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Independent injective domination of graphs

Research Article

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Abstract: An injective dominating set S in a graph G is called independent injective dominating set (IInj-dominating set) if for every $v \in S$, $N(v) \cap S = \phi$. The minimum cardinality of such injective dominating set is called independent injective domination number of G and denoted by $\gamma_{iin}(G)$. In this paper, we introduce the independent injective domination of graphs and we define the IID-graph. Exact values for some families of graphs, relations with the other domination parameters are obtained. Also, we introduce the independent injective frustration number of graphs. Bounds and some interesting results are established.

MSC: 05C69

Keywords: Independent injective domination • Injective domination • IID-graph • Independent injective frustration number

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected with no loops and multiple edges. we use $\langle X \rangle$ to denote the subgraph of G induced by the set of vertices X. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The distance between two vertices u and v in G is the number of edges in a shortest path connecting them, this is also known as the geodesic distance. The eccentricity of a vertex v is the greatest geodesic distance between v and any other vertex and denoted by $e(v)$.

A subset X of $V(G)$ is called an independent set of G if no two vertices of X are adjacent in G. An independent set is maximum if G has no independent set X' with $|X'| > |X|$. The number of vertices in a maximum independent set of G is called the independence number of G and is denoted by $\beta(G)$. A vertex and an edge are said to cover each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the vertex covering number of G and is denoted by $\alpha(G)$, and for any graph G, the equation $\alpha(G) + \beta(G) = |V(G)|$ is satisfied. A subset S of $V(G)$ is called dominating set if for every $v \in V - S$, there exists a vertex $u \in S$ such that v is adjacent to u. The minimum cardinality of a minimal dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S of G is called independent dominating set if there is no two adjacent vertices in S. The independent domination number of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. A complete subgraph or a clique is an induced subgraph such that there is an edge between each pair of vertices in the subgraph. A clique dominating set is a dominating set that induced a complete subgraph. A clique dominated graph is a graph that contains a clique dominating set. We denote to the smallest integer greater than or equal to x by $\lceil x \rceil$ and the greatest integer less than or equal to x by $\lfloor x \rfloor$.

Let G and H be any two graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively. Then the union $G \bigcup H$ is the graph whose vertex set is $V(G) \bigcup V(H)$ and edge set is $E(G) \bigcup E(H)$. The join $G + H$ is the graph union $G \cup H$ together with all the edges joining vertices of $V(G)$ and $V(H)$. The Cartesian product $G \times H$ is a graph with vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{((u, u'), (v, v')) : u = v \text{ and } (u', v') \in E(H), \text{ or } u' = v' \text{ and } (u, v) \in E(G)\}$.

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The Composition $G \cdot H$ or $G[H]$ has its vertex set $V(G) \times V(H)$, with (u, u') is adjacent to (v, v') if either u is adjacent to v in G or $u = v$ and u' is adjacent to v' in H. The Corona product $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \le i \le |V(G)|$.
A strongly regular graph (SRG) with parameters (n, k, λ, μ) is a graph with n vertices such that the number of common neighbors of two vertices u and v is k, λ or μ according to whether u and v are equal, adjacent or non-adjacent, respectively. When $\lambda = 0$ the strongly regular graph called strongly regular graph with no triangles(SRNT graph), A strongly regular graph G is called primitive if G and $\overline{G}$ are connected. For terminology and notations not specifically defined here we refer the reader to [1]. For more details about domination number and its related parameters, we refer to [2], [3] and [4].

The common neighborhood graph (congraph) of a graph G, denoted by $con(G)$, is the graph with the vertex set $V(G)$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. The common neighborhood (CN-neighborhood) of a vertex $u \in V(G)$ denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in V(G) : uv \in E(G)$ and $|\Gamma(u, v)| \ge 1\}$, where $|\Gamma(u, v)|$ is the number of common neighborhood between the vertices u and v, [5]. The common neighborhood domination in graph has introduced in [6]. A subset D of V is called a common neighborhood dominating set (CN-dominating set) if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $v \in N_{cn}(u)$. The minimum cardinality of such dominating set denoted by $\gamma_{cn}(G)$ and is called the common neighborhood domination number (CN-domination number) of G. The injective domination of graphs has introduced in [7]. For a graph G, a subset D of $V(G)$ is called an injective dominating set (Inj-dominating set) if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $|\Gamma(u, v)| \ge 1$. The minimum cardinality of such dominating set denoted by $\gamma_{in}(G)$ and is called the injective domination number (Inj-domination number) of G. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u, v)| \ge 1\}$. The cardinality of $N_{in}(u)$ is called the injective degree of the vertex u and denoted by $deg_{in}(u)$ in G, and $N_{in}[u] = N_{in}(u) \cup \{u\}$. The maximum and minimum injective degree of a vertex in G are denoted respectively by $\Delta_{in}(G)$ and $\delta_{in}(G)$. That is $\Delta_{in}(G) = \max_{u \in V} |N_{in}(u)|$, $\delta_{in}(G) = \min_{u \in V} |N_{in}(u)|$.

Proposition 3A. (Alwardi [7]) Let G be graph with p vertices. Then $\gamma_{in}(G) = p$ if and only if G is a forest with $\Delta(G) \le 1$.

Proposition 3B.(Alwardi [7]) Let G be a nontrivial connected graph. Then $\gamma_{in}(G) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cn}(v)$ and $e(v) \le 2$.

Theorem 3C.(Alwardi [7]) For any graph G with p vertices, $\lceil \frac{p}{1 + \Delta_{in}(G)} \rceil \le \gamma_{in}(G)$. Further, the equality hold if and only if for every minimum Inj-dominating set D in G the following conditions are satisfied:

(i) for any vertex v in D, $deg_{in}(v) = \Delta_{in}(G)$;

(ii) D is Inj-independent set in G;

(iii) every vertex in $V - D$ has common neighborhood with exactly one vertex in D.

As usual P_p, C_p and K_p are the p-vertex path, cycle and complete graph, respectively, $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices. In this paper, we introduce the independent injective dominating graph (IID-graph) and its independent injective domination number. Exact values for some families of graphs, relations with the other domination parameters are obtained. Also, we introduce the independent injective frustration number of a graph. Bounds and some interesting results are established.

2. An independent injective domination of graphs

In this section, we introduce the independent injective domination in graphs and define the independent injective dominating graph (IID-graph). Exact values of independent injective domination number for some families of IID-graphs are obtained, some bounds and the necessary and sufficient condition for a graph to be IID-graph are discussed.

Definition 2.1.
Let $G = (V, E)$ be a graph. An injective dominating set $S \subseteq V(G)$ is called an independent injective dominating set (IInj-dominating set) in G if the induced subgraph $\langle S \rangle$ is totally disconnected subgraph of G. The independent injective domination number of G is the minimum cardinality of an independent injective dominating set in G and denoted by $\gamma_{iin}(G)$.

Remark 2.1.
The independent injective domination number of graphs is not always defined. For example the cycle graph C_4 has no independent injective dominating set.

Definition 2.2.
A graph G is called independent injective dominating graph (IID-graph) if G has an independent injective dominating set.

Theorem 2.1.
A graph G is an IID-graph if and only if there exists an injective dominating set S such that $V - S$ is covering set.

Proof. Let G be an IID-graph. Then G has an IInj-dominating set say S. Suppose $V - S$ is not covering set in G. Then there exists at least one edge say $e = uv$ is not covered by $V - S$ that means $u, v \in S$ and this contradicts the independence of S. Then $V - S$ is a covering set in G.
Conversely, suppose for any Inj-dominating set of G, $V - S$ is not covering set. Then all the Inj-dominating sets in G are not independent which means that G is not IID-graph. Hence, there exists at least one Inj-dominating set S such that $V - S$ is a covering set of G. □

Lemma 2.1.
For any triangle-free graph G with diameter less than or equal two, $con(G) \cong \overline{G}$.

Proof. Let $f = uv$ be an edge in $con(G)$. Then u and v are not adjacent in G (G is a triangle-free). Therefore u and v are adjacent in $\overline{G}$. Similarly, if $e = xy$ be any edge in $\overline{G}$, then x and y are not adjacent in G and the distance between them is two ($diam(G) \leq 2$). Hence, e is an edge in $con(G)$. □

Theorem 2.2.
Let G be a triangle-free graph with diameter less than or equal two. Then G is an IID-graph if and only if $\overline{G}$ is a clique dominated graph.

Proof. Suppose G is an IID-graph. Then G has at least one minimum IInj-dominating set say S. As S is an Inj-dominating set of G, then S is a dominating set of $con(G)$. By Lemma 2.1, $con(G) \cong \overline{G}$, then S is a dominating set of $\overline{G}$. Because S is independent set of G, then $\langle S \rangle$ is a complete subgraph of $\overline{G}$. Hence, $\overline{G}$ is a clique dominated graph.
Similarly in the same way, if S is any clique dominating set in $\overline{G}$ it will be IInj-dominating set in G. Hence, G is an IID-graph. □

Proposition 2.1.
For any IID-graph G, $\gamma_{in}(G) \leq \gamma_{iin}(G) \leq \beta(G)$.

Proof. Let G be an IID-graph. Then G has a minimum IInj-dominating set S which is also Inj-dominating set, so $\gamma_{in}(G) \leq \gamma_{iin}(G)$. Since S is also an independent set of G, then $\gamma_{iin}(G) \leq \beta(G)$. □

Proposition 2.2.

1. *Let $G \cong P_p$, then G is an IID-graph if and only if $p = 3k + 1$.*
2. *Let $G \cong C_p$, then G is an IID-graph if and only if $p = 3k$.*

Where $k \in Z^+$.

Proof. Consider $V(G) = \{v_1, v_2, \ldots, v_p\}$.

1. Suppose $G \cong P_p$ is an IID-graph, then G has IInj-dominating set say S. It is clear that $S = \{v_1, v_4, v_7, v_{10}, \ldots, v_p\}$ which is unique because the end vertices v_1 and v_p should be belong to any IInj-dominating set of $G \cong P_p$. Now, if $p = 3k$, then S will not be Inj-dominating set and if $p = 3k+2$, then $V - S$ will not be a covering set in G. Finally, if $p = 3k + 1$, then S is an IInj-dominating set of G.
2. Suppose $G \cong C_p$ is an IID-graph, then any IInj-dominating set S of G has a form $S = \{v_i, v_{i+3}, \ldots, v_{p+i-3}\}$ for some $i = 1, 2, \ldots, p$. It is easy to see that if $p \neq 3k$, then $V - S$ will not be a covering set of G.

The converse is clear. □

Proposition 2.3.
Let $G \cong K_{r,m}$. Then G is not an IID-graph.

Proposition 2.4.
Let G be an IID-graph. Then $\gamma_{iin}(G) = 1$ *if and only if* $\gamma_{in}(G) = 1$.

Proposition 2.5.
Let $G = G_1 + G_2$ *is not complete bipartite graph. Then G is an IID-graph. Further,* $\gamma_{iin}(G) = 1$.

Proof. Since $G = G_1 + G_2$ is not complete bipartite graph, then either G_1 or G_2 has at least one edge. Without loss of generality, suppose G_1 has an edge $e = uv$. Then $N(u) = N_{cn}(u)$ and $e(u) \leq 2$, then by Proposition 3B, $\gamma_{iin}(G) = \gamma_{in}(G) = 1$. □

Proposition 2.6.
Let G be a graph with p vertices. Then $\gamma_{iin}(G) = p$ *if and only if G is totally disconnected.*

Proof. The proof comes immediately from Proposition 3A. □

Proposition 2.7.
Let G be an IID-graph.

1. *If* $G = P_p$, *then* $\gamma_{iin}(G) = \lceil \frac{p}{3} \rceil$.
2. *If* $G = C_p$, *then* $\gamma_{iin}(G) = \lceil \frac{p}{3} \rceil$.
3. *If G is the Peterson graph, then* $\gamma_{iin}(G) = 3$.

Theorem 2.3.
For any two positive integers a and b there exists a graph G with $|V(G)| = 3(a+b)$ *such that* $\gamma_{iin}(G) = \gamma_{in}(G) = a + b$.

Proof. Let G be a graph as in the Fig. 1, then G consists of a cycle C_n and a path P_{p-n+1}, where the vertex v_n is a common vertex between them. Suppose $n = 3a$ and $p - n + 1 = 3b + 1$ for some $a, b \in Z^+$. Then by Proposition 2.2, G is an IID-graph with $|V(G)| = p = 3(a+b)$. Also, by Proposition 2.7, $\gamma_{iin}(G) = \gamma_{in}(G) = \lceil \frac{n}{3} \rceil + \lceil \frac{p-n+1}{3} \rceil - 1$ (because v_n is taken twice). Hence, $\gamma_{iin}(G) = \gamma_{in}(G) = a + b$. □

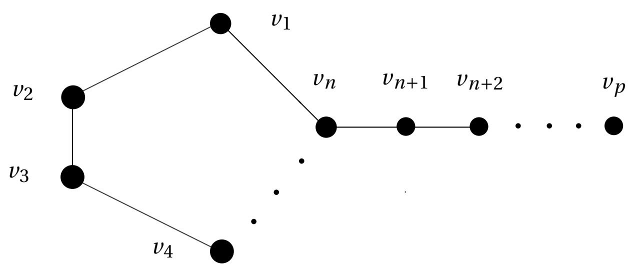

Fig. 1. Graph with $|V(G)| = 3(a+b)$ and $\gamma_{iin}(G) = a + b$.

Proposition 2.8.
Let $G \cong P_m \times P_2$ *such that* $m \geq 3$ *and* $m \neq 4$. *Then*

1. *G is an IID-graph.*
2. $\gamma_{iin}(G) \leq \lfloor \frac{m}{2} \rfloor + 1$.

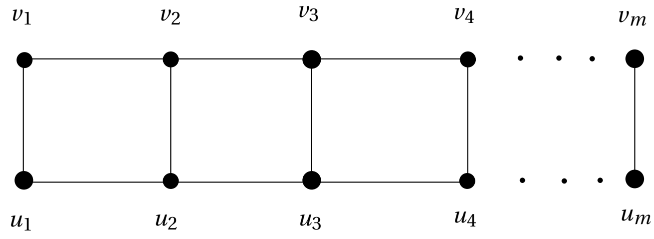

Fig. 2. Graph $G \cong P_m \times P_2$

Proof. Let $G \cong P_m \times P_2$ be labeling as in Fig. 2. If $m = 4$, then each Inj-dominating set of G contains at least one edge, then G is not IID-graph.
(i) Suppose $G \cong P_m \times P_2$, where $m \neq 4$. Then we have two cases:

Case 1. If $m = 2t+1$, then the set $S = \{v_1, u_3, v_5, u_7, \ldots, u_m\}$ or $S = \{v_1, u_3, v_5, u_7, \ldots, v_m\}$ is an IInj-dominating set of G depending on t is odd or even, respectively. Hence, G is an IID-graph.

Case 2. If $m = 2t$, where $t \geq 3$, then the set $S = \{v_1, u_3, v_4, u_6, v_8, \ldots, u_m\}$ or $S = \{v_1, u_3, v_4, u_6, v_8, \ldots, v_m\}$ is an IInj-dominating set of G depending on t is odd or even, respectively. Hence, G is an IID-graph.
(ii) The set S in Case 1 and Case 2, has cardinality $\lfloor \frac{m}{2} \rfloor + 1$, then the result. □

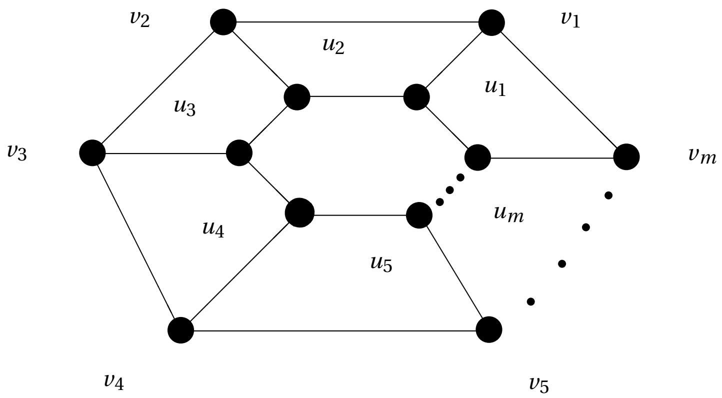

Fig. 3. Graph $G \cong C_m \times P_2$

Proposition 2.9.
Let $G \cong C_m \times P_2$ such that $m \neq 5$. Then

1. *G is an IID-graph.*
2. $\gamma_{iin}(G) \leq \begin{cases} \frac{2m}{3}, & \text{if } m = 6 \text{ or } 9; \\ \lfloor \frac{m}{2} \rfloor, & \text{if } m \equiv 3 \pmod 4; \\ \lceil \frac{m}{2} \rceil, & \text{otherwise.} \end{cases}$

Proof. Let $G \cong C_m \times P_2$ be labeling as in Fig. 3. It is easy to see that, if $m = 5$, then all the Inj-dominating sets of G are not independent, then G is not IID-graph.
(i) Suppose $G \cong C_m \times P_2$ such that $m \neq 5$. Then we have four cases:

Case 1. If $m \equiv 0 \ (mod\ 4)$, then the set $S = \{v_1, u_3, v_5, u_7, \ldots, u_{m-1}\}$ is an IInj-dominating set of G. Hence, G is an IID-graph.

Case 2. If $m \equiv 1 \ (mod\ 4)$ and $m \neq 5$, then the set $S = \{v_1, u_3, v_4, u_6, v_7, u_9, v_{11}, u_{13}, \ldots, v_{m-2}\}$ is an IInj-dominating set of G. Hence, G is an IID-graph.

Case 3. If $m \equiv 2 \ (mod\ 4)$, then the set $S = \{v_1, u_3, v_4, u_6, v_8, \ldots, v_{m-2}\}$ is an IInj-dominating set of G. Hence, G is an IID-graph.

Case 4. If $m \equiv 3 \ (mod\ 4)$, then the set $S = \{v_1, u_3, v_5, u_7, \ldots, v_{m-2}\}$ is an IInj-dominating set of G. Hence, G is an IID-graph.
(ii) The result is obvious from the cardinality of the set S of all Cases in (i). □

Lemma 2.2.
For any graph G with $N(v) = N_{cn}(v)$, $\forall v \in V(G)$, G is an IID-graph.

Proof. Suppose G is not IID-graph. Then each Inj-dominating set of G has at least one edge. Suppose now S is a minimum Inj-dominating set of G and $e = uv$ is an edge in S. Then uv is contained in a triangle in G, so $S - u$ and $S - v$ are Inj-dominating sets of G which contradicts the minimality of S. Hence, G is an IID-graph. □

Proposition 2.10.
For any connected graphs G_1 and G_2, $G_1 \cdot G_2$ is an IID-graph.

Proof. Let G_1 and G_2 be two connected graphs. Then from the definition of the Composition product $G_1 \cdot G_2$ we have $N(w) = N_{cn}(w)$, $\forall w \in V(G_1 \cdot G_2)$. Hence by Lemma 2.2, $G_1 \cdot G_2$ is an IID-graph. □

Proposition 2.11.
For any graph G isomorphic to $P_m \cdot P_n$ or $P_m \cdot C_n$ or $C_m \cdot P_n$ or $C_m \cdot C_n$, $\gamma_{iin}(G) = \lceil \frac{m}{5} \rceil$.

Proof. Since $N(w) = N_{cn}(w)$, $\forall w \in V(G)$, then each vertex $w = (u, v)$ in G Inj-dominates its neighbors and all the vertices of distance two of it, then $\gamma_{iin}(G) \leq \lceil \frac{m}{5} \rceil$, but in this graph $\Delta_{in}(G) = 5n - 1$, so by Theorem 3C, $\gamma_{in}(G) \geq \lceil \frac{mn}{5n} \rceil = \lceil \frac{m}{5} \rceil$. Hence by Proposition 2.1, $\gamma_{iin}(G) = \lceil \frac{m}{5} \rceil$. □

Proposition 2.12.
Let $G = \bigcup_{i=1}^{m} G_i$. Then G is an IID-graph if and only if G_i, $i = 1, 2, \ldots, m$ is an IID-graph.

Proof. Suppose G is an IID-graph. Then by Theorem 2.1, G has Inj-dominating set S such that $V - S$ is covering. Since $G = \bigcup_{i=1}^{m} G_i$, then $S = \bigcup_{i=1}^{m} S_i$. Thus each $S_i \subseteq V_i$ should be Inj-dominating set in G_i such that $V_i - S_i$ is covering. Hence, G_i, $i = 1, 2, \ldots, m$ is an IID-graph.
Conversely, suppose G_i, $i = 1, 2, \ldots, m$ is an IID-graph. Then by Theorem 2.1 there exist Inj-dominating sets $S_1, S_2, \ldots, S_m$ in $G_1, G_2, \ldots, G_m$, respectively and $V_i - S_i$ is covering set in G_i. Now, we have $\bigcup_{i=1}^{m} S_i$ is Inj-dominating set in G and $\bigcup_{i=1}^{m} (V_i - S_i)$ is covering set in G. Hence, G is an IID-graph. □

Proposition 2.13.
For any connected graph G with $p \geq 2$, $G \circ H$ is IID-graph.

Proof. Let G be a connected graph with $p \geq 2$. It is easy to chick that the set $S \subset V(G \circ H)$ which obtained by choosing one vertex from each copy of H is an IInj-dominating set of $G \circ H$. Hence, $G \circ H$ is an IID-graph. □

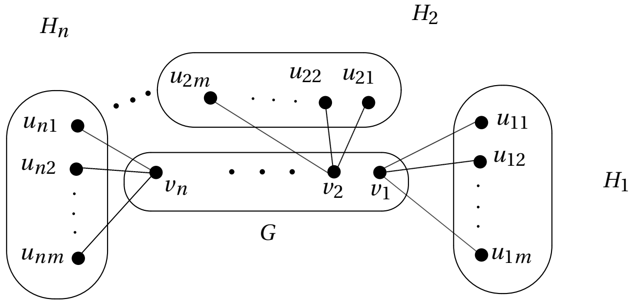

Fig. 4. $G \circ H$

Theorem 2.4.
Let G be a graph with $\delta(G) = 0$. Then $G \circ H$ is an IID-graph if and only if H is not totally disconnected graph.

Proof. Let $u \in V(G)$ such that $deg(u) = 0$ and $G \circ H$ is an IID-graph. We need to prove that H is not totally disconnected. Suppose that H is totally disconnected graph. Then one of the components of $G \circ H$ is a star with center vertex u and has no IInj-dominating set, then $G \circ H$ is not an IID-graph. A contradiction.
Conversely, let G be a graph with isolated vertex u and H is not totally disconnected graph. If G has components $G_1, G_2, \ldots, G_t, u$, it is easy to see that

$$G \circ H = \bigcup_{i=1}^{t} (G_i \circ H) \cup (K_1 \circ H).$$

By Proposition 2.12, the graph $\bigcup_{i=1}^{t} (G_i \circ H)$ is an IID-graph. To prove that $G \circ H$ is an IID-graph only we have to prove that $K_1 \circ H$ is an IID-graph. Since H is not totally disconnected,so at least there is one edge in H say $e = w_1 w_2$, then any vertex u or w_1 or w_2 is form IInj-dominating set of $K_1 \circ H$. Since $\bigcup_{i=1}^{t} (G_i \circ H)$ and $K_1 \circ H$ are IID-graphs, then again by Proposition 2.7, $G \circ H$ is an IID-graph. □

Proposition 2.14.
Let $G \circ H$ be an IID-graph, where H is a graph with $\delta(H) = 0$. Then $\gamma_{iin}(G \circ H) = |V(G)|$.

Proof. Let S be a minimum IInj-dominating set of $G \circ H$ and $u \in V(H)$ be an isolated vertex in H. Suppose S contains one vertex of G say $v \in V(G)$. Then v cannot Inj-dominates the isolated vertex u of the copy of H that joined to v and any other vertex of $V(G)$ can dominates u must be adjacent to v. Then S does not contain any vertex of G, so S contains one vertex from each copy H_i of H, $1 \le i \le |V(G)|$. Hence, $\gamma_{iin}(G \circ H) = |V(G)|$. □

Proposition 2.15.
Let G and $G \circ H$ be IID-graphs. Then $\gamma_{iin}(G) \le \gamma_{iin}(G \circ H)$.

Proposition 2.16.
For any connected graph H,

$$\gamma_{iin}(C_n \circ H) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \ (mod\, 3); \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. Consider, $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. We have the following cases:

Case 1. Suppose $n = 3k$. Then by Proposition 2.2 and Proposition 2.7, C_n is an IID-graph and $\gamma_{iin}(C_n) = \lceil \frac{n}{3} \rceil$. In this case, the subset $S = \{v_i, v_{i+3}, \ldots, v_{n+i-3}\}$ is a minimal IInj-dominating set of $C_n \circ H$ of order $\lceil \frac{n}{3} \rceil$. Thus $\gamma_{iin}(C_n \circ H) \le \gamma_{iin}(C_n)$. Hence by Proposition 2.15, $\gamma_{iin}(C_n \circ H) = \lceil \frac{n}{3} \rceil$.

Case 2. Suppose $n = 3k + 1$. In this case, to obtain an IInj-dominating set of $C_n \circ H$ we have only the following

forms:
i. $S_1 = \{u_{1j}, u_{2j}, \dots, u_{nj}\}$ for some $j = 1,2,\dots,m$, which has cardinality n.
ii. $S_2 = \{u_{ij}, v_{i+1}, u_{(i+2)j}, v_{i+3}, \dots, v_{n+i-2}, u_{(n+i-1)j}\}$ or $S_2 = \{u_{ij}, v_{i+1}, u_{(i+2)j}, v_{i+3}, \dots, u_{(n+i-2)j}, v_{n+i-1}\}$, where $1 \le i \le n$, for some $j = 1,2,\dots,m$ depending on n is odd or even, respectively, which also has cardinality n.
iii. $S_3 = \{u_{ij}, v_{i+2}, v_{i+5}, \dots, v_{n+i-2}\}$ for some $j = 1,2,\dots,m$, which has cardinality $\lceil \frac{n}{3} \rceil$.
Without loss of generality, we can get IInj-dominating sets of $C_n \circ H$ consist of a mixed of some of the three forms above, but it is clear that the cardinality of such sets are greater than the cardinality of S_3. Hence, $\gamma_{iin}(C_n \circ H) = \lceil \frac{n}{3} \rceil$.

Case 3. Suppose $n = 3k+2$. In this case, $C_n \circ H$ has no IInj-dominating set of order $\lceil \frac{n}{3} \rceil$ because there are only two forms of IInj-dominating sets of $C_n \circ H$ as well as S_1 and S_2 in Case 2:
i. $S_3 = \{u_{ij}, v_{i+1}, v_{i+4}, \dots, v_{n+i-1}\}$, where $1 \le i \le n$, for some $j = 1,2,\dots,m$, which has cardinality $\lceil \frac{n}{3} \rceil + 1$.
ii. $S_4 = \{u_{ij}, v_{i+2}, v_{i+5}, \dots, v_{n+i-3}, u_{(n+i-1)j}\}$ for some $j = 1,2,\dots,m$, which also has cardinality $\lceil \frac{n}{3} \rceil + 1$.

Note that, as in Case 2, we can get IInj-dominating sets of $C_n \circ H$ consist of a mixed of some of S_1 and S_2 with either S_3 or S_4, but clearly that the cardinality of such sets are not less than $\lceil \frac{n}{3} \rceil + 1$. Hence, $\gamma_{iin}(C_n \circ H) = \lceil \frac{n}{3} \rceil + 1$. □

Theorem 2.5.
Let G be an IID-graph with p vertices such that $G \neq \overline{K_p}$. Then $\gamma_{iin}(G) \le p-2$ and the equality holds if and only if G is one of the following graphs P_4, C_3, a union of a null graph with P_4 or C_3.

Proof. Since $G \neq \overline{K_p}$, then by Proposition 2.6, $\gamma_{iin}(G) \neq p$. Suppose now $S \subseteq V(G)$ is an Inj-dominating set of G, where $|S| = p-1$. Then $\langle S \rangle$ contains at least one edge, so S is not IInj-dominating set, then $\gamma_{iin}(G) \neq p-1$. Hence, $\gamma_{iin}(G) \le p-2$.

Now, let $\gamma_{iin}(G) = p-2$. Suppose $S \subseteq V(G)$ is a minimum IInj-dominating set of G, where $|S| = p-2$. Then $V-S$ contains only two vertices say u and v. Note that u and v must be adjacent because if they don't, either $\gamma_{iin}(G) \neq p-2$ or S is not independent set, then u and v are Inj-dominated by either one vertex i.e. u and v lie on C_3 or two vertices i.e. u and v lie on P_4, and by Proposition 2.6, all the other vertices in S are totally disconnected. The other side is clear. □

Proposition 2.17.
Let G be a connected IID-graph on p vertices. Then $\gamma_{iin}(G) \le \frac{p}{2}$ with the equality holds if $G \cong H \circ K_1$, where H is a connected graph.

Proof. Since G is a connected IID-graph, then $\beta(G) \le \frac{p}{2}$. Suppose $p \le 3$. Hence by Proposition 2.1, $\gamma_{iin}(G) \le \frac{p}{2}$. Now, if $G \cong H \circ K_1$, then by Proposition 2.14, $\gamma_{iin}(G) = |V(H)| = \frac{p}{2}$. □

Proposition 2.18.
For any strongly regular graph G with $(n,k,0,\mu)$ if G is an IID-graph, then $\gamma_{iin}(G) \le \mu+2$.

Proof. Let u and v be any two vertices in G which they are not adjacent. Then there are μ common vertices between u and v. Since $diam(G) = 2$, then the vertex u Inj-dominates all the vertices of G except its neighbors which are k vertices and the vertex v Inj-dominates $k-\mu$ vertices from the neighbors of u, so in the maximum case we can choose at most μ vertices of G which are independent with u and v to Inj-dominate the common neighbors between u and v. Hence, $\gamma_{iin}(G) \le \mu+2$. □

3. Independent injective frustration of graphs

In this section, we investigate the smallest number of edges which can be removed from any non IID-graph to becomes an IID-graph, which we called it the independent injective frustration number of graphs.

Definition 3.1.
The smallest number of edges that have to be deleted from a graph G to becomes an IID-spaning subgraph is called the independent injective frustration number of G, and denoted by $\varphi_{in}(G)$.

Note that: If the graph G is an IID-graph, then $\varphi_{in}(G) = 0$.

Proposition 3.1.
If $G \cong K_{r,m}$, where r and m deferent from one, then $\varphi_{in}(K_{r,m}) = 1$.

Proof. Suppose $G \cong K_{r,m}$ and let the partite sets are $V_1(G)$, $V_2(G)$, where $|V_1(G)| = r$ and $|V_2(G)| = m$. Choose $u \in V_1(G)$ and $v \in V_2(G)$. Then u Inj-dominates all the other vertices of $V_1(G)$, but it cannot Inj-dominates any vertex of $V_2(G)$ and v Inj-dominates all the other vertices of $V_2(G)$, but it cannot Inj-dominates any vertex of $V_1(G)$. But the minimum injective dominating set $\{u, v\}$ of G ($\gamma_{in}(G) = 2$) is not independent. Thus by removing the edge that joins u and v, we obtain an IID-spaning subgraph of G. Hence, $\varphi_{in}(K_{r,m}) = 1$. □

Proposition 3.2.
For any cycle graph C_p,

$$\varphi_{in}(C_p) = \begin{cases} 0, & \text{if } p \equiv 0 \pmod 3; \\ 1, & \text{if } p \equiv 1 \pmod 3; \\ 2, & \text{if } p \equiv 2 \pmod 3). \end{cases}$$

Proof. We have the following cases.

Case 1. If $p \equiv 0$ (mod 3), then G is an IID-graph. Hence, $\varphi_{in}(C_p) = 0$.

Case 2. If $p \equiv 1$ (mod 3), then by Proposition 2.2, G is non IID-graph. Thus by removing one edge of $E(G)$ the graph G will become a path of order $p = 3k + 1$, then again by Proposition 2.2, G is an IID-graph. Hence, $\varphi_{in}(G) = 1$.

Case 3. If $p \equiv 2$ (mod 3), then again by Proposition 2.2, G is non IID-graph. So by removing two edges join any vertex of C_p with its neighbors the cycle C_p will become a path of order $3k + 1$ union K_1 which is by Proposition 2.2, an IID-graph. Hence, $\varphi_{in}(G) = 2$. □

Proposition 3.3.
For any path P_p,

$$\varphi_{in}(P_p) = \begin{cases} 2, & \text{if } p \equiv 0 \pmod 3; \\ 0, & \text{if } p \equiv 1 \pmod 3; \\ 1, & \text{if } p \equiv 2 \pmod 3). \end{cases}$$

Proof. The proof is same as in Proposition 3.2. □

Proposition 3.4.
Let G be a graph isomorphic to $P_4 \times P_2$ or $C_5 \times P_2$. Then $\varphi_{in}(G) = 1$.

Proof. Suppose $G \cong P_4 \times P_2$ or $G \cong C_5 \times P_2$. Then by Proposition 2.8 and Proposition 2.9 G is not IID-graph.

Case 1. If $G \cong P_4 \times P_2$, then from Fig. 2, by removing the edge joins the vertices v_2 and u_2 the set $S = \{v_2, u_2\}$ becomes an IInj-dominating set of G. Hence, $\varphi_{in}(G) = 1$.

Case 2. If $G \cong C_5 \times P_2$, then from Fig. 3, by removing the edge $e = v_1v_5$ the set $S = \{v_1, u_3, v_5\}$ becomes an IInj-dominating set of G. Hence, $\varphi_{in}(G) = 1$. □

Theorem 3.1.
Let G be an isolated-free graph on p vertices and q edges. Then $\varphi_{in}(G) = q$ if and only if $G \cong \bigcup_{i=1}^{n} K_{1,m_i}$, where $p = n + \sum_{i=1}^{n} m_i$ and $q = \sum_{i=1}^{n} m_i$.

Proof. Suppose $\varphi_{in}(G) = q$ and let G consists of only one component. Then by Proposition 3.2 and Proposition 3.3, G does not contain any cycle or a path of order greater than three as induced subgraphs. Thus G is a tree of diameter less than or equal two. Since the only trees of diameter less than or equal two are the star graphs, then $G \cong K_{1,m}$ for some positive integer m. Since G is a star, then the center vertex is an Inj-isolated vertex, thus it should be contained in each

Inj-dominating set of G. Then G is not IID-graph. Since the center vertex has a full degree in G, then to obtain an IID-spaning subgraph of G we have to remove all the edges of G. Hence, $\varphi_{in}(G) = m$. Now, if G consists of n components, then clearly that $G \cong \bigcup_{i=1}^{n} K_{1,m_i}$.

Conversely, suppose that $G \cong \bigcup_{i=1}^{n} K_{1,m_i}$. Then G consists of n star components K_{1,m_i}, $1 \le i \le n$. Hence, $\varphi_{in}(G) = \sum_{i=1}^{n} m_i = q$. □

Corollary 3.1.
For any graph G with p vertices and q edges, $0 \le \varphi_{in}(G) \le q$.

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