Int. J. Adv. Appl. Math. and Mech. 3(4) (2016) 1 – 8 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

A fixed point approach to orthogonal stability of an Additive - Cubic functional equation

Research Article

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Received 09 March 2016; accepted (in revised version) 05 April 2016

Abstract: Using fixed point method, we prove the Hyers-Ulam stability of the orthogonally additive-cubic functional equation

$$f(2x+y)+f(2x-y)-f(4x)=2f(x+y)+2f(x-y)-8f(2x)+10f(x)-2f(-x)$$

for all x, y with $x \perp y$.

MSC: 39B52 • 39B55 • 39B82 • 46H25 • 47H10

Keywords: Hyers-Ulam stability • Additive and cubic functional equations • Fixed point method • Orthogonality space

1. Introduction

A basic equation in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, then we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] and affirmatively solved by Hyers [2]. The result of Hyers was generalized by Aoki [3] for approximatively additive mappings and by Rassias [4] for approximate linear mappings by allowing the Cauchy difference operator $Df(x,y)=f(x+y)-f(x)-f(y)$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of Rassias's theorem was obtained by Gavruta [5], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\phi(x,y)$.

In addition, Rassias et al. [6], [7] generalized the Hyers stability result by introducting two weaker conditions controlled by a norms respectively. Recently, several further interesting discussions, modifications, extensions and generalizations of the original problem of Ulam have been proposed see [8], [9], [10].

Let us recall the orthogonality in the sense of Ratz see [11]. Let X be a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(a) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(b) independence: if $x, y \in X - \{0\}$, then x, y are linearly independent;

(c) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in X$;

(d) the Thalesian property: Let P be a 2- dimensional subspace of X. If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (in the sense of Ratz). By an orthogonality normed space, we mean an orthogonality space equipped with a norm. Some examples of special interest are

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(i) The trivial orthogonality on a vector space X defined by (a), and for non-zero elements $x,y\in X$, $x\perp y$ if and only if x,y are linearly independent,

(ii) The ordinary orthogonality on an inner product space $(X,(.,.))$ given by $x\perp y$ if and only if $(x,y)=0$,

(iii) The Birkhoff-James orthogonality on a normed space $(X,\|.\|)$ defined by $x\perp y$ if and only if $\|x+y\|\geq\|x\|$ for all $\lambda\in\mathbb{R}$.

The relation $\perp$ is called symmetric if $x\perp y$ implies that $y\perp x$ for all $x,y\in X$. Clearly conditions (i) and (ii) are symmetric but (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. The orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y),\quad x\perp y$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D.Strawther [12]. The orthogonally quadratic equation $f(x+y)+f(x-y)=2f(x)+2f(y)$, $x\perp y$, was first investigated by Vajzovic [13] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljevic [14], Fochi [15], Moslehian [16], Szabo [17], Moslehian and Th. M. Rassias [18], [19] and Paganoni and Ratz [20] have investigated the orthogonal stability of functional equations. Ashish and Renu chugh [21] proved the Hyers-Ulam-Rassias stability of the orthogonally cubic and quartic functional equation in the sense of Ratz orthogonality.

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$ for all $x,y\in X$;
3. $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1 ([22]).
Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either

$$d(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(i) $d(J^nx,J^{n+1}x)<\infty$ for all $n\geq n_0$;
(ii) The sequence $\{J^nx\}$ is convergent to a fixed point y^* of J;
(iii) y^* is the unique fixed point of T in the set $Y=\{y\in X: d(J^{n_0}x,y)<\infty\}$;
(iv) $d(y,y^*)\leq\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$..

In 1996, Isac and Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors. see [24], [25], [26].
Recently, Choonkil Park [27], proved the Hyers-Ulam stability of orthogonally additive-quadratic functional equation in orthogonality spaces by using the fixed point method. Sung jin Lee, Choonkil Park and Reza Saadati [28] proved the orthogonal stability of an additive quadratic functional equation by using fixed point method.

This paper, we prove the Hyers-Ulam stability of the orthogonally additive-cubic functional equation

$$f(2x+y)+f(2x-y)-f(4x)=2f(x+y)+2f(x-y)-8f(2x)+10f(x)-2f(-x).$$

in orthogonality space of an odd mapping.
Throughout this paper, assume that $(X,\perp)$ is an orthogonality space and that $(Y,\|.\|_Y)$ is a real Banach space.

2. Hyers-Ulam stability of the orthogonally Additive-Cubic functional equation : Fixed Point Method

For a given mapping $f:X\to Y$, we define

$$Df(x,y)=f(2x+y)+f(2x-y)-f(4x)-2f(x+y)-2f(x-y)+8f(2x)-10f(x)+2f(-x).$$

Theorem 2.1.
Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with

$$\phi(x,y) \le 2\alpha\phi\left(\frac{x}{2},\frac{y}{2}\right) \tag{1}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying $f(0) = 0$ and

$$\|Df(x,y)\|_Y \le \phi(x,y) \tag{2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\|_Y \le \frac{\alpha}{1-\alpha}\phi\left(\frac{x}{2},0\right) \tag{3}$$

Proof. Letting $y = 0$ in (2), we get

$$\|-f(4x) + 10f(2x) - 16f(x)\|_Y \le \phi(x,0)$$

for all $x \in X$, since $x \perp 0$.
Now letting $g(x) = f(2x) - 8f(x)$ in above equation, we get

$$\|g(2x) - 2g(x)\| \le \phi(x,0)$$

$$\|2g(x) - g(2x)\| \le \phi(x,0) \tag{4}$$

$$\left\|g(x) - \frac{1}{2}g(2x)\right\|_Y \le \frac{1}{2}\phi(x,0)$$

$$\left\|g(x) - \frac{1}{2}g(2x)\right\|_Y \le \frac{1}{2}2\alpha\phi\left(\frac{x}{2},0\right) \tag{5}$$

for all $x \in X$.
Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S.

$$m(g,h) = \inf\left\{\mu \in \mathbb{R}_+, \|g(x) - h(x)\|_Y \le \mu\phi\left(\frac{x}{2},0\right), x \in X\right\}$$

where, as usual $\inf\phi = +\infty$. It is easy to show that (S,m) is complete. see [10]
Now consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.
Let $g, h \in S$ be given such that $m(g,h) = \epsilon$. Then

$$\|g(x) - h(x)\|_Y \le \phi\left(\frac{x}{2},0\right)$$

for all $x \in X$. Hence,

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y \le \alpha\phi\left(\frac{x}{2},0\right)$$

for all $x \in X$. So $m(g,h) = \epsilon$. implies that $m(Jg, Jh) \le \alpha\epsilon$. This means that

$$m(Jg, Jh) \le \alpha m(g,h)$$

for all $g, h \in S$. It follows from (5) that $m(g, Jg) \le \alpha$.
By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:
(1) A is a fixed point of J, i.e.,

$$A(2x) = 2A(x) \tag{6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : m(g,h) < \infty\}.$$

This implies that A is a unique mapping satisfying (6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - A(x)\|_Y \leq \mu\phi(x,0)$$

for all $x \in X$.
(2) $m(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} g(2^n x) = A(x)$$

forall $x \in X$.
(3) $m(g,A) \leq \frac{1}{1-\alpha} m(g,Jg)$, which implies the inequality $m(g,A) \leq \frac{\alpha}{1-\alpha}$.
This implies that the inequality (3) holds.
It follows from (1) and (2) that

$$\begin{aligned}\|DA(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{2^n} \|Dg(2^n x, 2^n y)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \phi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} \frac{2^n \alpha^n}{2^n} \phi(x,y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$DA(x,y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $A : X \to Y$ is an orthogonally additive mapping. □

Corollary 2.1.
Assume that $(X,\perp)$ is an orhogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying f(0)= 0 and

$$\|Df(x,y)\|_Y \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orhogonally additive mapping $A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{1}{2-2^p}\theta(\|x\|^p).$$

Proof. Taking $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{p-1}$ in Theorem 2.1, we get the result. □

Theorem 2.2.
Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with

$$\phi(x,y) \leq \frac{\alpha}{2}\phi(2x,2y) \tag{7}$$

for all $x, y \in X$ with $x \perp y$ and satisfying (2). Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{1}{1-\alpha}\phi\left(\frac{x}{2},0\right) \tag{8}$$

Proof. Let (S,m) be the generalized metric space defined in the proof of Theorem 2.1.
Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) = 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.
It follows from (4) that $m(g,Jg) \leq 1$.
The rest of the proof is similiar to the proof of Theorem 2.1. □

Corollary 2.2.
Assume that $(X,\perp)$ is an orhogonality normed space. Let θ be a positive real number and p a real number with $p>1$. Let $f:X\to Y$ be an odd mapping satisfying f(0)= 0 and

$$\left\|Df(x,y)\right\|_Y \le \theta(\|x\|^p + \left\|y\right\|^p)$$

for all $x,y\in X$ with $x\perp y$. Then there exists a unique orhogonally additive mapping $A:X\to Y$ such that

$$\left\|f(2x)-8f(x)-A(x)\right\|_Y \le \frac{1}{2^p-2}\theta(\|x\|^p).$$

Proof. Taking $\phi(x,y)=\theta(\|x\|^p+\left\|y\right\|^p)$ for all $x,y\in X$ with $x\perp y$ and choosing $\alpha=2^{1-p}$ in Theorem 2.2, we get the result. □

Theorem 2.3.
Let $\phi:X^2\to[0,\infty)$ be a function such that there exists an $\alpha<1$ with

$$\phi(x,y)\le 8\alpha\phi\left(\frac{x}{2},\frac{y}{2}\right) \tag{9}$$

or all $x,y\in X$ with $x\perp y$ and satisfying (2). Then there exists a unique orthogonally cubic mapping $C:X\to Y$ such that

$$\left\|f(2x)-2f(x)-C(x)\right\|_Y \le \frac{\alpha}{1-\alpha}\phi\left(\frac{x}{2},0\right) \tag{10}$$

for all $x\in X$.

Proof. Letting y = 0 in (2), we get

$$\left\|-f(4x)+10f(2x)-16f(x)\right\|_Y \le \phi(x,0)$$

for all $x\in X$, since $x\perp 0$.
Now, letting $h(x)=f(2x)-2f(x)$ in above equation, We get

$$\|h(2x)-8h(x)\|\le\phi(x,0).$$

$$\|8h(x)-h(2x)\|_Y\le\phi(x,0). \tag{11}$$

$$\left\|h(x)-\frac{1}{8}h(2x)\right\|_Y \le \frac{1}{8}\phi(x,0)$$

$$\left\|h(x)-\frac{1}{8}g(2x)\right\|_Y \le \frac{1}{8}8\alpha\phi\left(\frac{x}{2},0\right) \tag{12}$$

for all $x\in X$.
Consider the set

$$S:=\{h:X\to Y\}$$

and introduce the generalized metric on S:

$$m(g,h)=\inf\left\{\mu\in\mathbb{R}_+,\left\|g(x)-h(x)\right\|_Y\le\mu\phi\left(\frac{x}{2},0\right),x\in X\right\}$$

where, as usual $\inf\phi=+\infty$. It is easy to show that (S,m) is complete [10].
Now consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{8}g(2x)$$

for all $x\in X$.
Let $g,h\in S$ be given such that $m(g,h)=\epsilon$. Then,

$$\left\|g(x)-h(x)\right\|_Y\le\phi\left(\frac{x}{2},0\right)$$

for all $x \in X$. Hence,

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{8}g(2x) - \frac{1}{8}h(2x)\right\|_Y \leq \alpha\phi\left(\frac{x}{2}, 0\right)$$

for all $x \in X$. $m(g,h) = \epsilon$ implies that $m(Jg, Jh) \leq \alpha\epsilon$. This means that

$$m(Jg, Jh) \leq \alpha m(g,h)$$

for all $g, h \in S$.
It follows from (12) that $m(g, Jg) \leq \alpha$.
By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:
(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x) \tag{13}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : m(g,h) < \infty\}.$$

This implies that C is a unique mapping satisfying (13) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|h(x) - C(x)\|_Y \leq \mu\phi(x,0)$$

for all $x \in X$.
(2) $m(J^n h, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} h(2^n x) = C(x)$$

forall $x \in X$.
(3) $m(h,C) \leq \dfrac{1}{1-\alpha} m(h, Jh)$, which implies the inequality

$$m(h,C) \leq \frac{\alpha}{1-\alpha}.$$

This implies that the inequality (10) hlods.
It follows from (9) and (2) that

$$\begin{aligned}\|DC(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{8^n}\|Dg(2^n x, 2^n y)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{8^n}\phi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} \frac{8^n\alpha^n}{8^n}\phi(x,y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So $DC(x,y) = 0$ for all $x, y \in X$ with $x \perp y$. Hence $C : X \to Y$ is an orthogonally cubic mapping. □

Corollary 2.3.
Assume that $(X, \perp)$ is an orhogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying f(0)= 0 and

$$\|Df(x,y)\|_Y \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orhogonally cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{1}{8-2^p}\theta(\|x\|^p).$$

Proof. Taking $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{p-3}$ in Theorem 2.3, we get the result. □

Theorem 2.4.
Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with

$$\phi(x, y) \le \frac{\alpha}{8}\phi(2x, 2y) \tag{14}$$

or all $x, y \in X$ with $x \perp y$ and satisfying (2). Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\left\| f(2x) - 2f(x) - C(x) \right\|_Y \le \frac{1}{1-\alpha}\phi\left(\frac{x}{2}, 0\right) \tag{15}$$

for all $x \in X$.

Proof. Let (S, m) be the generalized metric space defined in the proof of Theorem 2.3.
Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) = 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.
It follows from (11) that $m(h, Jh) \le 1$.
The rest of the proof is similiar to the proof of Theorem 2.3. □

Corollary 2.4.
Assume that $(X, \perp)$ is an orhogonality normed space. Let θ be a positive real number and p a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying f(0)= 0 and

$$\left\| Df(x, y) \right\|_Y \le \theta(\|x\|^p + \left\| y \right\|^p)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orhogonally cubic mapping $C : X \to Y$ such that

$$\left\| f(2x) - 2f(x) - C(x) \right\|_Y \le \frac{1}{2^p - 8}\theta(\|x\|^p).$$

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \left\| y \right\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = 2^{3-p}$ in Theorem 2.4, we get the result. □

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