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Hyers-Ulam stability of the isometric Cauchy-Jenson mapping in generalized quasi-banach spaces

Research Article

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Abstract: In this article, we investigate the generalized Hyers-Ulam stability of the isometric Cauchy-Jenson mapping in generalized quasi-Banach spaces, and generalized $p-$ Banach spaces.

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Keywords: Cauchy-Jenson mapping • Generalized Hyers-Ulam stability • Generalized quasi-Banach space • Isometry • Generalized $p-$ Banach space

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyer's theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call generalized Hyers-Ulam-Rassias stability of functional equations. In 1990, Rassias [5] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [6] and Rassias and Semrl [7] that one cannot prove an analogous theorem when p=1. In 1994, a generalization was obtained by Gavruta [8], who replaced the bound $\epsilon\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\phi(x,y)$. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors, and there are many interesting results concerning this problem.
We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1.
Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisying the following:

1. $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ iff $x = 0$.
2. $||\lambda x|| = |\lambda|||x||$ forall $\lambda \in \mathbb{R}$ and for all $x \in X$.
3. $\|x+y\| \leq k(||x|| + ||y||)$ where $k \geq 1$ is constant and for all $x, y \in X$.

The pair $(X, ||.||)$ is called a quasi- normed space if $||.||$ is a quasi-norm on X. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a *p-norm* $(0 < p \leq 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-banach space is called a $p-$ Banach space.

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Definition 1.2.
Let X be a linear space. A *generalized quasi-norm* is a real valued function on X satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $||x|| = 0$ iff $x = 0$.
2. $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{R}$ and for all $x \in X$.
3. There is a constant $k \geq 1$ such that $|| \sum_{j=1}^{\infty} x_j || = \sum_{j=1}^{\infty} k||x_j||$ for all $x_1, x_2, \ldots \in X$.

The pair $(X, ||.||)$ is called a *generalized quasi- normed space* if $||.||$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $||.||$. A generalized quasi-Banach space is a complete generalized quasi-normed space.

A generalized quasi-norm $\|.\|$ is called a *p-norm* $(0 < p \leq 1)$ if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a generalized quasi-Banach space is called a *generalized p-Banach space.*

Definition 1.3.
Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$d_Y(f(x), f(y)) = d_X(x, y)$$

forall $x, y \in X$, where $d_X(.,.)$ and $d_Y(.,.)$ denote the metrics in the spaces X and Y respectively. For some fixed number $r > 0$, suppose that f preserves distance "r"; i.e., forall $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y((f(x), f(y)) = r$. Then r is called a conservative(or preserved) distance for the mapping "f". Let $(X, ||.||)$ and $(Y, ||.||)$ be normed spaces. A mapping $L : X \to Y$ is called an *isometry* if

$$||L(x) - L(y)|| = ||x - y|| \quad \text{for all} \quad x, y \in X.$$

Aleksandrov posed the following problem: Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry. The isometric problems have been investigated in several papers. see ([9], [10], [11], [12]).

Recently, Chun-Gil Park and Th.M. Rassias [13], investigated the generalized Hyers-Ulam stability of the isometric Cauchy additive mappings in generalized quasi-Banach spaces. In this paper, we prove the generalized Hyers-Ulam stability of the isometric Cauchy-Jenson mapping in generalized quasi-Banach spaces, and prove the generalized Hyers-Ulam stability of the isometric Cauchy-Jenson mapping in generalized $p-$ Banach spaces.

2. Stability of the isometric Cauchy-Jenson mapping in generalized quasi-banach spaces

Throughout this section, assume that A is a generalized quasi-normed vector space with generalized quasi-norm $\|.\|$ and that B is a generalized quasi-banach space with generalized quasi-norm $\|.\|$. Let K be the modulus of concavity of $\|.\|$.

Theorem 2.1.
Let $r > 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - 2f(z)\right\| \leq \theta \left(\|x\|^r + \|y\|^r + \|z\|^r\right) \tag{1}$$

$$\left|\|f(x)\| - \|x\|\right| \leq 3\,\|x\|^r \tag{2}$$

for all $x, y \in A$. Then there exists a unique isometric Cauchy-Jenson additive mapping $T : A \to B$ such that

$$\|f(x) - T(x)\| \leq \frac{3K\theta}{2^r - 2}\,\|x\|^r \tag{3}$$

for all $x \in A$.

Proof. Letting $y = z = x$ in (1), we get

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{3\theta}{2^r}\|x\|^r \tag{4}$$

for all $x \in A$. Therefore

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \le K\frac{3\theta}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{jr}}\|x\|^r \tag{5}$$

for all non-negative integers m and l with $m > l$ and for all $x \in A$. It follows from (5) that the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ converges. So one can define the mapping $T : A \to B$ by

$$T(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. It follows from (1) that

$$\begin{aligned}
&\left\| 2T\left(\frac{x+y}{2} + z\right) - T(x) - T(y) - 2T(z)\right\| \\
&\le \lim_{n\to\infty} 2^n \left\| 2f\left(\frac{x+y}{2.2^n} + \frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}\left(\|x\|^r + \|y\|^r + \|z\|^r\right) \\
&= 0.
\end{aligned}$$

for all $x, y, z \in A$. So

$$2T\left(\frac{x+y}{2} + z\right) = T(x) + T(y) + T(z).$$

for all $x, y, z \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5), we get (3).
Now let $T' : A \to B$ be another Cauchy-Jenson mapping satisfying (3). Then we have

$$\begin{aligned}
\|T(x) - T'(x)\| &= 2^n \left\| T\left(\frac{x}{2^n}\right) - T'\left(\frac{x}{2^n}\right)\right\| \\
&\le K2^n\left(\left\| T(\frac{x}{2^n}) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\
&= \frac{2^{n+1}3K^2\theta}{(2^r - 2)2^{nr}}\|x\|^r
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $T(x) = T'(x)$ for all $x \in A$. This proves the uniqueness of T.
It follows from (2) that

$$\begin{aligned}
\left| \left\| 2^n f(\frac{x}{2^n})\right\| - \|x\|\right| &\le 2^n \left| \left\| f(\frac{x}{2^n})\right\| - \left\| (\frac{x}{2^n})\right\|\right| \\
&\le 3\theta\frac{2^n}{2^{nr}}\|x\|^r
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So

$$\begin{aligned}
\|T(x)\| &= \lim_{n\to\infty}\left\| 2^n f(\frac{x}{2^n})\right\| \\
&= \|x\|
\end{aligned}$$

for all $x \in A$. Since T is additive,

$$\|T(x) - T(y)\| = \|T(x-y)\| = \|x - y\|$$

for all $x \in A$, as desired. □

Theorem 2.2.
Let $r < 1$ and θ be a positive real numbers, and let $f : A \to B$ be a mapping satisfying (1) and (2). Then there exists a unique isometric Cauchy-Jenson additive mapping $T : A \to B$ such that

$$\|f(x) - T(x)\| \le \frac{3K\theta}{2 - 2^r}\|x\|^r \tag{6}$$

for all $x \in A$.

Proof. It follows from (4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{3}{2} \theta \|x\|^r \tag{7}$$

for all $x \in A$. So

$$\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \frac{3K}{2} \sum_{j=l}^{m-1} \frac{2^{jr}\theta}{2^j} \|x\|^r \tag{8}$$

for all non-negative integers m and l with $m > 1$ and all $x \in A$. It follows from (6) that the sequence $\left\{ \frac{1}{2^n} f(2^n x) \right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{ \frac{1}{2^n} f(2^n x) \right\}$ converges. So, one can define the mapping $T : A \to B$ by

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$.
The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of the isometric Cauchy-Jenson mappings in generalized $p-$Banach spaces

Throughout this section, assume that A is a generalized quasi-normed vector space with generalized quasi-norm $\|.\|$ and that B is a generalized $p-$Banach space with generalized quasi-norm $\|.\|$.

Theorem 3.1.
Let $r > 1$ and θ be positive real numbers and let $f : A \to B$ be a mapping (1) and (2). Then there exist a unique isometric Cauchy-Jenson additive mapping $T : A \to B$ such that

$$\|f(x) - T(x)\| \leq \frac{3\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}} \|x\|^r \tag{9}$$

for all $x \in A$.

Proof. It follows from (4)

$$\left\| f(x) - 2f(\frac{x}{2}) \right\| \leq \frac{3\theta}{2^r} \|x\|^r \tag{10}$$

for all $x \in A$. Since B is a generalized $p-$Banach space,

$$\left\| 2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}) \right\|^p \leq \frac{3^p \theta^p}{2^{pr}} \sum_{j=l}^{m-1} \frac{2^{pj}}{2^{prj}} \|x\|^{pr} \tag{11}$$

for all non-negative integers m and l with $m > l$ and for all $x \in A$. It follows from (11) that the sequence $\left\{ 2^n f(\frac{x}{2^n}) \right\}$ is Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{ 2^n f(\frac{x}{2^n}) \right\}$ converges. So one can define the mapping $T : A \to B$ by

$$T(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.
By (1),

$$\begin{aligned}
&\left\| 2T\left(\frac{x+y}{2} + z \right) - T(x) - T(y) - 2T(z) \right\| \\
&= \lim_{n\to\infty} 2^n \left\| 2f\left(\frac{x+y}{2.2^n} + \frac{z}{2^n} \right) - f(\frac{x}{2^n}) - f(\frac{y}{2^n}) - 2f(\frac{z}{2^n}) \right\| \\
&\leq \lim_{n\to} \frac{2^n \theta}{2^{nr}} \left(\|x\|^r + \|y\|^r + \|z\|^r \right) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$2T\left(\frac{x+y}{2}+z\right) = T(x)+T(y)+2T(z).$$

for all $x, y, z \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (11), we get (9). Now, let $T' : A \to B$ be another Cauchy-Jenson additive mapping satisfying (9). Then we have

$$\begin{aligned}\|T(x)-T'(x)\|^p &= 2^{pn}\left\|T(\frac{x}{2^n})-T'(\frac{x}{2^n})\right\|^p\\ &\le 2^{pn}\left(\left\|T(\frac{x}{2^n})-f(\frac{x}{2^n})\right\|^p+\left\|T'(\frac{x}{2^n})-f(\frac{x}{2^n})\right\|^p\right)\\ &= 2.\frac{2^{pn}}{2^{pnr}}\left(\frac{3^p.\theta^p}{2^{pr}-2^p}\right)\|x\|^{pr}\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $T(x) = T'(x)$ for all $x \in A$. This proves the uniqueness of T.
The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 3.1.
The result for the case $K = 1$ in Theorem 2.1 is the same as the result for the case $p = 1$ in Theorem 3.1.

Theorem 3.2.
Let $r < 1$ and θ be positive real number and let $f : A \to B$ be a mapping satisfying (1) and (2). Then there exists a unique isometric Cauchy-Jenson additive mapping $T : A \to B$ such that

$$\|f(x)-T(x)\| \le \frac{3\theta}{(2^p-2^{pr})^{\frac{1}{p}}}\|x\|^r \tag{12}$$

for all $x \in A$.

Proof. It follows (7) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{3}{2}\theta\|x\|^r$$

for all $x \in A$. Since B is a generalized $p-$Banach space.

$$\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\|^p \le \frac{3^p\theta^p}{2^p}\sum_{j=l}^{m-1}\frac{2^{prj}}{2^{pj}}\|x\|^{pr} \tag{13}$$

for all non-negative integers m and l with $m > l$ and for all $x \in A$. It follows from (12) that the sequence $\left\{\frac{1}{2^n}f(2^nx)\right\}$ is Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{\frac{1}{2^n}f(2^nx)\right\}$ converges. So one can define the mapping $T : A \to B$ by

$$T(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x \in A$. The rest of the proof is similar to the proofs of Theorem 2.2 and Theorem 3.1. □

Remark 3.2.
The result for the case $K = 1$ in Theorem 2.2 is same as the result for the case $p = 1$ in Theorem 3.2.

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