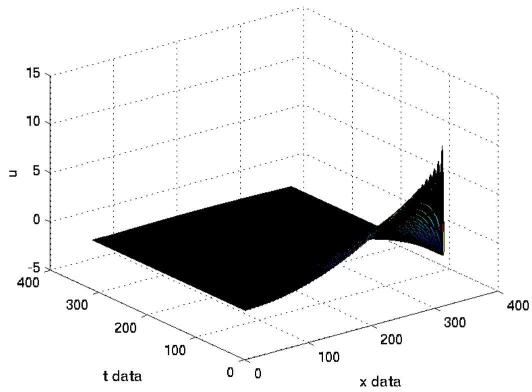

Fig. 3. Solution of heat equation using modified Adomian method

Now we consider an another example of heat equation.

Example 1.2.
The heat equation is

$$U_t = U_{xx},\ 0 < x < \pi,\ t > 0 \tag{25}$$

with boundary conditions

$$U(0,t) = U(\pi,t) = 0,\ t > 0 \tag{26}$$

and initial condition

$$U(x,0) = x. \tag{27}$$

A solution of the equation using separation of variables method is given as follows:

$$U(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-n^2 t} \sin(nx). \tag{28}$$

The solution (28) is shown in the Fig. 4. Using Adomian decomposition method, we get the following solution.

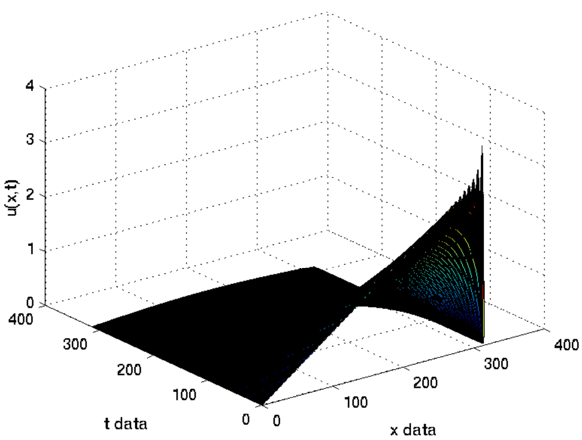

Fig. 4. Solution of heat equation using separation of variables method

$$U(x,t) = x \tag{29}$$

The solution (29) is shown in Fig. 5. Fig. 4 and Fig. 5 shows that solutions of heat equation (25) with boundary condi-

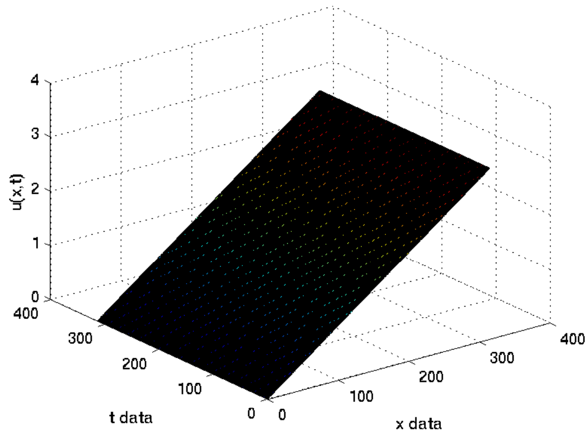

Fig. 5. Solution of heat equation using Adomian decomposition method

tions (26) and initial condition (27) are different. The solution using Adomian method is same as the initial solution and it does not satisfy the specified boundary conditions.

We modify the approach of Adomian method to find the correct solution. We express the initial condition $U(x,0) = x$ by Fourier series. That is, by half-range sine series

$$U(x,0) = x = \sum_{n=1}^{\infty} B_n \sin(nx), \tag{30}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx. \tag{31}$$

Integrating by parts, we get

$$B_n = \left\{\frac{2(-1)^{n+1}}{n}\right\} \tag{32}$$

Now by applying Adomian method using the initial condition in the form (30), we get the following solution.

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-tn^2}, \tag{33}$$

where B_n are given by equation (32). The solution using modified Adomian method is shown in the Fig. 6. By comparing the solution of Fig. 4 and Fig. 6, we see that separation of variables method and modified Adomian decomposition method gives same solution.

We give one more example of heat equation with homogeneous boundary conditions and constant initial temperature.

Example 1.3.
The heat equation is

$$U_t = U_{xx},\ 0 < x < \pi,\ t > 0 \tag{34}$$

with boundary conditions

$$U(0,t) = U(\pi,t) = 0,\ t > 0 \tag{35}$$

and initial condition

$$U(x,0) = K. \tag{36}$$

A solution of the equation using separation of variables method is given as follows:

$$U(x,t) = \sum_{n=1}^{\infty} \frac{2K}{n\pi}[1-(-1)^n]e^{-n^2 t}\sin(nx). \tag{37}$$

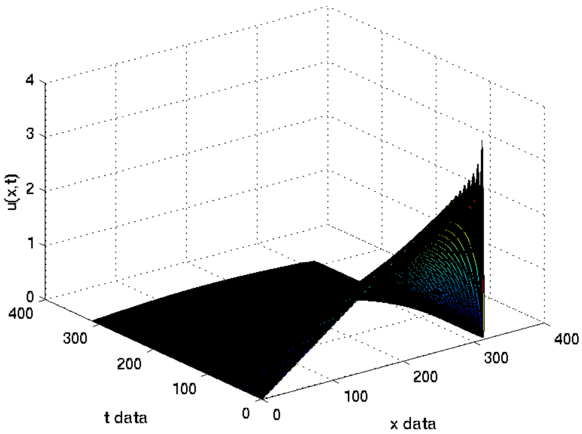

Fig. 6. Solution of heat equation using modified Adomian decomposition method

Using Adomian decomposition method, we get the following solution.

$$U(x,t) = K \tag{38}$$

Equations (37) and (38) show that the solutions of heat equation (34) with boundary conditions (35) and initial condition (36) are different. The solution using Adomian method is same as the initial solution and it does not satisfy the specified boundary conditions. Therefore, the solution (38) using Adomian method is not correct.

We express the initial condition $U(x,0) = K$ by Fourier series. That is, by half-range sine series

$$U(x,0) = K = \sum_{n=1}^{\infty} B_n \sin(nx), \tag{39}$$

where

$$B_n = \frac{2}{\pi} \int_0^{\pi} K \sin(nx) dx. \tag{40}$$

Integrating by parts, we get

$$B_n = \left\{ \frac{2K}{n\pi} [1 - (-1)^n] \right\} \tag{41}$$

By applying Adomian method using the initial condition in the form (39), we get the following solution.

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-tn^2}, \tag{42}$$

where B_n are given by equation (41). The solution (42) is same as solution given by the separation of variables method in equation (38).

Remark 1.1.
We can apply this modified approach of Adomian decomposition method to solve the heat equation with homogeneous boundary conditions and initial condition in the polynomial form $U(x,0) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, $n \geq 1$ is an integer. We can find the Fourier series for the polynomial expression and express the initial condition as a Fourier series. Then we apply the usual Adomian method with the initial condition in the Fourier series form instead of original polynomial form to get the desired correct solution.

2. Conclusions

We have solved one dimensional heat equation with homogeneous boundary conditions and initial condition in polynomial form using the Adomian decomposition method and compared the solution by the separation of variables method. We observed that while solving heat equation with the initial condition in the form $U(x,0) = x$, $U(x,0) = x^2$ or $U(x,0) = K$, K is constant, the usual Adomian method does not give correct solution. In this paper, we have expressed these initial conditions by Fourier series (half range sine series) and then solved by Adomian method which gives correct solution. The obtained solution matches with the solution of heat equation by the separation of variables method.

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