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On general Gamma-Taylor operators on weighted spaces

Research Article

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Abstract: In the present paper we consider new operators by combining general Gamma type operators and Taylors polynomials. We establish convergence properties of these operators in weighted spaces.

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1. Introduction

In 2007, Mao [17] defined the following Gamma type linear and positive operators

$$\begin{aligned} M_{n,k}(f;x) &= \int_0^\infty \int_0^\infty g_n(x,u) g_{n-k}(u,t) f(t)\,du\,dt \\ &= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t)\,dt, \ x>0. \end{aligned}$$

We can rewrite the operators $M_{n,k}(f;x)$ as

$$M_{n,k}(f;x) = \int_0^\infty K_{n,k}(x,t) f(t)\,dt, \tag{1}$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x,t \in (0,\infty).$$

The rate of convergence of these operators for functions with derivatives of bounded variation was studied in [12]. Some approximation results for these operators based on $q-$integers were obtained in [15]. The Voronovskaja type theorem and the local rate of convergence for the operators $M_{n,k}$ were given in [9]. In [10] global approximation theorems for these operators were obtained.

In this paper, we consider new operators by combining general Gamma type operators and Taylor polynomials of r times differentiable function f in weighted space on $(0,a_n]$ which expands to $(0,\infty)$ when $n\to\infty$. We study the convergence of these new operators.

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By $C^r(0,\infty)$, we denote the set of all real valued functions f such that $r^{th}(r=0,1,2,...)$ order derivatives are continuous.
For any $f \in C^r(0,\infty)$ and $t \in (0,\infty)$, we consider Taylor polynomials of order r

$$T_r(f;x) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j. \tag{2}$$

Combine (1) and (2), we obtain

$$M_{n,k,r}(f;x) = \int_0^\infty K_{n,k}(x,t) \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j dt. \tag{3}$$

It is clear that $M_{n,k,0}(f;x) = M_{n,k}(f;x)$.
Let $\nu(x) = 1+x^2, -\infty < x < \infty$ and B_ν be the set of all functions f defined on the real axis satisfying the condition $|f(x)| \leq \mathscr{C}_f \nu(x)$, where $\mathscr{C}_f$ is a constant depending only on f.
B_ν is a normed space with the norm

$$\| f \|_\nu = \sup_{x\in(-\infty,\infty)} \frac{|f(x)|}{\nu(x)}, f \in B_\nu.$$

C_ν denotes the subspace of all continuous functions in B_ν and C_ν^k denotes the subspace of all functions $f \in C_\nu$ for which

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\nu(x)} < \infty.$$

$B_{\nu,(0,a_n]}$, $C_{\nu,(0,a_n]}$ and $C^k_{\nu,(0,a_n]}$ are defined as B_ν, C_ν and C_ν^k respectively, only with the domain $(0,a_n]$ instead of real axis R and the norm is taken as

$$\| f \|_{\nu,(0,a_n]} = \sup_{x\in(0,a_n]} \frac{| f(x) |}{\nu(x)}.$$

In the sequel it will be assumed that $\lim_{n\to\infty} a_n = \infty$.

2. Auxiliary results

In this section we give some preliminary results which will be used in the main part of this paper.
Let us consider

$$e_m(t) = t^m, \ \varphi_{x,m}(t) = (t-x)^m, \ m \in N_0, \ x, t \in (0,\infty).$$

Lemma 2.1 ([12]).
For any $m \in N_0$ (set of non-negative integers), $m \leq n-k$

$$M_{n,k}(t^m;x) = \frac{[n-k+m]_m}{[n]_m} x^m \tag{4}$$

where $n,k \in N$ and $[x]_m = x(x-1)...(x-m+1), [x]_0 = 1, x \in R$.
In particular for m = 0, 1, 2... in (4) we get

(i) $M_{n,k}(1;x) = 1$,

(ii) $M_{n,k}(t;x) = \dfrac{n-k+1}{n} x$,

(iii) $M_{n,k}(t^2;x) = \dfrac{(n-k+2)(n-k+1)}{n(n-1)} x^2$.

Lemma 2.2 ([12]).
Let $m \in N_0$ and fixed $x \in (0,\infty)$, then

$$M_{n,k}(\varphi_{x,m};x) = \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n-m+j)!(n-k+m-j)!}{n!(n-k)!} \right) x^m.$$

Lemma 2.3.
For $m = 0,1,2,3,4$, one has

(i) $M_{n,k}(\varphi_{x,0};x) = 1$,

(ii) $M_{n,k}(\varphi_{x,1};x) = \dfrac{1-k}{n}x$,

(iii) $M_{n,k}(\varphi_{x,2};x) = \dfrac{k^2-5k+2n+4}{n(n-1)}x^2$,

(iv) $M_{n,k}(\varphi_{x,3};x) = \dfrac{-k^3+12k^2-17k+n(18-12k)+24}{n(n-1)(n-2)}x^3$,

(v) $M_{n,k}(\varphi_{x,4};x) = \dfrac{k^4-22k^3+k^2(143+12n)-k(314+108n)+12n^2+268n+192}{n(n-1)(n-2)(n-3)}x^4$,

(vi) $M_{n,k}(\varphi_{x,m};x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof. Using Lemma 2.2, we get Lemma 2.3. □

Remark 2.1.
Using Lemma 2.3, we get

$$M_{n,k}\left((t-x)^{2m};x\right) \le \lambda_m \frac{x^{2m}}{n^m},$$

where λ_m is a constant depending only on m.

Lemma 2.4.
Let $a_{n,k,m} = \dfrac{(n-m)!(n-k+m)!}{n!(n-k)!}$. Then, for all n we have

$$a_{n,k,m} \le e.$$

Proof.

$$\begin{aligned} a_{n,k,m} &= \frac{(n-k+m)(n-k+m-1)...(n-k+m-m+1)}{(n-m+1)...n} \\ &\le \left(1+\frac{m}{n}\right)^{m+k} < e. \end{aligned}$$

□

Lemma 2.5.
For sufficiently large n, the following inequalities holds:

(i) $M_{n,k}(|t-x|^m;x) \le \sqrt{\lambda_m}\dfrac{x^m}{n^{m/2}}$,

(ii) $M_{n,k}(|t-x|^m t^l;x) \le \sqrt{\lambda_m} e \dfrac{x^{m+l}}{n^{m/2}}$,

(iii) $M_{n,k}(|t-x|^m (t-x)^j;x) \le \sqrt{\lambda_m\lambda_j}\dfrac{x^{m+j}}{n^{(m+j)/2}}$,

(iv) $M_{n,k}(|t-x|^m t^l (t-x)^j;x) \le \left(\lambda_m^2\lambda_{2j}e\right)^{1/4}\dfrac{x^{m+l+j}}{n^{(m+j)/2}}$,

where $l, m, j \in N$.

Proof. (i) and (iii) follow by the using Hölder's inequality and Remark 2.1. Also by Hölder's inequality, Remark 2.1 and Lemma 2.4

$$M_{n,k}(|t-x|^m t^l;x) \le \sqrt{M_{n,k}((t-x)^{2m};x}\sqrt{M_{n,k}(t^{2l};x)} \le \sqrt{\lambda_m a_{n,k,2l}}\frac{x^{m+l}}{n^{m/2}} \le \sqrt{\lambda_m e}\frac{x^{m+l}}{n^{m/2}}$$

and

$$M_{n,k}(|t-x|^m t^l (t-x)^j;x) \le \sqrt{M_{n,k}((t-x)^{2m};x)}\sqrt{M_{n,k}(t^{2l}(t-x)^{2j};x)} \le \sqrt{\lambda_m}\frac{x^m}{n^{m/2}}\left(\sqrt{M_{n,k}((t-x)^{4j};x)}\sqrt{M_{n,k}(t^{4l};x)}\right)^{1/2} \le \left(\lambda_m^2\lambda_{2j}e\right)^{1/4}\frac{x^{m+l+j}}{n^{(m+j)/2}}.$$

□

Let $\{b_n\}$ be a sequence with positive terms, $b_{n+1} > b_n$,

$$\lim_{n\to\infty} b_n = \infty,\ \lim_{n\to\infty}\frac{b_n^2}{n} = 0.$$

Theorem 2.1.
For every $f \in C^k_{\nu,(0,b_n]}$, we have

$$\lim_{n\to\infty} \| M_{n,k}(f) - f \|_{\nu,(0,b_n]} = 0.$$

Proof. From [3], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| M_{n,k}(t^m;x) - x^m \|_{\nu,(0,b_n]} = 0,\ m = 0,1,2. \tag{5}$$

Since $M_{n,k}(1;x) = 1$, the condition in (5) holds for $m = 0$.
By using Lemma 2.1, we have

$$\| M_{n,k}(t;x) - x \|_{\nu,(0,b_n]} = \sup_{x\in(0,b_n]}\frac{|M_{n,k}(t;x)-x|}{1+x^2} \le \left|\frac{1-k}{n}\right|\sup_{x\in(0,b_n]}\frac{x}{1+x^2} \le \left|\frac{1-k}{n}\right|,$$

which implies that the condition in (5) holds for $m = 1$.
Similarly, we can write for $n > 1$

$$\| M_{n,k}(t^2;x) - x^2 \|_{\nu,(0,b_n]} = \sup_{x\in(0,b_n]}\frac{|M_{n,k}(t^2;x)-x^2|}{1+x^2} \le \left|\frac{k^2-3k+4n-2nk+2}{n(n-1)}\right|,$$

which implies that $\lim_{n\to\infty} \| M_{n,k}(t^2;x) - x^2 \|_{\nu,(0,b_n]} = 0$, the equation (5) holds for $m = 2$.
This completes the proof of theorem. □

3. Rate of convergence of $M_{n,k}(f;x)$ and $M_{n,k,r}(f;x)$ in weighted spaces

Now we want to find the rate of convergence of the operators $\{M_{n,k}\}$ and $\{M_{n,k,r}\}$.
It is well known that the first order modulus of continuity

$$\omega(f;\delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[a,b]} |f(t)-f(x)|$$

does not tend to zero, as $\delta \to 0$, on any infinite interval.

A weighted modulus of continuity $\Omega_n(f;\delta)$ was defined in [5], which tends to zero as $\delta \to 0$ on an infinite interval. A similar definition of the modulus of continuity can be found in [1].
For each $f \in C^k_{\nu,(0,b_n]}$ it is given by

$$\Omega_n(f;\delta) = \sup\left\{\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)} : |h| \le \delta,\ x \in (0,b_n]\right\}. \tag{6}$$

For every $f \in C^k_{\nu,(0,b_n]}$, following properties of $\Omega_n(f;\delta)$ were shown in [5]

$$\lim_{\delta\to 0}\Omega_n(f;\delta) = 0, \tag{7}$$

$$|f(t)-f(x)| \le 2(1+\delta_n^2)(1+x^2)\Omega_n(f;\delta_n)S_n(t;x), \tag{8}$$

where $S_n(t;x) = \left(1+(t-x)^2\right)\left(1+\dfrac{|t-x|}{\delta_n}\right)$.
It is easy to see that

$$S_n(t;x) \le \begin{cases} 2(1+\delta_n^2), & |t-x| \le \delta_n, \\ 2(1+\delta_n^2)\dfrac{(t-x)^4}{\delta_n^4}, & |t-x| \ge \delta_n. \end{cases} \tag{9}$$

Theorem 3.1.
Let $f \in C^k_{\nu,(0,b_n]}$. Then for all sufficiently large n

$$\| M_{n,k}(f) - f \|_{\nu,(0,b_n]} \le \mathscr{C}\Omega_n\left(f;\sqrt{\frac{b_n^2}{n}}\right),$$

where $\mathscr{C}$ is a positive constant.

Proof. Using Lemma 2.1, we get

$$\begin{aligned} | M_{n,k}(f;x) - f(x) | &\le M_{n,k}(| f(t) - f(x) |; x) \\ &\le 2\left(1+\delta_n^2\right)(1+x^2)\Omega_n\left(f;\delta_n\right) M_{n,k}\left(S_n(t;x);x\right). \end{aligned}$$

Using (9), we get

$$S_n(t;x) \le 2\left(1+\delta_n^2\right)\left(1+\frac{(t-x)^4}{\delta_n^4}\right),$$

for all $x \in (0,b_n]$ and $t \in (0,\infty)$. Thus, for $n > 3$, $x \in (0,b_n]$, using Lemma 2.3, we get

$$\begin{aligned} | M_{n,k}(f;x) - f(x) | &\le 4\left(1+\delta_n^2\right)^2(1+x^2)\left(1+\frac{1}{\delta_n^4}\frac{c_{n,k}}{n(n-1)(n-2)(n-3)}x^4\right)\Omega_n\left(f;\delta_n\right) \\ &\le 4\left(1+\delta_n^2\right)^2(1+x^2)\left(1+\frac{36}{\delta_n^4}\frac{b_n^4}{n^2}\right)\Omega_n\left(f;\delta_n\right). \end{aligned}$$

where $c_{n,k} = k^4 - 22k^3 + k^2(143+12n) - k(314+108n) + 12n^2 + 268n + 192$ and $\delta_n = \sqrt{\dfrac{b_n^2}{n}}$. Since $\lim\limits_{n\to\infty}\dfrac{b_n^2}{n} = 0$, we have that $\delta_n \le 1$ for sufficiently large n and the statement of the theorem follows. □

Now, we need the following modified Taylor formula. By Taylor's theorem ([19], p.391-392) we have

$$\begin{aligned} f(x) &= \sum_{j=0}^{r-1}\frac{f^{(j)}(t)}{j!}(x-t)^j + \int_t^x \frac{f^{(r)}(s)}{(r-1)!}(x-s)^{r-1}ds \\ &= \sum_{j=0}^{r-1}\frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}\int_0^1\left(\frac{x-s}{x-t}\right)^{r-1}\frac{f^{(r)}(s)}{(x-t)}ds. \end{aligned}$$

Let $s = t + u(x-t)$, then

$$\begin{aligned} f(x) &= \sum_{j=0}^{r-1}\frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1} f^{(r)}(t+u(x-t))\,du \\ &= \sum_{j=0}^{r-1}\frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\left(f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right)du. \end{aligned}$$

Theorem 3.2.
Let $f, f^{(r)} \in C^k_{\nu,(0,b_n]}$. Then for all sufficiently large n

$$\| M_{n,k,r}(f) - f \|_{\nu,(0,b_n]} \leq \mathscr{C}_r \left(\frac{b_n^2}{n}\right)^{r/2} \Omega_n\left(f^{(r)}; \sqrt{\frac{b_n^2}{n}}\right),$$

where $\mathscr{C}_r$ is a positive constant depends only on $r\,(r = 0, 1, 2, \ldots)$.

Proof. Using modified Taylor's formula, Lemma 2.1 and (3), we get

$$| M_{n,k,r}(f; x) - f(x) | \leq \int_0^\infty K_{n,k}(x,t) \frac{|x-t|}{(r-1)!} \Theta(r,t)\,dt$$

where $\Theta(r,t) = \int_0^1 (1-u)^{r-1} |f^{(r)}(t + u(x-t)) - f^{(r)}(t)|\,du.$
Using (8), we get

$$\left| f^{(r)}(t + u(x-t)) - f^{(r)}(t) \right| \leq 2(1+\delta_n^2)(1+t^2) S_n(x,u,t) \Omega_n\left(f^{(r)}, \delta_n\right),$$

where $x \in (0, b_n]$, $t \in (0,\infty)$, $u \in [0,1]$ and

$$S_n(x,u,t) = \left(1 + u^2(t-x)^2\right)\left(1 + \frac{u}{\delta_n}|t-x|\right).$$

It is easy to see that

$$S_n(x,u,t) \leq \begin{cases} (1+u)(1+u^2\delta_n^2), & |t-x| \leq \delta_n, \\ (1+u)(1+u^2\delta_n^2)\dfrac{(t-x)^4}{\delta_n^4}, & |t-x| \geq \delta_n. \end{cases}$$

So, for all $x \in (0, b_n]$, $t \in (0,\infty)$ and $u \in [0,1]$

$$S_n(x,u,t) \leq (1+u)\left(1+u^2\delta_n^2\right)\left(1 + \frac{(t-x)^4}{\delta_n^4}\right).$$

Thus,

$$| M_{n,k,r}(f;x) - f(x) | \leq \mathscr{C}(r,\delta_n) \Omega_n\left(f^{(r)}, \delta_n\right) M_{n,k}\left((1+t^2)|t-x|^r \left(1 + \frac{(t-x)^4}{\delta_n^4}\right); x\right)$$

where $\mathscr{C}(r,\delta_n) = 2(1+\delta_n^2)\left(\frac{1}{r!} + \frac{1}{(r+1)!} + \frac{\delta_n^2}{(r+2)!} + \frac{\delta_n^3}{(r+3)!}\right)$.
Using Lemma 2.4, we get

$$\begin{aligned} | M_{n,k,r}(f;x) - f(x) | &\leq \mathscr{C}(r,\delta_n)\Omega_n\left(f^{(r)},\delta_n\right)\left(\frac{\sqrt{\lambda_r}x^r}{n^{r/2}} + \frac{\sqrt{\lambda_r e}x^{r+2}}{n^{r/2}} + \frac{\sqrt{\lambda_r\lambda_4}}{\delta_n^4}\frac{x^{r+4}}{n^{(r+4)/2}} + \frac{\left(\lambda_r^2\lambda_8 e\right)^{1/4}}{\delta_n^4}\frac{x^{r+6}}{n^{(r+4)/2}}\right) \\ &\leq \mathscr{C}(r,\delta_n)\Omega_n\left(f^{(r)},\delta_n\right)(1+x^2)\left(\frac{\sqrt{\lambda_r}x^r}{n^{r/2}} + \frac{\sqrt{\lambda_r e}x^{r+2}}{n^{r/2}} + \frac{\sqrt{\lambda_r\lambda_4}}{\delta_n^4}\frac{x^{r+4}}{n^{(r+4)/2}} + \frac{\left(\lambda_r^2\lambda_8 e\right)^{1/4}}{\delta_n^4}\frac{x^{r+6}}{n^{(r+4)/2}}\right). \end{aligned}$$

Thus, we have

$$\sup_{x\in(0,b_n]} \frac{|M_{n,k,r}(f;x) - f(x)|}{1+x^2} \leq \mathscr{C}(r,\delta_n)\Omega_n(f^{(r)},\delta_n)\left(\frac{b_n^2}{n}\right)^{r/2}\left(A_r + \frac{B_r}{\delta_n^4}\left(\frac{b_n^2}{n}\right)^2\right),$$

where $A_r = \sqrt{\lambda_r} + \sqrt{\lambda_r e}$, $B_r = \sqrt{\lambda_r\lambda_4} + (\lambda_r^2\lambda_8 e)^{1/4}$.
Choosing $\delta_n = \sqrt{\frac{b_n^2}{n}}$ and taking into account that $\frac{b_n^2}{n} \leq 1$ for sufficiently large n, since $\lim_{n\to\infty}\frac{b_n^2}{n} = 0$, we obtain

$$\mathscr{C}(r,\delta_n) \leq 4\left(\sum_{j=0}^3 \frac{1}{(r+j)!}\right) := \xi_r$$

and

$$\| M_{n,k,r}(f) - f \|_{\nu,(0,b_n]} \leq \xi_r(A_r + B_r)\left(\frac{b_n^2}{n}\right)^{r/2}\Omega_n\left(f^{(r)}; \sqrt{\frac{b_n^2}{n}}\right).$$

Hence, the statement of the theorem follows with $\mathscr{C}_r = \xi_r(A_r + B_r)$. □

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